The Rényi-Ulam Pathological Liar Game with a Fixed Number of Lies

Robert B. Ellis [a],[*],[1] Vadim Ponomarenko [b] Catherine H. Yan [a],[2]

[a] *Department of Mathematics, Texas A&M University, College Station, Texas 77843-3368*

[b] *Mathematics Department, Trinity University, San Antonio, Texas 78212-7200*

Abstract

The q-round Rényi-Ulam pathological liar game with k lies on the set $[n] := \{1, \ldots, n\}$ is a 2-player perfect information zero sum game. In each round Paul chooses a subset $A \subseteq [n]$ and Carole either assigns 1 lie to each element of A or to each element of $[n] \setminus A$. Paul wins if after q rounds there is at least one element with k or fewer lies. The game is dual to the original Rényi-Ulam liar game for which the winning condition is that at most one element has k or fewer lies. We prove the existence of a winning strategy for Paul to the existence of a covering of the discrete hypercube with certain relaxed Hamming balls. Defining $F_k^*(q)$ to be the minimum n such that Paul can win the q-round pathological liar game with k lies and initial set $[n]$, we find $F_1^*(q)$ and $F_2^*(q)$ exactly. For fixed k we prove that $F_k^*(q)$ is within an absolute constant (depending only on k) of the sphere bound, $2^q / \binom{q}{\leq k}$; this is already known to hold for the original Rényi-Ulam liar game due to a result of J. Spencer.

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* Author to whom Correspondence should be sent.
  Email addresses: rellis@math.tamu.edu, vadim@trinity.edu, cyan@math.tamu.edu (Catherine H. Yan).

1 Introduction

In this paper we consider the following 2-player perfect information zero-sum game, which we call the *Rényi-Ulam pathological liar game*, first defined in [4]. The players Paul and Carole play a q-round game on a set of n elements, $[n] := \{1, \ldots, n\}$. Each round, Paul splits the set of elements by choosing a *question* set $A \subseteq [n]$; Carole then completes the round by choosing to assign one *lie* either to each of the elements of A, or to each of the elements of $[n] \setminus A$. A given element is removed from play, or *disqualified*, if it accumulates $k + 1$ lies, where k is a predetermined nonnegative constant; in choosing the question set A, we may consider the game to be restricted to the *surviving* elements, which have $\leq k$ lies. The game starts with each element having no associated lies. If after q rounds at least one element survives, Paul wins; otherwise Carole wins. Thus Paul plays a strategy to preserve at least one element for q rounds, and Carole answers adversely. We think of a capricious or contrary Carole lying "pathologically" in order to disqualify elements as quickly as possible. Our main result, stated as Theorem 3 in Section 2 and proved in Section 4, is a tight asymptotic characterization of the minimum n for which Paul has a winning strategy for the q-round game with a fixed number, k, of lies.

This game arises as the dual to the Rényi-Ulam liar game, originating in [9] and [12], which we refer to as the *original liar game*. The simplest version of the original game is the "20 questions" game in which Paul may ask 20 Yes-No questions in order to identify a distinguished element x from a set $[n]$, where Carole answers "Yes" or "No" without lying. Here, Paul has a winning strategy iff $\log_2 n \leq 20$. In the general version, the number of rounds q and number of elements n are predetermined, as is the number, k, of times Carole is allowed to lie. We take the equivalent viewpoint that the distinguished element is not chosen ahead of time by Carole, but rather that she must answer consistently with there being at least one candidate for the distinguished element at each round. Thus a candidate element $y \in [n]$ cannot be the distinguished element if it would cause Carole to have lied about it $k + 1$ times. Paul's strategy in the original game, therefore, is to win by forcing Carole to associate $k + 1$ lies with all but one element within q rounds, and Carole's strategy is to answer questions adversely so that at least two candidate elements remain after q rounds. Recently, Pelc thoroughly surveyed what is known about the original liar game and many of its variants [7].

The duality between the pathological liar game and the original liar game arises from the choice of Paul's condition to win. In the pathological liar game at least one element must survive for Paul to win, but in the original game at most one element may survive for him to win. The remaining mechanics of the two games are the same, in that each round Paul chooses a question subset $A \subseteq [n]$ and Carole decides to assign lies either to A or to $[n] \setminus A$.

In Section 2, we describe how each stage of the pathological game can be encoded in a $(k+1)$-tuple state vector which keeps track of the number of lies associated with each element. In Section 3 we discuss the Berlekamp weight function on a state vector and how a winning strategy by Paul corresponds to maximizing (minimizing) the weight of the state vector after q rounds in the pathological (original) liar game. In Section 4, we give the value of n, up to a constant independent of q, for which Paul can win the q-round game with a fixed number, k, of lies. In Sections 5 and 6, we give the exact minimum n for which Paul can win the q-round 1-lie and 2-lie games, respectively. Finally, in Section 7, we prove the equivalence of the existence of a winning strategy for Paul in the pathological (original) liar game to the existence of a covering (packing) of the hypercube with certain relaxed Hamming spheres, and discuss the connection to covering codes and error-correcting codes.

2 The vector game format

The mechanics of both the pathological liar game and the original liar game are encapsulated in the following vector framework due to Berlekamp [1]. Given that the game parameters are n elements, q rounds, and k lies, the initial state of the game is the $(k+1)$-vector $(n, 0, \ldots, 0)$. An intermediate stage of the game after some number of rounds is encoded by the *state vector* $\vec{x} = (x_0, x_1, \ldots, x_k)$, where x_i denotes the number of elements of $[n]$ associated with i lies (disqualified elements, with $k+1$ lies, are not tracked by the state vector). The state vector completely encodes a stage of the game because an element of $[n]$ is distinguished only by the number of lies associated with it. Paul chooses a question set $A \subseteq [n]$ corresponding to an integer *question vector* $\vec{a} = (a_0, a_1, \ldots, a_k)$ which must be *legal*, that is, $0 \le a_i \le x_i$ for each $i \in \{0, \ldots, k\}$. Carole answers either "Yes" or "No." By answering "Yes," Carole assigns an additional lie to each element in $[n] \setminus A$, so that the next state vector $Y(\vec{x}, \vec{a})$ is obtained from $\vec{x}$ by moving elements corresponding to $[n] \setminus A$ to the right one position. Analogously, by answering "No," Carole causes the next state vector $N(\vec{x}, \vec{a})$ to arise from moving elements corresponding to A to the right one position. Therefore the subsequent state chosen by Carole is either

$$
\begin{aligned}
Y(\vec{x}, \vec{a}) &:= (\qquad a_0, \ a_1 + x_0 - a_0, \ \ldots, \ a_k + x_{k-1} - a_{k-1}) \quad \text{or} \\
N(\vec{x}, \vec{a}) &:= (\ x_0 - a_0, \ x_1 - a_1 + a_0, \ \ldots, \ \ x_k - a_k + a_{k-1}).
\end{aligned}
\tag{1}
$$

Elements which become associated with $k+1$ lies are considered to be shifted out of the state vector to the right, and so we may consider the question set A and the set of elements $[n]$ to be restricted at any given stage to the surviving elements. In the pathological liar game, Paul wins iff after q rounds

$\sum_{i=0}^{k} x_i \geq 1$ (at least one element survives). In the original liar game, Paul wins iff after q rounds $\sum_{i=0}^{k} x_i \leq 1$.

More generally, we may consider a game starting with an arbitrary nonnegative state vector $\vec{x} = (x_0, \ldots, x_k)$. We will use the following shorthand.

Definition 1 *(i)* *The* $(\vec{x}, q, k)^*$-game *is the q-round pathological liar game with k lies and initial state $\vec{x}$.*
(ii) *The* $(\vec{x}, q, k)$-game *is the q-round original liar game with k lies and initial state $\vec{x}$.*
In either game, the initial state $\vec{x} = (x_0, \ldots, x_k)$ encodes for $0 \leq i \leq k$ the number x_i of elements which are initially associated with i lies.

The k is redundant when $\vec{x}$ is specified. Both games are monotonic in the following sense. Suppose $\vec{x} = (x_0, \ldots, x_k)$, $\vec{y} = (y_0, \ldots, y_k)$, and $0 \leq y_i \leq x_i$ for all $0 \leq i \leq k$; i.e., $\vec{x}$ *covers* $\vec{y}$. If Paul has a strategy to win the $(\vec{y}, q, k)^*$-game (the $(\vec{x}, q, k)$-game), then he has a strategy to win the $(\vec{x}, q, k)^*$-game (the $(\vec{y}, q, k)$-game). The new strategy is obtained from the winning strategy in the pathological game by arbitrarily choosing whether the extra elements corresponding to $x_i - y_i$ are in A or $[n] \setminus A$, and in the original game by restricting all questions A by intersection with the set of all elements represented by $y_0, \ldots, y_k$. In fact, the same monotonicity holds if $\vec{x}$ *majorizes* $\vec{y}$; i.e., if for all $0 \leq j \leq k$, $\sum_{i=0}^{j} y_i \leq \sum_{i=0}^{j} x_i$. Empirically, an element lasts longer in the game if it starts with fewer associated lies. Monotonicity under majorization is an immediate result of Theorem 19, as we will describe in Section 7. We may now define $F_k^*(q)$ to be the minimum number n such that Paul has a winning strategy for the $((n, 0, \ldots, 0), q, k)^*$-game. The previously defined maximum n such that Paul can win the $((n, 0, \ldots, 0), q, k)$-game is $F_k(q)$. Pelc determined $F_1(q)$ exactly in [6], Guzicki determined $F_2(q)$ in [5], Deppe determined $F_3(q)$ in [3], and Spencer determined $F_k(q)$ for fixed k to within a constant independent of q. Of particular importance to this paper is the following result of Spencer, given implicitly in Section 3 of [10].

Theorem 2 (Spencer) *For any fixed nonnegative integer k there exist constants q_k, C_k such that for all $q \geq q_k$,*

$$\frac{2^q}{\binom{q}{\leq k}} - C_k \ \leq \ F_k(q) \ \leq \ \frac{2^q}{\binom{q}{\leq k}}.$$

Here, $\binom{q}{\leq k} := \sum_{i=0}^{k} \binom{q}{i}$ is the size of a radius k Hamming ball in the q-dimensional discrete hypercube Q_q (Section 7 explores this further). The main result of this paper, which we prove in Section 4, is the following dual of Theorem 2.

Theorem 3 *For any fixed nonnegative integer k there exist constants q_k^*, C_k^**

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such that for all $q \geq q_k^*$,

$$\frac{2^q}{\binom{q}{\leq k}} \leq F_k^*(q) \leq \frac{2^q}{\binom{q}{\leq k}} + C_k^*.$$

3 The Berlekamp weight function

For a nonnegative integer q and a state vector $\vec{x} = (x_0, \ldots, x_k)$, the q-weight of $\vec{x}$ is defined to be

$$wt_q(\vec{x}) := \sum_{i=0}^{k} x_i \binom{q}{\leq k - i}. \tag{2}$$

This is the *Berlekamp weight function* introduced in [1]. The number of ways to select positions for at most $k - i$ lies in a sequence of Y/N responses by Carole of length q is $\binom{q}{\leq k-i}$, which motivates the weight of an element counted by x_i. We will abuse notation and denote $wt_q((x_0, \ldots, x_k))$ by $wt_q(x_0, \ldots, x_k)$. We will see that Carole can always win the $(\vec{x}, q, k)^*$-game when $wt_q(\vec{x}) < 2^q$. Intuitively, elements with fewer associated lies are worth more toward a win by Paul. To borrow an analogy from [10], we can think of the x_i's as representing *coins* of various denominations, where we call the coins with smallest weight, counted by x_k, *pennies*. We now present a well-known conservation lemma concerning the weight function, previously appearing in [1].

Lemma 4 (Conservation of weight) *Let $q \geq 1$, let $\vec{x}$ be a state vector, and let $\vec{a}$ be a legal question for $\vec{x}$. Then*

$$wt_q(\vec{x}) = wt_{q-1}(Y(\vec{x}, \vec{a})) + wt_{q-1}(N(\vec{x}, \vec{a})).$$

PROOF. Using (1) and (2), we compute

$$wt_{q-1}(Y(\vec{x}, \vec{a})) + wt_{q-1}(N(\vec{x}, \vec{a})) = x_0 \binom{q-1}{\leq k} + \sum_{i=1}^{k} (x_i + x_{i-1}) \binom{q-1}{\leq k - i}$$

$$= \sum_{i=0}^{k} x_i \left(\binom{q-1}{\leq k - i} + \binom{q-1}{\leq k - i - 1} \right) = wt_q(\vec{x}),$$

by repeated use of the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. $\quad\square$

The lemma illustrates that Carole's choice in answering "Yes" or "No" to a question by Paul induces a choice of weight of the resulting state vector. In particular, Carole might always choose the resulting state with lower weight, giving a constraint on Paul's ability to win the $(\vec{x}, q, k)^*$-game which holds for

any k. We call the following lemma the *sphere bound* because of a connection to the sphere bound of coding theory to be made clear after Theorem 19.

Lemma 5 (Sphere bound) *Let $q, k \geq 0$ and let $\vec{x} = (x_0, \ldots, x_k)$ be a non-negative vector. If $wt_q(\vec{x}) < 2^q$, then Carole can win the q-round pathological liar game with k lies and initial state $\vec{x}$. Consequently, $F_k^*(q) \geq 2^q / \binom{q}{\leq k}$.*

PROOF. Regardless of Paul's initial question, by Lemma 4 Carole may respond so that the resulting state has weight at most $wt_q(\vec{x})/2 < 2^{q-1}$. By induction, Carole may respond to Paul's remaining $q - 1$ questions to ensure the 0-weight of the final state is < 1. Since the state vector must always be integer, Carole can always force the vector $(0, \ldots, 0)$ in q rounds. $\square$

In the original game, the analog to the above lemma is that Carole has a strategy to win the $(\vec{x}, q, k)$-game when $wt_q(\vec{x}) > 2^q$. This is proved in [10] by showing that if Carole answers randomly at each stage, the probability that the final weight is > 1 is nonzero, and thus Carole has a winning strategy since it is a perfect information game. The proof of Lemma 5 could be rewritten from this randomized perspective.

Lemma 5 shows that a necessary condition for Paul to win the q-round pathological liar game with starting state $\vec{x}$ is that $wt_q(\vec{x}) \geq 2^q$, but in general this is not sufficient. Paul is not always able to choose a question which balances the weights of the possible next states. Given some intermediate state $\vec{x}$ with $j + 1$ rounds remaining and a question $\vec{a}$, the resulting *weight imbalance* between possible next states is defined as (cf. Section 2 of [10])

$$\Delta_j(\vec{x}, \vec{a}) := wt_j(Y(\vec{x}, \vec{a})) - wt_j(N(\vec{x}, \vec{a})). \tag{3}$$

The following is a counterexample to the converse of Lemma 5.

Example 6 *Let $\vec{x} = (3, 1)$ be the initial state of a $((3, 1), 4, 1)^*$-game. Note that $wt_4((3, 1)) = 3 \cdot 5 + 1 \cdot 1 = 16$, and so Paul could possibly have a winning strategy. But any first-round question $\vec{a}$ by Paul will satisfy $|\Delta_3(\vec{x}, \vec{a})| \geq 2$. One question minimizing $|\Delta_3(\vec{x}, \vec{a})|$ is $\vec{a} = (1, 1)$, for which $Y(\vec{x}, \vec{a}) = (1, 3)$, $N(\vec{x}, \vec{a}) = (2, 1)$, and $\Delta_3(\vec{x}, \vec{a}) = 7 - 9 = -2$. In any event, Carole responds so that the next state has 3-weight at most 7, guaranteeing herself to win the game.*

Paul's goal in the pathological liar game, in terms of the weight function, corresponds to maximizing the 0-weight of the game state after q rounds. The capability to identify situations in which he can choose "perfectly balancing" questions at every stage so that $\Delta_j(\vec{x}, \vec{a}) = 0$ would provide a partial con-

verse to Lemma 5; however, this is sometimes impossible (cf. Example 6), and difficult to know if it is possible when initially the q-weight is close to 2^q.

4 Asymptotics of the k-lie game

Since the full converse to Lemma 5 is impossible, we instead wish to identify the states $\vec{x}$ having $wt_q(\vec{x})$ close to 2^q for which Paul can win the $(\vec{x}, q, k)^*$-game. As Spencer proved in [10], there is a large category of states $\vec{x} = (x_0, \ldots, x_k)$ such that if $wt_q(\vec{x}) = 2^q$ and x_k is large enough, then Paul can find q questions which make the weight imbalance vanish *at each stage*. Intuitively two processes are at work. If there are enough "pennies," counted by x_k, then $\vec{a}$ can be chosen so that the weights of the two possible next states $Y(\vec{x}, \vec{a})$ and $N(\vec{x}, \vec{a})$ are exactly equal. The number of pennies in the next state is maintained sufficiently by drawing from x_{k-1} and x_k. To employ Spencer's result, it will suffice to begin with $\vec{x}$ having q-weight slightly more than 2^q and reduce in k rounds to a state $\vec{y}$ with $(q-k)$-weight exactly 2^{q-k} for which Spencer's theorem holds. Here now is Spencer's result, essentially appearing as the "Main Theorem" in Section 2 of [10], in a form convenient for our purposes.

Theorem 7 (Spencer) *Let k be fixed. There are constants c, q_0 (dependent on k) so that the following holds for all $q \geq q_0$: if $wt_q(x_0, \ldots, x_k) = 2^q$ and $x_k > cq^k$, then Paul has a strategy to reach a state $\vec{z}$ with $wt_0(\vec{z}) = 1$ in exactly q rounds such that every intermediate state $(u_0, \ldots, u_k)$ after playing j rounds satisfies $wt_{q-j}(u_0, \ldots, u_k) = 2^{q-j}$.*

Theorem 8 *Let k be fixed. There are constants c_1, q_k^* (dependent on k) so that the following holds for all $q \geq q_k^*$: if $wt_q(x_0, \ldots, x_k) \geq 2^q + c_1 \binom{q}{k}$, then Paul can win the q-round pathological liar game with k lies and initial state $\vec{x} = (x_0, \ldots, x_k)$.*

PROOF. The proof proceeds in three main stages. First, the first k rounds of the game are played with a "floor-ceiling" question strategy which ensures that the resulting state $\vec{y'}$ satisfies $wt_{q-k}(\vec{y'}) \geq 2^{q-k}$. Second, coins are removed from $\vec{y'}$ to obtain $\vec{y}$ with $(q-k)$-weight exactly 2^{q-k}. Finally, Theorem 7 is applied to $\vec{y}$ to reach a state $\vec{z}$ with $wt_0(\vec{z}) = 1$ after an additional $q-k$ rounds.

Paul plays the first k rounds of the game, reaching the state $\vec{y'} = (y'_0, \ldots, y'_k)$, according to the following strategy which is oblivious to Carole's responses. If $\vec{u}(j) = (u_0(j), \ldots, u_k(j))$ is the state when j rounds remain, then for $q \geq j > q - k$, Paul's next question $\vec{a}(j) = (a_0(j), \ldots, a_k(j))$ is defined by letting $a_i(j) = \lfloor u_i(j)/2 \rfloor$ or $\lceil u_i(j)/2 \rceil$, so that the least i for which $u_i(j)$ is odd results

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in choosing $a_i(j) = \lceil u_i(j)/2 \rceil$, and the overall choice of floors and ceilings for the odd $u_i(j)$'s alternates.

By combining (1) and (2) with the definition of Δ_j in (3), the weight imbalance of the two possible next states when $j + 1$ rounds remain is at most

$$\Delta_j(\vec{u}(j+1), \vec{a}(j+1)) = \sum_{i=0}^{k}(2a_i(j+1) - u_i(j+1))\binom{j}{k-i} \leq \binom{j}{k}, \quad (4)$$

where we know the value is nonnegative by definition of $\vec{a}(j+1)$. By Lemma 4 and (4), we have for each intermediate state $\vec{u}(j+1)$ (with indexes $j+1$ suppressed for clarity)

$$wt_j(Y(\vec{u}, \vec{a})) \geq wt_j(N(\vec{u}, \vec{a})) \geq \frac{wt_{j+1}(\vec{u}) - \binom{j}{k}}{2}.$$

Therefore with an initial state of weight

$$wt_q(\vec{x}) \geq 2^q + c_1\binom{q}{k} \geq 2^q + \sum_{j=q-1}^{q-k} 2^{q-1-j}\binom{j}{k},$$

for some constant c_1 and $q \geq q_1$ large enough, Paul can guarantee a state $\vec{y'}$ with $wt_{q-k}(\vec{y'}) \geq 2^{q-k}$ after k rounds.

The number of pennies y'_k after k rounds is large, by the following argument. Since $wt_q(\vec{x}) \geq 2^q$ and the largest weight of an element is $\binom{q}{\leq k} \leq q^k$, then $\sum_{i=0}^{k} x_i \geq 2^q/q^k$. Thus there exists a coordinate i_0 for which $x_{i_0} \geq 2^q/\big((k+1)q^k\big)$. By definition of the first k questions,

$$y'_k = u_k(q-k) \geq \lfloor 2^{-1}u_k(q-k+1) \rfloor \geq \cdots \geq \lfloor 2^{-i_0}u_k(q-k+i_0) \rfloor$$
$$\geq \lfloor 2^{-i_0-1}u_{k-1}(q-k+i_0+1) \rfloor \geq \cdots \geq \lfloor 2^{-k}u_{i_0}(q) \rfloor = x_{i_0}$$
$$\geq \left\lfloor 2^{-k} \cdot \frac{2^q}{(k+1)q^k} \right\rfloor \geq c_2 q^k.$$

The first line is true because $u_k(j)$ is at least $\lfloor u_k(j+1)/2 \rfloor$, the second line is true because $u_i(j)$ is at least $\lfloor u_{i-1}(j+1)/2 \rfloor$, and the last inequality is true for any choice of c_2 and $q \geq q_2$ provided q_2 is taken to be large enough. We note that the choice of c_1 does not affect the choice of c_2 in this analysis.

Now obtain the state $\vec{y} = (y_0, \ldots, y_k)$ with $(q-k)$-weight 2^{q-k} from $\vec{y'}$ by greedily removing coins of decreasing weight, so that either only 2^{q-k} pennies are left, or fewer than $\binom{q-k}{\leq k}$ pennies were removed. In the first case Paul trivially can make the game last another $q - k$ rounds; in the second case at

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least

$$y_k \geq c_2 q^k - \binom{q-k}{\leq k} \geq c_3 (q-k)^k$$

pennies remain. The constant c_3 can be chosen to be at least $c_2 - 1$, for instance, provided that $q \geq q_3$ for q_3 large enough. Choose c_3 and $q_k^* \geq \max\{q_1, q_2, q_3\}$ large enough so that c_3 and $q_k^* - k$ satisfy the requirements of Theorem 7 for the $(\vec{y}, q-k, k)$-game. Therefore Paul can win the $(\vec{x}, q, k)^*$-game. □

PROOF. (Proof of Theorem 3.) From Lemma 5, $F_q^*(k) \geq 2^q / \binom{q}{\leq k}$. Now suppose $q \geq q_k^*$ and let $n = \lceil (2^q + c_1 \binom{q}{k}) / \binom{q}{\leq k} \rceil$, where c_1 and q_k^* are as in Theorem 8. Then $wt_q(n, 0, \ldots, 0) \geq 2^q + c_1 \binom{q}{k}$ and $F_k^*(q) \leq n \leq \lceil (2^q + c_1 \binom{q}{k}) / \binom{q}{\leq k} \rceil \leq 2^q / \binom{q}{\leq k} + C_k^*$ for $q \geq q_k^*$ and some constant C_k^*. □

We remark that the excess weight above 2^q in Theorem 8 is needed so that Paul can guarantee a $(q-k)$-weight of 2^{q-k} after the first k rounds and go on to win when q is large enough. The exact excess required is difficult to compute for general k. However, in the next two sections we will compute the exact amount required for $k = 1$ and 2 for any q, not just when q is large enough.

5 Exact result for the 1-lie game

We now consider the q-round pathological liar game with 1 lie and initial state $(n, 0)$. For this section, define the *character* $ch(x_0, x_1)$ of a state (x_0, x_1) to be the maximum q such that $wt_q(x_0, x_1) \geq 2^q$. Furthermore, denote by (y_0, y_1) the game state immediately following the state (x_0, x_1) and Paul's question (a_0, a_1), so that $(y_0, y_1) = (a_0, a_1 + x_0 - a_0)$ or $(x_0 - a_0, a_0 + x_1 - a_1)$, depending on Carole's response of "Y" or "N," respectively. The next theorem completely characterizes the values of n for which Paul can win the $((n, 0), q, 1)^*$-game.

Theorem 9 *Let $q \geq 0$. Paul has a winning strategy for the q-round pathological liar game with 1 lie and initial state $(n, 0)$ iff*

$$2^q \leq \begin{cases} n(q+1) & \text{if } n \text{ is even,} \\ n(q+1) - (q-1) & \text{if } n \text{ is odd.} \end{cases} \tag{5}$$

The difference in the even and odd cases reflects the fact that when n is odd, Paul's first question is forced to be inefficient, as there is no way to balance

a_0 with $x_0 - a_0$. By considering the possibilities for $\lceil 2^q/(q+1) \rceil \bmod 2$ and $2^q \bmod q + 1$, it is not difficult to obtain the following.

Corollary 10 *Let $SB_1^* := \lceil 2^q/(q+1) \rceil$ be the sphere bound for the $((n,0),q,1)^*$-game. Then*

$$F_1^*(q) = \begin{cases} SB_1^*, & \text{if } SB_1^* \text{ is odd and } (2^q \bmod q + 1) \in \{1,2\}, \\ 2\lceil SB_1^*/2 \rceil, & \text{otherwise.} \end{cases}$$

The proof of Theorem 9 follows in one direction by Lemma 11, and the other direction will be proved after Lemmas 12 and 13. This proof technique is based on that of Pelc's theorem in Section 2 of [6], which states that the characterization for Paul having a winning strategy for the $((n,0),q,1)$-game is obtained from (5) by reversing the inequality.

Lemma 11 *Let $q \geq 0$. Carole can win the q-round pathological liar game with 1 lie and initial state $(n,0)$ provided*

$$2^q > \begin{cases} n(q+1) & \text{if } n \text{ is even,} \\ n(q+1) - (q-1) & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. The case of n even follows directly from Lemma 5, since $wt_q(n,0) = n(q+1)$. If n is odd, observe that whatever Paul's first question is, Carole may respond so that in the resulting state (y_0, y_1), $y_0 < y_1$, and so

$$wt_{q-1}(y_0, y_1) \leq \frac{n-1}{2}q + \frac{n+1}{2} = \frac{n(q+1)-(q-1)}{2} < 2^{q-1}.$$

Now apply Lemma 5 to show that Carole can win the $((y_0, y_1), q-1, 1)^*$-game. $\square$

The next lemma handles the late rounds of the game for which there is at most 1 element with no accumulated lies.

Lemma 12 *Paul can win the q-round pathological liar game with 1 lie and initial state (x_0, x_1) provided $0 \leq x_0 \leq 1$ and $q \leq ch(x_0, x_1)$.*

PROOF. Without loss of generality, assume $q = ch(x_0, x_1)$. We prove the lemma by induction on q, by exhibiting a question Paul can ask that will not reduce the character by more than one. Since $q = ch(x_0, x_1)$, $wt_q(x_0, x_1) = (q+1)x_0 + x_1 \geq 2^q$.

If $x_0 = 0$, then $x_1 = wt_q(x_0, x_1) \geq 2^q$; if Paul chooses the question $\vec{a} = (0, \lfloor \frac{x_1}{2} \rfloor)$, then $y_1 = \lfloor \frac{x_1}{2} \rfloor$ or $\lceil \frac{x_1}{2} \rceil$. In either case, $wt_{q-1}(y_0, y_1) \geq \lfloor \frac{2^q}{2} \rfloor \geq 2^{q-1}$, and so $ch(y_0, y_1) \geq q - 1$.

If $x_0 = 1$, set $a_0 = 1$ and $a_1 = \lfloor \frac{x_1 + 1 - q}{2} \rfloor$. Observe that $a_1 \geq 0$, since otherwise $q > x_1 + 1$ and $2q > q + 1 + x_1 = wt_q(1, x_1) \geq 2^q$, which is impossible. Paul then asks $\vec{a} = (1, a_1)$, and Carole can choose between $(1, a_1)$ or $(0, x_1 + 1 - a_1)$. The weight imbalance is $|\Delta_{q-1}(\vec{x}, \vec{a})| = |(q + a_1) - (x_1 + 1 - a_1)| = |q - x_1 - 1 + 2a_1| \leq 1$. By Lemma 4 and because 2^q is even, we have $wt_{q-1}(y_0, y_1) \geq 2^{q-1}$. Hence $ch(y_0, y_1) \geq q - 1$. $\square$

We now show that certain state vectors (x_0, x_1) in the game allow Paul a question which guarantees that the next state has three narrow constraints, including a character reduced by at most one.

Lemma 13 *Let (x_0, x_1) be a state with $ch(x_0, x_1) \geq 1$ and $x_1 \geq x_0 - 1 \geq 1$. Then there exists a question (a_0, a_1) such that regardless of Carole's answer the next state (y_0, y_1) will satisfy:*

$$\left\lfloor \frac{x_0}{2} \right\rfloor \leq y_0 \leq \left\lceil \frac{x_0}{2} \right\rceil \tag{6}$$

$$y_1 \geq y_0 - 1 \tag{7}$$

$$ch(y_0, y_1) \geq ch(x_0, x_1) - 1. \tag{8}$$

PROOF. Without loss of generality, assume $q = ch(x_0, x_1)$. The proof depends on whether x_0 is even or odd. *Case 1 (x_0 is even)*. Paul chooses the legal question $\vec{a} = (\frac{x_0}{2}, \lfloor \frac{x_1}{2} \rfloor)$ so that $(y_0, y_1) = (\frac{x_0}{2}, \frac{x_0}{2} + \lfloor \frac{x_1}{2} \rfloor)$ or $(\frac{x_0}{2}, \frac{x_0}{2} + \lceil \frac{x_1}{2} \rceil)$. Regardless of Carole's response, $y_0 = \frac{x_0}{2}$, satisfying condition (6); also, $y_1 \geq \frac{x_0}{2} + \lfloor \frac{x_1}{2} \rfloor \geq y_0 - 1$, satisfying condition (7). Finally, since 2^q is even and $|\Delta_{q-1}(\vec{x}, \vec{a})| = \lceil \frac{x_1}{2} \rceil - \lfloor \frac{x_1}{2} \rfloor \leq 1$, we have $wt_{q-1}(y_0, y_1) \geq 2^{q-1}$, and so condition (8) is satisfied.

Case 2 (x_0 is odd). Paul chooses $\vec{a} = (\frac{x_0 + 1}{2}, \lceil \frac{x_1 - q + 1}{2} \rceil)$, so that $(y_0, y_1) = (\frac{x_0 + 1}{2}, \frac{x_0 - 1}{2} + \lceil \frac{x_1 - q + 1}{2} \rceil)$ or $(\frac{x_0 - 1}{2}, \frac{x_0 + 1}{2} + x_1 - \lceil \frac{x_1 - q + 1}{2} \rceil)$. To show the question is legal, we require $a_1 = \lceil \frac{x_1 - q + 1}{2} \rceil \geq 0$. Otherwise, $x_1 - q + 1 < -1$, or $x_1 \leq q - 3$, and so $x_0 \leq q - 2$. With this assumption on x_0 and x_1, $2^q \leq wt_q(x_0, x_1) \leq (q + 1)(q - 2) + q - 3 = q^2 - 5$, which is impossible for $q \geq 0$, and so the question is legal. Continuing, clearly condition (6) holds. If Carole answers "Y," $y_1 - y_0 + 1 = \lceil \frac{x_1 - q + 1}{2} \rceil$, which is at least 0. If Carole answers "N," $y_1 - y_0 + 1 = x_1 - \lceil \frac{x_1 - q + 1}{2} \rceil + 2$, which is clearly nonnegative. Thus condition (7) holds. Again, 2^q is even, and $|\Delta_{q-1}(\vec{x}, \vec{a})| = |2\lceil \frac{x_1 - q + 1}{2} \rceil - (x_1 - q + 1)| \leq 1$; therefore $wt_{q-1}(y_0, y_1) \geq 2^{q-1}$ and condition (8) holds. $\square$

We now finish the proof of the theorem by handling the first round, applying Lemma 13 until $x_0 \leq 1$, and by applying Lemma 12 until $ch(x_0, x_1) = 0$.

PROOF. (Proof of Theorem 9.) By Lemma 11, we may assume that n satisfies (5). For even $n = 2m$, Paul chooses $\vec{a} = (m, 0)$ for his first question so that the next state is forced to be $(y_0, y_1) = (m, m)$. By Lemma 4 and the hypothesis, $wt_{q-1}(m, m) \geq 2^{q-1}$, and so $ch(m, m) \geq q - 1$. If $m = 1$, we apply Lemma 12 to have Paul ask $q - 1$ more questions. Otherwise, $m > 1$, and (m, m) satisfies the requirements of Lemma 13. We apply it repeatedly until we reach a state of the form $(1, u)$. The lemma assures us that this will happen in t steps, where $\lfloor \log_2(m) \rfloor \leq t \leq \lceil \log_2(m) \rceil$. At the conclusion, we will have $ch(1, u) \geq q - 1 - t$. Then, applying Lemma 12, Paul can ask at least $q - 1 - t$ further questions. Therefore, altogether he has asked $1 + t + (q - 1 - t) = q$ questions.

For odd $n = 2m + 1$, Paul chooses $\vec{a} = (m + 1, 0)$ for his first question. Carole can then choose $(y_0, y_1) = (m + 1, m)$ or $(m, m + 1)$ as the next state. We see that $wt_{q-1}(y_0, y_1) \geq mq + m + 1 = \frac{2mq + 2m + 2}{2} = \frac{n(q+1) - (q-1)}{2} \geq 2^{q-1}$, by hypothesis. Hence regardless of Carole's response, $ch(y_0, y_1) \geq q - 1$. The rest of the proof mimics the case for even n. □

6 Exact result for the 2-lie game

We now consider the q-round pathological liar game with 2 lies and initial state $(n, 0, 0)$. The next theorem completely characterizes the values of n for which Paul can win the $((n, 0, 0), q, 2)^*$-game. Its proof follows some definitions and two lemmas focusing on the first two rounds and then the rest of the game.

Theorem 14 *Let $q \geq 0$. Paul has a winning strategy for the q-round pathological liar game with 2 lies and initial state $(n, 0, 0)$ iff*

$$2^q \leq n\binom{q}{\leq 2} - A\binom{q-1}{2} - B\binom{q-2}{1}, \tag{9}$$

where $A = n \bmod 2$ and

$$B = \begin{cases} 0, & \text{if } n \equiv 0 \mod 4, \\ 2 \cdot (q \bmod 2), & \text{if } n \equiv 1 \mod 4, \\ (1 - q^3) \bmod 4, & \text{if } n \equiv 2 \mod 4, \\ (1 + q^3) \bmod 4, & \text{if } n \equiv 3 \mod 4. \end{cases}$$

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We say that Paul *survives* the first two rounds of the $((n,0,0),q,k)^*$-game provided he has a strategy which guarantees that the $(q-2)$-weight of the state after two rounds is at least 2^{q-2} regardless of Carole's responses. Let $\vec{a}$ be Paul's first question, and if Carole's response is "Y" ("N"), then let Paul's second question be $\vec{b^Y}$ ($\vec{b^N}$). Then Paul can survive the first two rounds iff

$$2^{q-2} \leq \max_{\vec{a},\vec{b^Y},\vec{b^N}} \min \left\{ wt_{q-2}(Y(Y((n,0,0),\vec{a}),\vec{b^Y}), wt_{q-2}(N(Y((n,0,0),\vec{a}),\vec{b^Y}), \right.$$

$$\left. wt_{q-2}(Y(N((n,0,0),\vec{a}),\vec{b^N}), wt_{q-2}(N(N((n,0,0),\vec{a}),\vec{b^N}) \right\},$$

where $\vec{a}$, $\vec{b^Y}$, and $\vec{b^N}$ must be legal questions when they are asked. Now define weight imbalances

$$\Delta_{q-1} := \Delta_{q-1}((n,0,0),\vec{a}),$$
$$\Delta_{q-2}^Y := \Delta_{q-2}(Y((n,0,0),\vec{a}),\vec{b^Y}), \quad \text{and}$$
$$\Delta_{q-2}^N := \Delta_{q-2}(N((n,0,0),\vec{a}),\vec{b^N});$$

where without loss of generality, we choose the questions $\vec{a}$, $\vec{b^Y}$, and $\vec{b^N}$ so that Δ_{q-1}, Δ_{q-2}^Y, and Δ_{q-2}^N are *nonnegative* (for instance, by replacing $\vec{a}$ with $\vec{x} - \vec{a}$). By Lemma 4 and (3), Paul can survive the first two rounds of the $((n,0,0),q,2)^*$-game iff

$$2^q \leq wt_q(n,0,0) + \Delta, \qquad \text{where}$$
$$\Delta := \max_{\vec{a},\vec{b^Y},\vec{b^N}} \min \left\{ \Delta_{q-1} + 2\Delta_{q-2}^Y, \Delta_{q-1} - 2\Delta_{q-2}^Y, \right.$$
$$\left. -\Delta_{q-1} + 2\Delta_{q-2}^N, -\Delta_{q-1} - 2\Delta_{q-2}^N \right\}. \tag{10}$$

We have reduced the problem to finding the value of Δ because, given a fixed first question $\vec{a}$, we may refer to Section 5 of [5] to compute $\vec{b^Y}$ and $\vec{b^N}$ minimizing Δ_{q-2}^Y and Δ_{q-2}^N, respectively.

Lemma 15 *Let* $q \geq 19$, $n \geq 2^q / \binom{q}{\leq 2}$, *and let* Δ *be defined as in (10) for the q-round pathological liar game with 2 lies and initial state* $(n,0,0)$. *Then*

$$\Delta = -A\binom{q-1}{2} - B\binom{q-2}{1},$$

where A and B are defined as in Theorem 14. Furthermore, Paul's strategy achieving Δ guarantees at least $(q-2)^2 + \binom{q-2}{\leq 2}$ pennies after the first two rounds.

PROOF. Write $n = 4p + r$ and $q - 2 = 4l + s$, where $0 \leq r, s < 4$. We consider cases of the initial state $(n,0,0)$ based on the values of r and s. In

each case, there is only one choice of $\vec{a}$ achieving Δ because any other choice of $\vec{a}$ results in $-\Delta_{q-1} - 2\Delta_{q-2}^{\mathrm{N}} < \Delta$ (recall that, without loss of generality, $\vec{a}$, $\vec{b^{\mathrm{Y}}}$ and $\vec{b^{\mathrm{N}}}$ are chosen to make Δ_{q-1}, $\Delta_{q-2}^{\mathrm{Y}}$ and $\Delta_{q-2}^{\mathrm{N}}$ nonnegative). We give Paul's strategy for achieving Δ by listing the questions $\vec{a}$, $\vec{b^{\mathrm{Y}}}$ and $\vec{b^{\mathrm{N}}}$ in each case explicitly. Guzicki proved that the choices below of $\vec{b^{\mathrm{Y}}}$ and $\vec{b^{\mathrm{N}}}$ minimize $\Delta_{q-2}^{\mathrm{Y}}$ and $\Delta_{q-2}^{\mathrm{N}}$; we omit the details and refer the interested reader in Section 5 of [5]. The calculations for the minimum q for which all questions are legal and for which the resulting states have at least $(q-2)^2$ pennies are tedious but straightforward, and thus omitted.

Case $n = 4p$. Set $\vec{a} = (2p, 0, 0)$ and $\vec{b^{\mathrm{Y}}} = \vec{b^{\mathrm{N}}} = (p, p, 0)$ to achieve $\Delta = 0$ with unique possible resulting state $(p, 2p, p)$. The resulting state has $p \geq (q-2)^2 + \binom{q-2}{\leq 2}$ pennies when $q \geq 19$.

Case $n = 4p + 1$. Set $\vec{a} = (2p+1, 0, 0)$ and $\vec{b^{\mathrm{Y}}} = (p+1, p, 0)$ in each subcase, so that $\Delta_{q-1} = \binom{q-1}{2}$ and two possible resulting states are $(p+1, 2p, p)$ and $(p, 2p+1, p)$. *Subcase $2 \nmid (q-2)$.* Set $\vec{b^{\mathrm{N}}} = (p, p+1, 0)$ to achieve $\Delta = -\binom{q-1}{2} - 2\binom{q-2}{1}$ with additional possible resulting state $(p, 2p, p+1)$. *Subcase $2 \mid (q-2)$.* Set $\vec{b^{\mathrm{N}}} = (p+1, p - \frac{q-2}{2} + 1, 0)$ to achieve $\Delta = -\binom{q-1}{2}$ with additional possible resulting states $(p+1, 2p - \frac{q-2}{2}, p + \frac{q-2}{2})$ and $(p-1, 2p + \frac{q-2}{2} + 1, p - \frac{q-2}{2} + 1)$. All questions are legal when $q \geq 9$, and all resulting states have at least $(q-2)^2 + \binom{q-2}{\leq 2}$ pennies when $q \geq 19$.

Case $n = 4p + 2$. Set $\vec{a} = (2p+1, 0, 0)$ in each subcase. *Subcase $q - 2 = 4l$.* Set $\vec{b^{\mathrm{Y}}} = \vec{b^{\mathrm{N}}} = (p+1, p - \frac{q-2}{4} + 1, 0)$ to achieve $\Delta = -\binom{q-2}{1}$ with possible resulting states $(p+1, 2p - \frac{q-2}{4} + 1, p + \frac{q-2}{4})$ and $(p, 2p + \frac{q-2}{4} + 1, p - \frac{q-2}{4} + 1)$. *Subcase $q - 2 = 4l + 1$.* Set $\vec{b^{\mathrm{Y}}} = \vec{b^{\mathrm{N}}} = (p, p + \frac{q-3}{4} + 1, 0)$ to achieve $\Delta = -2\binom{q-2}{1}$ with possible resulting states $(p, 2p + \frac{q-3}{4} + 2, p - \frac{q-3}{4})$ and $(p+1, 2p - \frac{q-3}{4}, p + \frac{q-3}{4} + 1)$. *Subcase $q - 2 = 4l + 2$.* Set $\vec{b^{\mathrm{Y}}} = \vec{b^{\mathrm{N}}} = (p, p + \frac{q-4}{4} + 1, 0)$ to achieve $\Delta = -\binom{q-2}{1}$ with possible resulting states $(p, 2p + \frac{q-4}{4} + 2, p - \frac{q-4}{4})$ and $(p+1, 2p - \frac{q-4}{4}, p + \frac{q-4}{4} + 1)$. *Subcase $q - 2 = 4l + 3$.* Set $\vec{b^{\mathrm{Y}}} = \vec{b^{\mathrm{N}}} = (p, p + \frac{q-5}{4} + 1, 0)$ to achieve $\Delta = 0$ with possible resulting states $(p, 2p + \frac{q-5}{4} + 2, p - \frac{q-5}{4})$ and $(p+1, 2p - \frac{q-5}{4}, p + \frac{q-5}{4} + 1)$. All questions are legal when $q \geq 8$, and all resulting states have at least $(q-2)^2 + \binom{q-2}{\leq 2}$ pennies when $q \geq 19$.

Case $n = 4p + 3$. Set $\vec{a} = (2p+2, 0, 0)$ and $\vec{b^{\mathrm{Y}}} = (p+1, p+1, 0)$ in each subcase so that two possible resulting states are always $(p+1, 2p+2, p)$ and $(p+1, 2p+1, p+1)$. *Subcase $q - 2 = 4l$.* Set $\vec{b^{\mathrm{N}}} = (p, p + \frac{q-2}{4} + 1, 0)$ to achieve $\Delta = -\binom{q-1}{2} - \binom{q-2}{1}$ with additional possible resulting states $(p, 2p + \frac{q-2}{4} + 2, p - \frac{q-2}{4} + 1)$ and $(p+1, 2p - \frac{q-2}{4} + 1, p + \frac{q-2}{4} + 1)$. *Subcase $q - 2 = 4l + 1$.* Set

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$b\vec{N} = (p, p+\frac{q-3}{4}+1, 0)$ to achieve $\Delta = -\binom{q-1}{2}$ with additional possible resulting states $(p, 2p+\frac{q-3}{4}+2, p-\frac{q-3}{4}+1)$ and $(p+1, 2p-\frac{q-3}{4}+1, p+\frac{q-3}{4}+1)$. *Subcase* $q-2 = 4l+2$. Set $b\vec{N} = (p+1, p-\frac{q-4}{4}+1, 0)$ to achieve $\Delta = -\binom{q-1}{2} - \binom{q-2}{1}$ with additional possible resulting states $(p+1, 2p-\frac{q-4}{4}+1, p+\frac{q-4}{4}+1)$ and $(p, 2p+\frac{q-4}{4}+2, p-\frac{q-4}{4}+1)$. *Subcase* $q-2 = 4l+3$. Set $b\vec{N} = (p, p+\frac{q-5}{4}+2, 0)$ to achieve $\Delta = -\binom{q-1}{2} - 2\binom{q-2}{1}$ with additional possible resulting states $(p, 2p+\frac{q-5}{4}+3, p-\frac{q-5}{4})$ and $(p+1, 2p-\frac{q-5}{4}, p+\frac{q-5}{4}+2)$. All questions are legal when $q \geq 8$, and all resulting states have at least $(q-2)^2 + \binom{q-2}{\leq 2}$ pennies when $q \geq 19$. $\square$

Lemma 16 *Let $q \geq 23$. If $wt_q(x_0, x_1, x_2) = 2^q$ and $x_2 \geq q^2$, then Paul has a strategy to reach a state $\vec{z}$ with $wt_0(\vec{z}) = 1$ in exactly q rounds such that every intermediate state (u_0, u_1, u_2) after playing $q - j$ rounds satisfies $wt_j(u_0, u_1, u_2) = 2^j$.*

PROOF. The proof proceeds by showing how Spencer's "Main Theorem" of [10, Section 2], quoted here as Theorem 7, can be tightened in the case $k = 2$ so that we may take $c = 1$ and $q_0 = 23$. Spencer's technique is to relax the game to allow the pennies position to take on negative integer values in both questions and resulting states in *fictitious play*, and then to show in fact that this position never goes negative for a given c and q_0.

Before stating and proving the three claims which tighten Spencer's result, we recall the necessary notation and results from [10] for the case $k = 2$. Assume there are $j + 1$ rounds remaining, and the current position is $\vec{P} = (p_0, p_1, p_2)$ with weight 2^{j+1}.

Fictitious play: Paul selects the next question vector (v_0, v_1, v_2) according to the parity of p_0 and p_1 as follows. If p_0 is odd, then $v_0 = \frac{p_0+1}{2}$ and $v_1 = \lfloor \frac{v_1}{2} \rfloor$; otherwise if p_0 is even, then $v_0 = \frac{p_0}{2}$ and $v_1 = \lceil \frac{v_1}{2} \rceil$. Let v_2 be the unique integer that makes the weight imbalance $\Delta_j(\vec{P}, \vec{v}) = 0$. In other words, in fictitious play the weight of the states is exactly halved after each round. Note that by the choices of v_0 and v_1, $\Delta_j(\vec{P}, \vec{v}) - (2v_2 - p_2) \geq 0$. Hence, $v_2 \leq p_2$, and so (v_0, v_1, v_2) is legal whenever $v_2 \geq 0$.

In fictitious play Paul and Carole continue to play formally even though the last entry of the states may turn negative. Let

$$fic(j) = (fic_0(j), fic_1(j), fic_2(j))$$

be the state of the game when there are j rounds remaining. Note that $fic(q) = (x_0, x_1, x_2)$ is simply the initial state of the game, and $fic_0(j), fic_1(j)$ are always non-negative.

Perfect play: When the state is $\vec{P}$, Paul selects $\vec{v} = \vec{P}/2$. This results in $Y(\vec{P}, \vec{v}) = N(\vec{P}, \vec{v})$ and uniquely determines the state $pp(j) = (pp_0(j), pp_1(j), pp_2(j))$ when j rounds remain in the game. When the initial state $pp(q)$ is $\vec{x}$, it is easy to compute that

$$pp_0(j) = \frac{x_0}{2^{q-j}}, \qquad pp_1(j) = \frac{x_1 + x_0\binom{q-j}{1}}{2^{q-j}}.$$

$$pp_2(j) = \frac{x_2 + x_1\binom{q-j}{1} + x_2\binom{q-j}{2}}{2^{q-j}}.$$

Defining $e_i(j) = |pp_i(j) - fic_i(j)|$, Spencer proves $e_0(j) \leq 1$ and $e_1(j) \leq 3$. By replacing the j^k in Spencer's calculations with $\frac{1}{2}\binom{j}{k}$ for $k = 2$, it follows that $|fic_2(j) - \frac{1}{2}(fic_2(j+1) + fic_1(j+1))| \leq \frac{1}{2}\binom{j}{2} + \frac{1}{2}$. Hence $e_2(j) \leq \frac{1}{2}e_2(j+1) + \frac{1}{2}\binom{j}{2} + 2$. By induction, $e_2(j) \leq \binom{j}{2} + 5$.

Now we describe the strategy for Paul: starting from the state (x_0, x_1, x_2) with q-weight 2^q and $x_2 \geq q^2$, Paul plays fictitious play in all rounds. Our analysis now deviates from that of Spencer. We argue that Paul can win by seeing that no entries turn negative and by examining the state Paul reaches at $j = 6$, i.e., when 6 rounds remain. Explicitly, we prove the following claims for $q \geq 23$.

1. $fic_0(6) \leq 1$.
2. $fic_2(j) > 1$ for $j \geq 6$. (Fictitious play questions are legal when $j \geq 6$.)
3. When $j = 6$, the state of the game is not $(1, 5, 7)$, or $(1, 4, 14)$.

If the above claims are true, then the possible states at $j = 6$ are $(1, 3, 21)$, $(1, 2, 18)$, $(1, 1, 35)$, $(1, 0, 42)$, $(0, 8, 8)$, $(0, 7, 15)$, $(0, 6, 22)$, $(0, 5, 29)$, $(0, 4, 36)$, $(0, 3, 43)$, $(0, 2, 50)$, $(0, 1, 57)$, and $(0, 0, 64)$. It is easy to check that in all these states, Paul can split the weight evenly until he reaches a state $\vec{z}$ with $wt_0(\vec{z}) = 1$.

Proof of Claim 1. Since $e_0(j) \leq 1$, it suffices to show that $pp_0(6) < 1$, i.e., $x_0 < 2^{q-6}$. This is true because $x_0\binom{q}{\leq 2} \leq wt_q(x_0, x_1, x_2) = 2^q$. Hence $x_0 \leq 2^q/\binom{q}{\leq 2}$, which is less than 2^{q-6} when $q \geq 12$.

Proof of Claim 2. We show that $pp_2(j) > e_2(j) + 1$ for $6 \leq j \leq q - 1$. It is enough to show that $\min\left\{\left(x_2 + x_1\binom{q-j}{1} + x_0\binom{q-j}{2}\right)/2^{q-j}\right\} \geq \binom{j}{2} + 6$ for all x_0, x_1, x_2 satisfying $wt_q(x_0, x_1, x_2) = 2^q$ and $x_2 \geq q^2$. The minimum of $\left(x_2 + x_1(q - j) + x_0\binom{q-j}{2}\right)/2^{q-j}$ is achieved at one of the vertices of the

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feasible region, that is, when (x_0, x_1, x_2) is

$$(0, 0, 2^q), \quad \left(0, \frac{2^q - q^2}{\binom{q}{\leq 1}}, q^2\right), \quad \text{or} \quad \left(\frac{2^q - q^2}{\binom{q}{\leq 2}}, 0, q^2\right).$$

For $6 \leq j \leq q - 1$, direct computation shows that the minimum is greater than $\binom{j}{2} + 6 > e_2(j) + 1$ for $q \geq 16$. The case $j = q - 1$ is special, for which $\binom{q-j}{2} = 0$; the inequality remains true here since $x_2 \geq q^2$.

Proof of Claim 3. We show that when $q \geq 23$, $fic_0(8) \leq 1$ and $fic_0(8) + fic_1(8) \leq 16$. Then by definition of fictitious play, $fic(6)$ could not be $(1, 5, 7)$ or $(1, 4, 14)$.

To show $fic_0(8) \leq 1$, note that $pp_0(8) = x_0/2^{q-8} \leq 2^8/\binom{q}{2}$, which is less than 1 when $q \geq 23$. To show $fic_0(8) + fic_1(8) \leq 16$, define $e_{01}(j) := |pp_0(j) + pp_1(j) - (fic_0(j) + fic_1(j))|$. By definition of fictitious play, $|fic_0(j) + fic_1(j) - (fic_0(j+1) + \frac{1}{2}fic_1(j+1))| \leq \frac{1}{2}$, which implies $e_{01}(j) \leq \frac{1}{2}e_{01}(j+1) + 1$. By induction with base case $e_{01}(q) = 0$, we have $e_{01}(j) < 2$. Now assume that $fic_0(8) + fic_1(8) \geq 17$. Then $pp_0(8) + pp_1(8) > 15$, that is, $x_1 + x_0\binom{q-8}{\leq 1} > 15 \cdot 2^{q-8}$. However, the maximum of $x_1 + x_0\binom{q-8}{\leq 1}$ is reached when (x_0, x_1) is either $(0, (2^q - q^2)/\binom{q}{\leq 1})$, or $((2^q - q^2)/\binom{q}{\leq 2}, 0)$. For the first one, $x_1 + x_0\binom{q-8}{\leq 1} > 15 \cdot 2^{q-8}$ iff $q \leq 16$, for the second one, $x_1 + x_0\binom{q-8}{\leq 1} > 15 \cdot 2^{q-8}$ iff $q \leq 22$. This contradicts the fact that $q \geq 23$. $\square$

PROOF. (Proof of Theorem 14.) The values of $F_2^*(q)$ for $1 \leq q \leq 24$, found by exhaustive computation, are listed in Table 1. In each case, $F_2^*(q)$ is the first value of n which satisfies the inequality in (9). These values were generated by a dynamic programming algorithm based on the recurrence

$$r^*(\vec{x}) = 1 + \max_{\vec{a}}\{\min\{r^*(Y(\vec{x}, \vec{a})), r^*(N(\vec{x}, \vec{a}))\}\},$$

where $r^*(\vec{x})$ is defined to be the maximum number of rounds for which Paul can win the pathological liar game with initial state $\vec{x}$.

Now suppose $q \geq 25$. If n satisfies (9), then by Lemma 15, Paul can survive two rounds with all possible resulting states having at least $(q-2)^2 + \binom{q-2}{\leq 2}$ pennies. If after the first two rounds the $(q-2)$-weight of the resulting state is $> 2^{q-2}$, greedily remove coins as large as possible so that the $(q-2)$-weight is exactly 2^{q-2}. Either the resulting state has only pennies remaining, or at most $\binom{q-2}{\leq 2}$ pennies were removed. Since $q - 2 \geq 23$, Lemma 16 shows that Paul can win the $((n, 0, 0), q, 2)^*$-game. If n fails to satisfy (9), then by (10) and Lemma 15, Paul cannot survive the first two rounds and therefore has no winning strategy for the $((n, 0, 0), q, 2)^*$-game. $\square$

7 Winning strategies and hypercube coverings and packings

The pathological liar game has an important natural reformulation in terms of coverings of the hypercube Q_k with certain adaptive Hamming balls. For our purposes, we think of the q-dimensional hypercube Q_q as the set of vertices $\{Y, N\}^q$ in which two vertices are adjacent iff they differ in exactly one position. Instead of the usual 0's and 1's, the bits are Y's and N's, and so "bit" complementation is defined by $\overline{Y} = N$ and $\overline{N} = Y$. A Hamming ball of radius k in Q_q consists of a center $\omega \in Q_q$ and all $w' \in Q_q$ which differ from ω in at most k positions. A *covering (packing)* of Q_q usually refers to a collection of Hamming balls of a fixed radius whose union is Q_q (disjoint in Q_q), but there are many variations. We refer the interested reader to the literature for further information [2,8]. It happens that a winning strategy for Paul in the pathological liar game can be converted to a covering of Q_q with these adaptive Hamming balls, and vice versa. We now formalize this relationship.

Noting that $2^{[q]}$ is the power set of $[q]$, define

$$\binom{[q]}{j} := \{J \in 2^{[q]} : |J| = j\} \qquad \text{and} \qquad \binom{[q]}{\leq i} := \bigcup_{j=0}^{i} \binom{[q]}{j}.$$

We have the following definition of an adaptive Hamming ball, which we call a *quasiball*, followed by an example for $q = 4$ and radius $i = 2$.

Definition 17 (i-quasiball) *Let $q, i \geq 0$. An i-quasiball is the image $f\left(\binom{[q]}{\leq i}\right)$ of an injective function*

$$f : \binom{[q]}{\leq i} \to Q_q,$$

such that whenever $A, B \in \binom{[q]}{\leq i}$ are of the form

$$A = \{p_1, \ldots, p_{|A|}\} \quad and \quad B = \{p_1, \ldots, p_{|A|}, p_{|A|+1}, \ldots, p_{|B|}\}, \qquad (11)$$

where $p_1 < \cdots < p_{|A|} < p_{|A|+1} < \cdots < p_{|B|}$, then $f(A)$ and $f(B)$ are of the form

$$f(A) = \omega_1 \cdots \omega_{p_{|A|}} \cdots \omega_q \quad and \quad f(B) = \omega_1 \cdots \omega_{p_{|A|}-1} \overline{\omega}_{p_{|A|}} \omega'_{p_{|A|}+1} \cdots \omega'_q,$$

where $\omega'_{p_{|A|}+1} \cdots \omega'_q \in Q_{q-p_{|A|}}$.

Example 18 (A 2-quasiball in Q_4) *Let $q = 4$ and $i = 2$. Define $f : \binom{[4]}{\leq 2} \to Q_4$ by $f(\emptyset) = NYNN$, $f(\{1\}) = YNNY$, $f(\{2\}) = NNYN$, $f(\{3\}) = NYYN$, $f(\{4\}) = NYNY$, $f(\{1,2\}) = YYYN$, $f(\{1,3\}) = YNYN$, $f(\{1,4\}) = YNNN$, $f(\{2,3\}) = NNNY$, $f(\{2,4\}) = NNYY$, and $f(\{3,4\}) = NYYY$. For instance, letting $A = \{2\}$ and $B = \{2,3\}$, we see that the first two coordinates of $f(A)$*

and $f(B)$ agree, and the third coordinate is opposite, satisfying the constraint on A and B given by the definition (the fourth coordinate happens to be opposite as well). After similar verification for all possible choices of A and B, we see that $f\left(\binom{[4]}{\leq 2}\right)$ is a 2-quasiball in Q_4. We assign a tree structure to $f\left(\binom{[4]}{\leq 2}\right)$ by defining the parent of $f(B)$, for any $B = \{p_1, \ldots, p_{|B|}\} \neq \emptyset$, to be $f(B \setminus \{p_{|B|}\})$, as illustrated in Figure 1.

Intuitively, for $i > 0$ an i-quasiball contains a stem, $f(\emptyset)$, and q children in $f\left(\binom{[q]}{1}\right)$ obtained from $f(\emptyset)$ by complementing one of its q bits and choosing the bits to the right arbitrarily. The child $f(\{p\})$ can be considered to be the stem of the $(i-1)$-quasiball obtained by deleting the first p bits from each of the vertices in $f(\{\{p\} \cup P : P \in \binom{[q] \setminus [p]}{\leq i-1}\})$. An i-quasiball is clearly a generalization of a Hamming ball of radius i, since for $A, B \in Q_q$ satisfying (11), we may choose $f(B)$ by complementing $f(A)$ in positions $p_{|A|+1}, \ldots, p_{|B|}$ and leaving the other positions unchanged. We note in passing that some i-quasiballs, for example $\{Y, N\}$ and otherwise whenever $i \geq q$, are obtained from more than one such function f.

In order to understand the relationship between winning strategies for Paul and coverings by i-quasiballs, recall that a *covering code* of length q and radius k is a set of Hamming balls of radius k whose union is Q_q. By relaxing Hamming balls to i-quasiballs and by allowing i to vary between 0 and k, we define an *$\vec{x}$-covering*, where $\vec{x} = (x_0, \ldots, x_k)$ to be a collection consisting of x_i $(k-i)$-quasiballs for each $0 \leq i \leq k$ whose union is Q_q. Similarly, an *$\vec{x}$-packing* is such a collection whose constituent members are pairwise disjoint, and whose union is not necessarily Q_q. Informally speaking, we may think of an $(n, 0, \ldots, 0)$-covering of Q_q as an adaptive covering code of length q and fixed radius. The following theorem is adapted from [11, Theorem 1.2] which is for an asymmetric version of the original game.

Theorem 19 *Let $q, k \geq 0$. Paul has a strategy for winning the q-round, k-lie pathological liar game with initial state $\vec{x}$ iff there exists an $\vec{x}$-covering of Q_q. Similarly, Paul has a strategy for winning the original game with the same parameters iff there exists an $\vec{x}$-packing of Q_q.*

PROOF. For the proof it is convenient to keep track of the sets of elements with a given number of lies, and not just their cardinalities. Without loss of generality, in a game with initial state $\vec{x} = (x_0, \ldots, x_k)$, let $n = \sum_{i=0}^{k} x_i$ and let $X_i \subseteq [n]$ be the x_i elements initially associated with i lies. We will abuse notation and let a state or question vector be given in either integer or set format; for example, $\vec{x} = (x_0, \ldots, x_k)$ or $(X_0, \ldots, X_k)$. We prove the statement about the pathological liar game and remark how to adapt the proof for the original game afterward.

For the forward implication, Paul's winning strategy corresponds to a decision tree which is a full binary tree of depth q. The root contains the initial state $\vec{x}$ and the first question. Each node contains a nonzero state, and each internal node contains a legal question for the state in the same node. A node containing state $\vec{P}$ and question $\vec{v}$ has left child containing state $N(\vec{P}, \vec{v})$ and right child containing state $Y(\vec{P}, \vec{v})$, corresponding to responses of "N" or "Y," respectively, by Carole. A game played under this strategy is a path from the root to a leaf of the decision tree, passing down q levels of questions by Paul and answers by Carole. We say that a leaf is *labeled* by each element of $[n]$ which survives in that leaf's state. A leaf labeled by $x \in [n]$ has a *response vertex* with respect to x, which is Carole's Yes/No response sequence $\omega_1 \cdots \omega_q \in Q_q$ read in order from the root to that leaf. If the context is clear, we will refer to a response vertex with respect to x simply as a response vertex. The leaves are in bijection with Q_q by considering the response sequence leading to each leaf.

Let $i \in \{0, \ldots, k\}$ and choose $x \in X_i$. Let $S \subseteq Q_q$ be the set of response vertices with respect to x of those leaves labeled with x. We define the function $f : \binom{[q]}{\leq i} \to Q_q$ certifying that S is a $(k-i)$-quasiball as follows. Set $f(\emptyset)$ equal to the unique $\omega \in S$ for which every response by Carole is truthful. In general a response vertex is completely determined by the positions $A \subseteq [q]$ corresponding to lies by Carole. Set $f(A)$ equal to this response vertex for all $A \in \binom{[q]}{\leq k-i}$. Two leaves α and β both labeled by x and having response vertices with lies in positions $A, B \subseteq [q]$, respectively, and satisfying (11), must have the same first $p_{|A|} - 1$ response sequence steps from the root and bifurcate at step $p_{|A|}$. Therefore S is a $(k-i)$-quasiball, and since every leaf is labeled by at least one element of $[n]$, there exists an $\vec{x}$-covering of Q_q.

For the reverse implication, the states and questions contained in the depth q full binary decision tree are determined by the $\vec{x}$-covering. The initial state at the root is $\vec{x} = (X_0, \ldots, X_k)$, where each $(k-i)$-quasiball is identified with a unique element $x \in X_i$ and is the image of a function $f_x : \left(\binom{[q]}{\leq k-i}\right) \to Q_q$ satisfying Definition 17. Paul constructs the first question vector $\vec{a} = (A_0, \ldots, A_k)$ by letting $x \in A_i$ whenever the stem of the $(k-i)$-quasiball identified with x begins with "Y." Thus every $x \in A_i$ will label a leaf whose response vertex with respect to x begins with "Y." Suppose Carole responds to $\vec{a}$ with "Y." If $x \in A_i$ for some i, no lie is associated with x by Carole's response, and $f_x \left(\binom{[q]\setminus\{1\}}{\leq k-i}\right) \subseteq YQ_{q-1}$ may be viewed as a $(k-i)$-quasiball in Q_{q-1} by restricting the domain of f_x to $\binom{[q]\setminus\{1\}}{\leq k-i}$ and deleting the first bit of each vertex in the image. The resulting state vector $Y(\vec{x}, \vec{a})$ counts x in the ith position. If x is not counted by $\vec{a}$, one lie is associated to x by Carole's response, and $f_x(\{\{1\}\cup P : P \in \binom{[q]\setminus\{1\}}{\leq k-i-1}\}) \subseteq YQ_{q-1}$ may be viewed as a $(k-i-1)$-quasiball in Q_{q-1} by restricting the domain of f_x to $\{\{1\}\cup P : P \in \binom{[q]\setminus\{1\}}{\leq k-i-1}\}$ and deleting the first bit of each vertex in the image. The resulting state vector $Y(\vec{x}, \vec{a})$

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counts x in the $(i+1)$st position (if $i+1 > k$, then the $(k-i-1)$-quasiball is empty and x does not appear in $Y(\vec{x}, \vec{a})$). In both cases, the rest of the domain of f_x is mapped to NQ_{q-1}. Therefore there exists a $Y(\vec{x}, \vec{a})$-covering of Q_{q-1}. Similarly, if Carole answers "N" there exists a $N(\vec{x}, \vec{a})$-covering of Q_{q-1}. The reverse implication follows by induction, since a covering of Q_0 must consist of at least one i-quasiball, which corresponds to a surviving element.

For the original liar game, the function f in the forward implication is defined in the same way; however, there is at most one surviving element labeling each leaf of the decision tree. This ensures that the collection of i-quasiballs which are the sets of response vertices of leaves with a given label are disjoint, and thus form a packing. For the reverse implication, the inductive step is the same, but for the base case a packing of Q_0 corresponds to at most one i-quasiball. $\quad\square$

Monotonicity under majorization, defined in Section 2, is now clear because an i-quasiball realized by a function $f : \binom{[q]}{\leq i} \to Q_q$ can be considered to contain an $(i-1)$-quasiball obtained by restricting f to $\binom{[q]}{\leq i-1}$. Theorem 19 allows Lemma 5, and its dual version for the original game, to be interpreted in terms of the sphere bound for coverings or packings, respectively, of the hypercube. A k-quasiball has size $\binom{q}{\leq k}$ in Q_q, and so there can exist neither a covering of Q_q with fewer than $2^q / \binom{q}{\leq k}$ k-quasiballs, nor a packing of Q_q with more than $2^q / \binom{q}{\leq k}$ k-quasiballs. A natural question is whether the asymptotic sizes of optimal coverings and packings, that is, covering codes and error-correcting codes, meet at the sphere bound. For Hamming balls, this is true for radius 1 [2, Theorem 12.4.11], and is unknown for larger radius. For k-quasiballs, this is now known to be true for fixed k by combining Theorems 2 and 3.

Acknowledgment

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q	1	2	3	4	5	6	7	8	9	10	11	12
$F_2^*(q)$	1	1	2	2	2	4	6	8	12	20	32	52

q	13	14	15	16	17	18	19	20	21	22	23	24
$F_2^*(q)$	90	156	272	480	852	1525	2746	4970	9040	16514	30284	55740

Table 1
Values of $F_2^*(q)$, the minimum number of elements n for which Paul can win the q-round pathological liar game with 2 lies and initial state $(n, 0, 0)$.
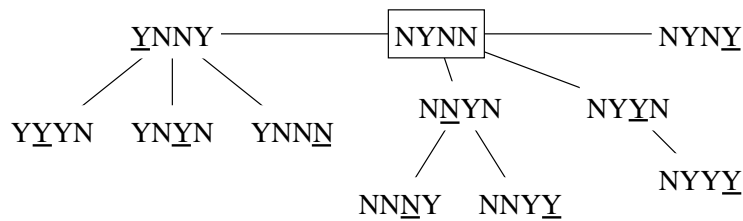

Fig. 1. A 2-quasiball in the hypercube Q_4 satisfying Definition 17 is given a tree structure with stem NYNN. A child agrees with its parent before the underlined position, is opposite at the underlined position, and has unconstrained relationship with its parent afterward.