Multivariate Delta Gončarov and Abel Polynomials

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Abstract

Classical Gončarov polynomials are polynomials which interpolate derivatives. Delta Gončarov polynomials are polynomials which interpolate delta operators, e.g., forward and backward difference operators. We extend fundamental aspects of the theory of classical bivariate Gončarov polynomials and univariate delta Gončarov polynomials to the multivariate setting using umbral calculus. After introducing systems of delta operators, we define multivariate delta Gončarov polynomials, show that the associated interpolation problem is always solvable, and derive a generating function (an Appell relation) for them. We show that systems of delta Gončarov polynomials on an interpolation grid $Z \subseteq \mathbb{R}^d$ are of binomial type if and only if $Z = A\mathbb{N}^d$ for some $d \times d$ matrix A. This motivates our definition of delta Abel polynomials to be exactly those delta Gončarov polynomials which are based on such a grid. Finally, compact formulas for delta Abel polynomials in all dimensions are given for separable systems of delta operators. This recovers a former result for classical bivariate Abel polynomials and extends previous partial results for classical trivariate Abel polynomials to all dimensions.

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1. Introduction

The main purpose of this paper is to extend some fundamental aspects of the theory of Gončarov and Abel polynomials to higher dimensions and to replace partial derivatives by systems of delta operators.

Such topics have an extensive history. In 1881, Abel [1] introduced a sequence $g_0, g_1, g_2, \ldots$ of polynomials, now carrying his name, to represent analytic functions. These polynomials are determined by the condition that they interpolate the derivatives of *any* given analytic function $f : \mathbb{R} \to \mathbb{R}$ at the nodes of an arithmetic progression through the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{g_n(x)}{n!} f^{(n)}(nb), \tag{1}$$

where $b \in \mathbb{R}$ is a fixed parameter. In particular, it can be shown that $g_n(x) = x(x - nb)^{n-1}$ for every n, and g_n satisfies the orthogonality condition

$$g_n^{(k)}(kb) = n! \delta_{k,n} \qquad \text{for all } k.$$

Abel polynomials count some basic combinatorial objects, for example, labeled trees and (generalized) parking functions [24]. The sequence $\{g_n(x)\}_{n \geq 0}$ is of binomial type, which leads to a connection with umbral calculus (or finite operator calculus), a branch of mathematics that studies analytic and algebraic-combinatorial properties of polynomials by a systematic use of operator methods.

The (classical) umbral calculus of polynomials in one variable was put onto a firm theoretical basis by Rota et al. in a series of papers [16, 21, 22]. It was extended to multivariate polynomials and applied to combinatorial problems in [2, 17, 18, 19]. In [3, 7] and [23], higher dimensional umbral calculus is used to derive various versions of the Lagrange inversion formula.

The operators considered in umbral calculus are delta operators, a family of linear operators, acting on the algebra of univariate polynomials with coefficients in a field. Delta operators share many properties in common with derivatives. Each delta operator $\mathfrak{d}$ is uniquely associated with a sequence $\{p_n\}$ of polynomials of binomial type, which interpolates the iterates of $\mathfrak{d}$, evaluated at 0, as

$$f(x) = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!} [\mathfrak{d}^n f(x)]_{x=0}.$$

In [4, 5], Gončarov allowed the interpolation grid to be arbitrary, obtaining that

$$f(x) = \sum_{n=0}^{\infty} g_n(x; a_0, a_1, \ldots, a_n) f^{(n)}(a_n), \tag{2}$$

where $a_n \in \mathbb{R}$ and $g_n(x; a_0, a_1, \ldots, a_n)$ are the Gončarov polynomials. Such polynomials have an extensive history in numerical analysis, even for interpolation of derivatives [14]. Gončarov polynomials in two or more variables have the unusual property that the interpolation problem is solvable for any choice of the nodes of interpolation. The uniqueness was shown in [12] for the bivariate case and in [6] for the multivariate case.

It is well known that certain values of (univariate) Gončarov polynomials are connected with order statistics (e.g., see [9]). This connection has been further developed [11] into a complete

correspondence between Gončarov polynomials and parking functions, a discrete structure lying at the heart of combinatorics. In [10], difference Gončarov polynomials were studied. Since difference operators are delta operators, this was another connection with umbral calculus.

Our basic goal here is to extend the theory of classical multivariate Gončarov polynomials by using umbral calculus and replacing partial derivatives with *delta operators*. This is a further development of our previous work [8, 15] on the analytic and combinatorial properties of bivariate Gončarov polynomials and [13] on the interpolation with general univariate delta operators. In addition, the theory of generating functions, polynomial recursion and approximation theory (the latter being Abel's original motivation for studying his Abel polynomials), just to mention a few, all play a role here.

The rest of the paper is organized as follows. Section 2 contains the definition and basic properties of a system of delta operators in d variables. In Section 3 we define the multivariate Gončarov polynomials associated with a system of delta operators and an interpolation grid Z, derive a generating function (Appell relation), and characterize the set of delta Abel polynomials, which are multivariate Gončarov polynomials of binomial type associated with delta operators. In the last two sections, we present closed formulas for multivariate delta Abel polynomials in the special case when these are associated with a *separable* system of delta operators. In particular, Section 4 deals with the bivariate case, and Section 5 contains the general formulas in an arbitrary dimension.

2. Systems of delta operators

A univariate delta operator $\mathfrak{d}$ is defined as a linear operator on the space of univariate polynomials in one variable x that is shift-invariant and for which $\mathfrak{d}(x)$ is a non-zero constant. In the multivariate case, we need to consider *systems of delta operators*. To state the definition, we first give some preliminary notations using multi-indices.

Let d be a fixed integer ≥ 1 (the space dimension). For a vector $\mathbf{v} \in \mathbb{R}^d$, we denote by v_j the j-th component of $\mathbf{v}$. We write the vector of space variables as $\mathbf{x} = (x_1, x_2, \ldots, x_d)$, and we do the same with the vectors of $\mathbb{R}^d$. In particular, for $\mathbf{y}, \mathbf{z} \in \mathbb{R}^d$ we let $\mathbf{y} \cdot \mathbf{z} = \sum_{i=1}^d y_i z_i$, and we use $\mathbf{0}$ for the zero vector of $\mathbb{R}^d$.

Given $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$, we set $\mathbf{n}! = n_1! n_2! \cdots n_d!$, $|\mathbf{n}| = \sum_{i=1}^d n_i$, and $\mathbf{x^n} = x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}$ (typically, the vectors of $\mathbb{N}^d$ will serve as indices for the Gončarov polynomials we will work with). For $\mathbf{k}, \mathbf{n} \in \mathbb{N}^d$, $\mathbf{k} \leq \mathbf{n}$ means $k_i \leq n_i$ for all i, and $\binom{\mathbf{n}}{\mathbf{k}} = \binom{n_1}{k_1} \cdots \binom{n_d}{k_d}$.

Let $\mathbb{R}[\mathbf{x}]$ and $\mathbb{R}[\![\mathbf{x}]\!]$ be, respectively, the algebra of polynomials (of finite degree) and the algebra of formal power series in the variables $x_1, x_2, \ldots, x_d$ with coefficients in $\mathbb{R}$. The (total) degree of a multivariate polynomial $p(\mathbf{x})$ is the maximum of $|\mathbf{n}|$ for all $\mathbf{n} \in \mathbb{N}^d$ such that $\mathbf{x^n}$ appears in $p(\mathbf{x})$ with a non-zero coefficient, while by the *(relative) degree* of $p(\mathbf{x})$ *with respect to* x_i we mean the highest power m for which x_i^m is a factor of a monomial $\mathbf{x^n}$ appearing in $p(\mathbf{x})$ with a non-zero coefficient.

We denote by $\mathcal{E}_{\mathbf{v}}$, for each $\mathbf{v} \in \mathbb{R}^d$, the shift operator $\mathbb{R}[\mathbf{x}] \to \mathbb{R}[\mathbf{x}] : f(\mathbf{x}) \mapsto f(\mathbf{x} + \mathbf{v})$. We say that a linear operator $\mathfrak{L} : \mathbb{R}[\mathbf{x}] \to \mathbb{R}[\mathbf{x}]$ is shift-invariant if $\mathfrak{L}\mathcal{E}_{\mathbf{v}} = \mathcal{E}_{\mathbf{v}}\mathfrak{L}$ for every $\mathbf{v} \in \mathbb{R}^d$. In particular, $\mathcal{E}_{\mathbf{v}}$ is shift-invariant.

Definition 1. *Let* $\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d$ *be shift-invariant linear operators on* $\mathbb{R}[\mathbf{x}]$. *Then* $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ *is a* system of delta operators *if for any linear polynomial* $p(\mathbf{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d$, *we have that* $\mathfrak{d}_i p(x) = c_i$ *for every* $i = 1, \ldots, d$, *where* $c_i \in \mathbb{R}$ *is a constant depending only on* i, *and at least one of* $c_1, c_2, \ldots, c_d$ *is not* 0.

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It is shown by Parrish [18, Theorem 4.1] that every shift-invariant linear operator $\mathfrak{L}$ on $\mathbb{R}[\mathbf{x}]$ is a formal power series in $D_1, D_2, \ldots, D_d$, where $D_i = D_{x_i}$ is the partial derivative with respect to the variable x_i. To wit, we can write

$$\mathfrak{L} = \sum_{\mathbf{n} \in \mathbb{N}^d} a_{\mathbf{n}} D_1^{n_1} D_2^{n_2} \cdots D_d^{n_d}. \tag{3}$$

We refer to the formal power series

$$f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}^d} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$$

as the indicator of $\mathfrak{L}$, and we write $\mathfrak{L} = f(D_1, D_2, \ldots, D_d)$. With this notation in hand, we have the following:

Theorem 2.1. *Let $\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d$ be shift-invariant linear operators with indicators $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x})$. Assume*

$$f_i(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}^d} a_{\mathbf{n}}^{(i)} \mathbf{x}^{\mathbf{n}}.$$

We form the $d \times d$ matrix $J_f = (m_{i,j})$ by letting $m_{i,j}$ be the constant term of the formal power series $\partial f_i / \partial x_j$. Then $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ is a system of delta operators if and only if $\det J_f \neq 0$ and $a_{\mathbf{0}}^{(i)} = 0$ for all i.

Proof. Assume $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ is a system of delta operators, and let $p(\mathbf{x}) = p_0 + \sum_{j=1}^d p_j x_j$, where $p_0, p_1, \ldots, p_d \in \mathbb{R}$. The representation of delta operators provided by Eq. (3) yields that

$$\mathfrak{d}_i p(\mathbf{x}) = a_{\mathbf{0}}^{(i)} p(\mathbf{x}) + \sum_{j=1}^d m_{i,j} p_j \qquad (i = 1, \ldots, d). \tag{4}$$

Also, $\mathfrak{d}_i f(\mathbf{x})$ being a constant for every linear polynomial $f(\mathbf{x})$ implies, for each i, that $a_{\mathbf{0}}^{(i)} = 0$, and hence $\mathfrak{d}_i p(\mathbf{x}) = \sum_{j=1}^d m_{i,j} p_j$. But then, the condition that $\mathfrak{d}_i p(\mathbf{x}) \neq 0$ for some i means that there is no non-trivial solution to the linear system $J_f \mathbf{v} = \mathbf{0}$. It follows that $\det J_f \neq 0$.

Conversely, if $a_{\mathbf{0}}^{(i)} = 0$, then $\mathfrak{d}_i p(\mathbf{x}) = \sum_{j=1}^d m_{i,j} p_j \in \mathbb{R}$ for all i. And since $\det J_f \neq 0$, we have that $J_f v \neq \mathbf{0}$ for any non-zero vector $\mathbf{v} = (p_1, p_2, \ldots, p_d) \in \mathbb{R}^d$. Thus any linear polynomial of the form $p_0 + \sum_{j=1}^d p_j x_j$ satisfies $\mathfrak{d}_i p(\mathbf{x}) \neq 0$ for some i. $\square$

Definition 2. *A set of polynomials $\{p_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ is said to be a basic sequence of a system of delta operators $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ if:*

(1) *the degree of $p_{\mathbf{n}}(\mathbf{x})$ is $|\mathbf{n}|$ for every $\mathbf{n} \in \mathbb{N}^d$;*

(2) *$p_{\mathbf{0}}(\mathbf{x}) = 1$ and $p_{\mathbf{n}}(\mathbf{0}) = 0$ for all $\mathbf{n} \in \mathbb{N}^d$ with $|\mathbf{n}| \geq 1$;*

(3) *$\mathfrak{d}_i(p_{\mathbf{n}}) = n_i p_{\mathbf{n} - \mathbf{e}_i}$ for all $\mathbf{n} \in \mathbb{N}^d$ and $1 \leq i \leq d$, where $\mathbf{e}_i$ is the i-th standard basis vector $(0, \ldots, 1, \ldots, 0)$.*

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From Parrish [18], each system of delta operators has a basic sequence (called normalized shift basis in [18]). Moreover, a sequence of polynomials $\{p_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ is the basic sequence of some system of delta operators if and only if it satisfies the binomial identity

$$p_{\mathbf{n}}(\mathbf{x} + \mathbf{y}) = \sum_{\mathbf{k} \leq \mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} p_{\mathbf{k}}(\mathbf{x}) p_{\mathbf{n}-\mathbf{k}}(\mathbf{y}) \qquad \text{for all } \mathbf{n} \in \mathbb{N}^d.$$

Notice that, for each $m \in \mathbb{N}$, the set $\{p_{\mathbf{n}}(\mathbf{x}) : |\mathbf{n}| \leq m\}$ is a basis of the vector space of polynomials of $\mathbb{R}[\mathbf{x}]$ of degree at most m.

Univariate delta operators reduce the degree of a polynomial by one. The next definition gives a generalization of this property.

Definition 3. *We say that the shift-invariant linear operators $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ form a* strict system *of delta operators if $\mathfrak{d}_i$ is degree reducing for each i, in the sense that for any polynomial $p(\mathbf{x})$ of degree m with respect to x_i, the degree of $\mathfrak{d}_i p$ with respect to x_i is $m - 1$ if $m \geq 1$, otherwise $\mathfrak{d}_i(p) = 0$.*

Strict systems of delta operators can be characterized as follows.

Theorem 2.2. *A system of shift-invariant delta operators $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ is a strict system if and only if for each i, the indicator $f_i(\mathbf{x})$ of $\mathfrak{d}_i$ can be written as $f_i(\mathbf{x}) = x_i g_i(\mathbf{x})$ for some formal power series $g_i(\mathbf{x})$ with $g_i(\mathbf{0}) \neq 0$.*

Proof. It is clear that if $f_i(\mathbf{x}) = x_i g_i(\mathbf{x})$ with $g_i(\mathbf{0}) \neq 0$, then the operator $\mathfrak{d}_i$ reduces the degree with respect to x_i by 1.

Conversely, assume that $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ is a strict system of delta operators. We will show that $f_1(\mathbf{x}) = x_1 g_1(\mathbf{x})$ with $g_1(\mathbf{0}) \neq 0$. The formulas for other i are similar. Assume

$$f_1 = \sum_{\mathbf{n} \in \mathbb{N}^d} c_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}.$$

We need to show that $c_{\mathbf{n}} = 0$ whenever $n_1 = 0$, and $c_{(1,0,\ldots,0)} \neq 0$.

First, notice that $\mathfrak{d}_1(1) = 0$, which implies $c_{\mathbf{0}} = 0$. Then, assume that $c_{\mathbf{n}} \neq 0$ for some $\mathbf{n}$ with $n_1 = 0$. Find in the set

$$\{\mathbf{n} \in \mathbb{N}^d : n = (0, n_2, \ldots, n_d) \text{ and } c_{\mathbf{n}} \neq 0\}$$

the minimal index $\mathbf{m} = (0, m_2, \ldots, m_d)$ under the lexicographic order. Apply $\mathfrak{d}_1$ to $x_1 \cdot \mathbf{x}^{\mathbf{m}}$. Note that $D_1^{n_1} D_2^{n_2} \cdots D_d^{n_d}(x_1 x_2^{m_2} \cdots x_d^{m_d})$ is 0 if $n_1 > 1$ or $n_i > m_i$ for any $2 \leq i \leq d$, and is a polynomial with no x_1 factor if $n_1 = 1$. With our choice of $\mathbf{m}$, we conclude that

$$\mathfrak{d}_1(x_1 \cdot \mathbf{x}^{\mathbf{m}}) = c_{\mathbf{m}} m_2! \cdots m_d! x_1 + g(x_2, \ldots, x_d),$$

which is of degree 1 with respect to x_1, a contradiction.

It remains to show that $c_{(1,0,\ldots,0)} \neq 0$, which can be seen by applying $\mathfrak{d}_1$ to x_1^2. Indeed, if $c_{(1,0,\ldots,0)} = 0$, then $\mathfrak{d}_1(x_1^2) = 2c_{(2,0,\ldots,0)}$, so that $\mathfrak{d}_1$ has reduced the degree with respect to x_1 by at least 2, which is a contradiction. $\square$

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REMARK 1. A system of formal power series $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x})$ is called *admissible* if and only if (i) each $f_i(\mathbf{x})$ has zero constant term and (ii) $\det J_f \neq 0$. (See e.g. [3, 7, 20].) Hence, a system of delta operators in our definition is precisely a system of shift-invariant linear operators whose indicators form an admissible system. Such systems of operators have been called "admissible differential operators" by Joni [7] and "umbral operators" by Garsia and Joni [3].

It is known that for any admissible system $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x})$ there exists a unique compositional inverse, i.e., an admissible system of formal power series $g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_d(\mathbf{x})$ such that

$$f_i(g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_d(\mathbf{x})) = x_i \text{ and } g_i(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x})) = x_i$$

for all i. Consequently a system of delta operators has compositional inverse.

REMARK 2. Another definition of system of delta operators is given by Niederhausen [17] for pairs of delta operators whose indicators are "delta multi-series", which is equivalent to a strict system of delta operators defined here. Clearly, strict systems are special cases of systems of delta operators. Conversely, not all admissible systems are strict. For example, the pair $(\mathfrak{d}_1, \mathfrak{d}_2) = (D_x + D_y, D_x - D_y)$ in the bivariate case has admissible indicators, but does not form a strict system.

3. Delta Gončarov polynomials

In the present section we introduce delta Gončarov polynomials starting from the biorthogonality condition in the Gončarov interpolation problem. The univariate version was discussed in [11] with the differentiation operator and in [13] with an arbitrary delta operator. Here we extend the theory to many variables.

To this end, let $\Delta = (\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ be a system of delta operators, which we assume as fixed for the rest of the section, and let $\{\Phi_\mathbf{n} : \mathbf{n} \in \mathbb{N}^d\}$ be a set of shift-invariant operators of the form

$$\Phi_\mathbf{n} = \mathfrak{L}_\mathbf{n} \mathfrak{d}_1^{n_1} \mathfrak{d}_2^{n_2} \cdots \mathfrak{d}_d^{n_d}, \tag{5}$$

where $\mathfrak{L}_\mathbf{n}$ is a shift-invariant linear operator whose indicator has a non-zero constant.

We say that a set of polynomials $\{q_\mathbf{n}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ is a sequence of polynomials if $\deg q_\mathbf{n} = |\mathbf{n}|$ for all $\mathbf{n}$. A sequence $\{q_\mathbf{n}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ of polynomials is then called biorthogonal to the set $\{\Phi_\mathbf{n} : \mathbf{n} \in \mathbb{N}^d\}$ if

$$\varepsilon(\mathbf{0})\Phi_\mathbf{n}(q_\mathbf{k}(\mathbf{x})) = \mathbf{n}!\delta_{\mathbf{n},\mathbf{k}}. \tag{6}$$

Theorem 3.1. *Given a set $\{\Phi_\mathbf{n} : \mathbf{n} \in \mathbb{N}^d\}$ of shift-invariant operators, there exists a unique sequence of polynomials that is biorthogonal to it.*

To prove Theorem 3.1 we need some additional notation and a couple lemmas.

Definition 4. *A subset S of $\mathbb{N}^d$ is a lower set if for any $\mathbf{n} \in S$, it follows that $\mathbf{k} \in S$ for any $\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}$.*

For any lower set S, we let

$$\Pi_S^d(\Delta) = \left\{ q(\mathbf{x}) : q(\mathbf{x}) = \sum_{\mathbf{k} \in S} a_\mathbf{k} p_\mathbf{k}(\mathbf{x}), a_\mathbf{k} \in \mathbb{R} \right\},$$

where $\{p_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}^d}$ is the basic sequence of $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$. In particular, we write

$$\Pi_{\mathbf{n}}^d(\Delta) = \left\{ q(\mathbf{x}) : q(\mathbf{x}) = \sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k}} p_{\mathbf{k}}(\mathbf{x}), a_{\mathbf{k}} \in \mathbb{R} \right\}.$$

Lemma 3.2. *Let $S \subseteq \mathbb{N}^d$ a lower set. For any $\mathbf{n} \in \mathbb{N}^d$, the set of functionals*

$$\{\Psi_{\mathbf{k}} = \varepsilon(\mathbf{0})\Phi_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d, \mathbf{k} \leq \mathbf{n}\} \tag{7}$$

is a linearly independent set over $\Pi_{\mathbf{n}}^d(\Delta)$.

Proof. Suppose there are real numbers $\lambda_{\mathbf{k}}$ such that

$$\Psi(p(\mathbf{x})) = \sum_{\mathbf{k} \leq \mathbf{n}} \lambda_{\mathbf{k}} \Psi_{\mathbf{k}}(p(\mathbf{x})) = 0 \tag{8}$$

for each $p(\mathbf{x}) \in \Pi_{\mathbf{n}}^d(\Delta)$. We will show that the coefficients $\lambda_{\mathbf{k}}$ all vanish.

The basic sequence $\{p_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ has the property that

$$\mathfrak{d}_1^{k_1} \mathfrak{d}_2^{k_2} \cdots \mathfrak{d}_d^{k_d} p_{\mathbf{n}} = (n_1)_{k_1} (n_2)_{k_2} \cdots (n_d)_{k_d} p_{\mathbf{n}-\mathbf{k}} \tag{9}$$

for each $\mathbf{k} \leq \mathbf{n}$, and

$$\mathfrak{d}_1^{k_1} \mathfrak{d}_2^{k_2} \cdots \mathfrak{d}_d^{k_d} p_{\mathbf{n}} = 0 \tag{10}$$

whenever $k_i > n_i$ for some i.

Substituting $p_{\mathbf{0}}(\mathbf{x}) = 1$ into Eq. (8), we have

$$\Psi(p_{\mathbf{0}}(\mathbf{x})) = \lambda_{\mathbf{0}} \varepsilon(\mathbf{0}) \mathfrak{L}_{\mathbf{0}}(1) = 0. \tag{11}$$

Since the indicator of $\mathfrak{L}_{\mathbf{0}}$ has non-zero constant term, this yields $\mathfrak{L}_{\mathbf{0}}(1) \neq 0$, and hence $\lambda_{\mathbf{0}} = 0$.

Next we show that for any index vector $\mathbf{k} \in \mathbb{N}^d$ with $\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}$, if $\lambda_{\mathbf{s}} = 0$ for all $\mathbf{0} \leq \mathbf{s} \leq \mathbf{k}$ and $\mathbf{s} \neq \mathbf{k}$, then $\lambda_{\mathbf{k}} = 0$. To see this, we substitute $p_{\mathbf{k}}(\mathbf{x})$ into Eq. (8) and use the assumption that $\lambda_{\mathbf{s}} = 0$ for $\mathbf{s} < \mathbf{n}$ and Eq. (10), so as to find that

$$\Psi(p_{\mathbf{k}}(\mathbf{x})) = \lambda_{\mathbf{k}} \varepsilon(\mathbf{0}) \mathfrak{L}_{\mathbf{k}}(\mathbf{k}!).$$

But the indicator of $\mathfrak{L}_{\mathbf{k}}$ has non-zero constant term, so $\mathfrak{L}_{\mathbf{k}}(\mathbf{k}!)$ is a non-zero constant. Evaluating at $\mathbf{0}$ does not change its value, and this implies $\lambda_{\mathbf{k}} = 0$.

The lemma then follows from an induction on $|\mathbf{k}|$. $\qquad \square$

Lemma 3.3. *Let V be a vector space of dimension m, and $\{\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_m\}$ an independent set of linear functionals on V. Then, for any $b_1, \ldots, b_m \in \mathbb{R}$ there is a unique vector $v \in V$ such that $\mathfrak{t}_i(v) = b_i$ for all i.*

Proof. Let $\{e_1, e_2, \ldots, e_m\}$ be a basis of V and let $a_{i,j} = \mathfrak{t}_i(e_j)$. Then the lemma just says that $Ax = b$ is uniquely solvable if and only if the matrix $A = (a_{i,j})$ has linearly independent rows. $\quad \square$

Combining Lemmas 3.2 and 3.3, we have the following:

Corollary 3.4. *Under the assumptions of Lemma 3.2, given any multiset of real numbers* $\{b_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d, \ \mathbf{k} \leq \mathbf{n}\}$, *there is a unique polynomial* $q(\mathbf{x}) \in \Pi_{\mathbf{n}}^d(\Delta)$ *such that, for all* $\mathbf{k} \leq \mathbf{n}$,

$$\varepsilon(\mathbf{0})\Phi_{\mathbf{k}}(q(\mathbf{x})) = \varepsilon(\mathbf{0})\mathfrak{L}_{\mathbf{k}}\mathfrak{d}_1^{k_1}\mathfrak{d}_2^{k_2}\cdots\mathfrak{d}_d^{k_d}(q(\mathbf{x})) = b_{\mathbf{k}}. \tag{12}$$

Theorem 3.1 follows from Corollary 3.4 by letting $q_{\mathbf{n}}(\mathbf{x})$ be the unique polynomial in $\Pi_{\mathbf{n}}^d(\Delta)$ that satisfies

$$\varepsilon(\mathbf{0})\Phi_{\mathbf{k}}(q_{\mathbf{n}}(\mathbf{x})) = \mathbf{n}!\delta_{\mathbf{n},\mathbf{k}} \tag{13}$$

for all $\mathbf{k} \leq \mathbf{n}$. By the definition of $\Pi_{\mathbf{n}}^d(\Delta)$, given any polynomial $r(\mathbf{x}) \in \Pi_{\mathbf{n}}^d(\Delta)$ and any $\mathbf{k}'$ with $k_i' > n_i$ for some i, we have

$$\mathfrak{d}_1^{k_1'}\mathfrak{d}_2^{k_2'}\cdots\mathfrak{d}_d^{k_d'}(r(\mathbf{x})) = 0.$$

In particular, $\varepsilon(\mathbf{0})\Phi_{\mathbf{k}'}(q_{\mathbf{n}}(\mathbf{x})) = 0$, and this shows that the biorthogonality condition (13) holds for all $\mathbf{k} \in \mathbb{N}^d$. On the other hand, it is clear that $q_{\mathbf{n}}(\mathbf{x})$ is of degree $m = |\mathbf{n}|$, otherwise $q_{\mathbf{n}}(\mathbf{x})$ would be a linear combination of $\{p_{\mathbf{k}} : |\mathbf{k}| < m\}$, and hence we would have $\varepsilon(\mathbf{0})\Phi_{\mathbf{n}}(q_{\mathbf{n}}(\mathbf{x})) = 0$, which is a contradiction.

Putting it all together, we see from the above that the sequence of polynomials $\{q_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ is biorthogonal to the set of operators $\{\Phi_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$.

In particular, if each $\mathfrak{L}_{\mathbf{n}}$ in Eq. (5) is a shift operator, then the set of polynomials biorthogonal to $\{\Phi_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ is just the set of the delta Gončarov polynomials in which we are interested. More precisely, using that $\varepsilon(\mathbf{0})\mathcal{E}_{\mathbf{v}} = \varepsilon(\mathbf{v})$, we have the following:

Definition 5. *Let* $\Delta = (\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ *be a system of delta operators, and* $Z = \{z_{\mathbf{k}} : \mathbf{k} \in S\} \subseteq \mathbb{R}^d$ *a multiset of nodes, which we refer to as an interpolation grid. The delta Gončarov polynomial* $t_{\mathbf{n}}(\mathbf{x}; Z)$ *is the unique polynomial in* $\Pi_{\mathbf{n}}^d(\Delta)$ *satisfying*

$$\varepsilon(z_{\mathbf{k}})\mathfrak{d}_1^{k_1}\mathfrak{d}_2^{k_2}\cdots\mathfrak{d}_d^{k_d}(t_{\mathbf{n}}(\mathbf{x}; Z)) = \mathbf{n}!\delta_{\mathbf{k},\mathbf{n}} \tag{14}$$

for all $\mathbf{k} \leq \mathbf{n}$.

From the preceding discussion, we see that $t_{\mathbf{n}}(\mathbf{x}; Z)$ only depends on the set of nodes in $Z_{\mathbf{n}} = \{z_{\mathbf{k}} : \mathbf{k} \leq \mathbf{n}\}$.

The delta Gončarov polynomials form a basis for the solutions of the following interpolation problem with a system of delta operators $\Delta = (\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$, which we call the *delta Gončarov interpolation problem in dimension* d:

Problem. *Let S be a lower set. Given a multiset of nodes* $Z = \{z_{\mathbf{k}} : \mathbf{k} \in S\} \subseteq \mathbb{R}^d$ *and a multiset of real numbers* $\{b_{\mathbf{k}} : \mathbf{k} \in S\}$, *find a polynomial* $P(\mathbf{x})$ *in* $\Pi_S^d(\Delta)$ *such that, for all* $\mathbf{k} \in S$,

$$\varepsilon(z_{\mathbf{k}})\mathfrak{d}_1^{k_1}\mathfrak{d}_2^{k_2}\cdots\mathfrak{d}_d^{k_d}(P(\mathbf{x})) = b_{\mathbf{k}}.$$

By Theorem 3.1, the above problem has the unique solution

$$P(\mathbf{x}) = \sum_{\mathbf{k} \in S} \frac{b_{\mathbf{k}}}{\mathbf{k}!} t_{\mathbf{k}}(\mathbf{x}; Z).$$

Conversely, any polynomial in $\Pi_S^d(\Delta)$ can be expanded as a linear combination of delta Gončarov polynomials. This is the content of the next proposition, which follows from the orthogonality relations.

Proposition 3.5. *For any $P \in \Pi_S^d(\Delta)$,*

$$P(\mathbf{x}) = \sum_{\mathbf{k} \in S} \frac{1}{\mathbf{k}!} \left[\varepsilon(z_{\mathbf{k}}) \mathfrak{d}_1^{k_1} \mathfrak{d}_2^{k_2} \cdots \mathfrak{d}_d^{k_d}(P(\mathbf{x})) \right] t_{\mathbf{k}}(\mathbf{x}; Z).$$

In particular, substituting $P(\mathbf{x})$ in Proposition 3.5 with the basic polynomial $p_{\mathbf{n}}(\mathbf{x})$, and noting that $\binom{\mathbf{n}}{\mathbf{k}} = 0$ unless $\mathbf{k} \leq \mathbf{n}$, we obtain a binomial relation between the basic sequence and the delta Gončarov polynomials, which generalizes the linear recurrence for univariate Gončarov polynomials [11, Eq. (2.5)].

Proposition 3.6. *Let $\{p_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ be the basic sequence of the system of delta operators $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$. Then*

$$p_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{k} \leq \mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} p_{\mathbf{n}-\mathbf{k}}(z_{\mathbf{k}}) t_{\mathbf{k}}(\mathbf{x}; Z). \tag{15}$$

4. Sequences of binomial type

The basic sequence of a system of delta operators is of binomial type. In general, this is not true for multivariate delta Gončarov polynomials. In the present section we give a necessary and sufficient condition under which a sequence of delta Gončarov polynomials is of binomial type.

The following definition is adapted from Parrish [18].

Definition 6. *A polynomial sequence $\{q_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ is said to be of binomial type if $q_{\mathbf{0}}, q_{\mathbf{e}_1}, \ldots, q_{\mathbf{e}_d}$ span the vector space of linear polynomials in $\mathbb{R}[\mathbf{x}]$ and, for any $\mathbf{n} \in \mathbb{N}^d$, $q_{\mathbf{n}}$ satisfies the identity*

$$q_{\mathbf{n}}(\mathbf{x} + \mathbf{y}) = \sum_{\mathbf{k} \leq \mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} q_{\mathbf{k}}(\mathbf{x}) q_{\mathbf{n}-\mathbf{k}}(\mathbf{y}).$$

In particular, Parrish showed in [18] that:

(1) A sequence of polynomials is of binomial type if and only if it is the basic sequence of a system of delta operators;

(2) A sequence of polynomials $\{p_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ is a sequence of binomial type if and only if

$$\sum_{\mathbf{n} \in \mathbb{N}^d} p_{\mathbf{n}}(\mathbf{x}) \frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!} = e^{\mathbf{x} \cdot \mathbf{g}(\mathbf{y})} \tag{16}$$

for an admissible system of formal power series $\mathbf{g} = (g_1, g_2, \ldots, g_d)$.

(3) If $\{p_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$ is the basic sequence of a system $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ of delta operators, then the formal power series $g_1, g_2, \ldots, g_d$ in point (2) above are, respectively, the compositional inverses of $f_1, f_2, \ldots, f_d$, where f_i is the indicator of $\mathfrak{d}_i$ for each i. Consequently, Eq. (16) is equivalent to the following *Appell relation* for the sequence $\{p_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in \mathbb{N}^d}$:

$$\sum_{\mathbf{n} \in \mathbb{N}^d} \frac{p_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!} \mathbf{f}(\mathbf{y})^{\mathbf{n}} = e^{\mathbf{x} \cdot \mathbf{y}}, \tag{17}$$

where $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_d(\mathbf{y}))$.

Now, fix a system of delta operators, and let their indicators and the corresponding compositional inverses be as in the above. Then, we can derive an Appell relation for multivariate delta Gončarov polynomials as follows:

Theorem 4.1 (Appell relation). *Let $\{t_{\mathbf{n}}(\mathbf{x}; Z)\}_{\mathbf{n} \in \mathbb{N}^d}$ be the sequence of delta Gončarov polynomials associated to a system $\Delta = (\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n)$ of delta operators and a grid $Z = \{z_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subseteq \mathbb{R}^d$. Then*

$$e^{\mathbf{x} \cdot \mathbf{g}(\mathbf{y})} = \sum_{\mathbf{k} \in \mathbb{N}^d} \frac{t_{\mathbf{k}}(\mathbf{x}; Z)}{\mathbf{k}!} \mathbf{y}^{\mathbf{k}} e^{z_{\mathbf{k}} \cdot \mathbf{g}(\mathbf{y})}. \tag{18}$$

Proof. Combining Eqs. (15) and (16), we have

$$e^{\mathbf{x} \cdot \mathbf{g}(\mathbf{y})} = \sum_{\mathbf{n} \in \mathbb{N}^d} p_{\mathbf{n}}(\mathbf{x}) \frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{1}{\mathbf{n}!} \left(\sum_{\mathbf{k} \leq \mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} p_{\mathbf{n}-\mathbf{k}}(z_{\mathbf{k}}) t_{\mathbf{k}}(\mathbf{x}; Z) \right) \mathbf{y}^{\mathbf{n}}$$

$$= \sum_{\mathbf{k} \in \mathbb{N}^d} \left(\frac{1}{\mathbf{k}!} t_{\mathbf{k}}(\mathbf{x}; Z) \mathbf{y}^{\mathbf{k}} \cdot \sum_{\mathbf{m}=\mathbf{n}-\mathbf{k}} \frac{p_{\mathbf{m}}(z_{\mathbf{k}})}{\mathbf{m}!} \mathbf{y}^{\mathbf{m}} \right)$$

$$= \sum_{\mathbf{k} \in \mathbb{N}^d} \frac{t_{\mathbf{k}}(\mathbf{x}; Z)}{\mathbf{k}!} \mathbf{y}^{\mathbf{k}} e^{z_{\mathbf{k}} \cdot \mathbf{g}(\mathbf{y})},$$

which concludes the proof. □

Note that an equivalent version of the Appell relation (18) is

$$e^{\mathbf{x} \cdot \mathbf{y}} = \sum_{\mathbf{k} \in \mathbb{N}^d} \left(\frac{t_{\mathbf{k}}(\mathbf{x}; Z)}{\mathbf{k}!} \mathbf{f}(\mathbf{y})^{\mathbf{n}} \cdot e^{z_{\mathbf{k}} \cdot \mathbf{y}} \right). \tag{19}$$

This will be used in the proof of the following theorem, which is the main result of the present section and characterizes the delta Gončarov polynomials that are of binomial type in terms of the "geometry" of the underlying interpolation grid, by showing that the latter is actually a linear transformation of the set of lattice points $\mathbb{N}^d$. We will refer to such Gončarov polynomials as *delta Abel polynomials*, as in the univariate case they are exactly the Abel polynomials $x(x - nb)^{n-1}$, for which the interpolation grid is an arithmetic progression of the form $0, b, 2b, \ldots$ for some $b \in \mathbb{R}$.

Theorem 4.2. *Given a system of delta operators $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ and a grid $Z = \{z_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\} \subseteq \mathbb{R}^d$. Then the associated set of delta Gončarov polynomials $\{t_{\mathbf{n}}(\mathbf{x}; Z)\}_{\mathbf{n} \in \mathbb{N}^d}$ is of binomial type if and only if there is a $d \times d$ matrix A for which $z_{\mathbf{k}} = A\mathbf{k}$ for all $\mathbf{k}$. (Here we view $\mathbf{k} \in \mathbb{N}^d$ as a column vector). In other words, the set of nodes is the image of the set of lattice points $\mathbb{N}^d$ under the matrix A.*

Proof. We split the proof into two parts.

Necessity. Assume the sequence of delta Gončarov polynomials $\{t_{\mathbf{k}}(\mathbf{x}; Z)\}$ is of binomial type. This is equivalent to the existence of an admissible system of formal power series $\mathbf{g} = (g_1, g_2, \ldots, g_d)$ such that

$$\sum_{\mathbf{k} \in \mathbb{N}^d} t_{\mathbf{k}}(\mathbf{x}; Z) \frac{\mathbf{y}^{\mathbf{k}}}{\mathbf{k}!} = e^{\mathbf{x} \cdot \mathbf{g}(\mathbf{y})}. \tag{20}$$

Let $\mathbf{h} = (h_1, h_2, \ldots, h_d) \in \mathbb{R}[\![x]\!]$ be the compositional inverse of $\mathbf{g}$. Substituting $\mathbf{y}$ with $\mathbf{h}(\mathbf{y}) = (h_1(\mathbf{y}), h_2(\mathbf{y}), \ldots, h_d(\mathbf{y}))$, we obtain

$$\sum_{\mathbf{k} \in \mathbb{N}^d} \frac{t_{\mathbf{k}}(\mathbf{x}; Z)}{\mathbf{k}!} \mathbf{h}(\mathbf{y})^{\mathbf{k}} = e^{\mathbf{x} \cdot \mathbf{y}}. \tag{21}$$

Comparing this with the alternate Appell relation (19) and using the uniqueness of the coefficients, we have

$$\mathbf{h}(\mathbf{y})^{\mathbf{k}} = \mathbf{f}(\mathbf{y})^{\mathbf{k}} \cdot e^{z_{\mathbf{k}} \cdot \mathbf{y}} \qquad \text{for all } \mathbf{k} \in \mathbb{N}^d. \tag{22}$$

Taking the same equation with index $\mathbf{k} + \mathbf{e}_j$ and dividing by the first we get

$$h_j(\mathbf{y}) = f_j(\mathbf{y}) \cdot e^{(z_{\mathbf{k}+\mathbf{e}_j} - z_{\mathbf{k}}) \cdot \mathbf{y}},$$

which holds for all $\mathbf{k} \in \mathbb{N}^d$ and $1 \le j \le d$. It follows that $z_{\mathbf{k}+\mathbf{e}_j} - z_{\mathbf{k}}$ is a constant vector independent of $\mathbf{k}$. Denote this constant vector by $\mathbf{t}_j$. It follows that

$$z_{\mathbf{k}} = z_{\mathbf{0}} + k_1 \mathbf{t}_1 + k_2 \mathbf{t}_2 + \cdots + k_d \mathbf{t}_d.$$

So, taking $\mathbf{k} = \mathbf{0}$ in Eq. (22), we get $e^{z_{\mathbf{0}} \cdot \mathbf{y}} = 1$, which implies $z_{\mathbf{0}} = \mathbf{0}$. Hence $z_{\mathbf{k}}$ is of the form $A\mathbf{k}$, where A is the matrix whose columns are $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_d$.

Sufficiency. Conversely, assume Z is a linear transformation of $\mathbb{N}^d$ by a $d \times d$ matrix A. Then

$$z_{\mathbf{k}} = k_1 \mathbf{t}_1 + k_2 \mathbf{t}_2 + \cdots + k_d \mathbf{t}_d,$$

where $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_d$ are the column vectors of the matrix A. The Appell relation for $t_{\mathbf{k}}(\mathbf{x})$ then becomes

$$e^{\mathbf{x} \cdot \mathbf{y}} = \sum_{\mathbf{k} \in \mathbb{N}^d} \left(\frac{t_{\mathbf{k}}(\mathbf{x}; Z)}{\mathbf{k}!} \mathbf{f}(\mathbf{y})^{\mathbf{k}} \cdot e^{(k_1 \mathbf{t}_1 + k_2 \mathbf{t}_2 + \cdots + k_d \mathbf{t}_d) \cdot \mathbf{y}} \right)$$

$$= \sum_{\mathbf{k} \in \mathbb{N}^d} \left(\frac{t_{\mathbf{k}}(\mathbf{x}; Z)}{\mathbf{k}!} \mathbf{h}(\mathbf{y})^{\mathbf{k}} \right),$$

where we set $\mathbf{h}(\mathbf{y}) = (h_1(\mathbf{y}), h_2(\mathbf{y}), \ldots, h_d(\mathbf{y}))$, with

$$h_i(\mathbf{y}) = f_i(\mathbf{y}) e^{\mathbf{t}_i \cdot \mathbf{y}}.$$

Since Eq. (23) agrees with Eq. (17), it only remains to show that the system $\mathbf{h}(\mathbf{y})$ is admissible. For this, note that

$$\frac{\partial h_i}{\partial y_j} = \frac{\partial f_i}{\partial y_j} \cdot e^{\mathbf{t}_i \cdot \mathbf{y}} + f_i(\mathbf{y}) t_{i,j} e^{\mathbf{t}_i \cdot \mathbf{y}}.$$

But $f_i(\mathbf{0}) = 0$, hence $\frac{\partial h_i}{\partial y_j}(\mathbf{0}) = \frac{\partial f_i}{\partial y_j}(\mathbf{0})$, and then $J_h = J_f$ is non-singular. Thus the sequence $\{t_{\mathbf{n}}(\mathbf{x}; Z)\}_{\mathbf{n} \in \mathbb{N}^d}$ is of binomial type. $\qquad \square$

Given a system of delta operators, Theorem 4.2 says that if the defining grid is a linear transformation of $\mathbb{N}^d$, then the associated sequence of delta Gončarov polynomials is of binomial type. Therefore, these polynomials are the basic sequence of some system of delta operators. More explicitly, we have:

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Proposition 4.3. *Let $\Delta = (\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ be a system of delta operators. Let Z be a linear transformation of $\mathbb{N}^d$ by a $d \times d$ matrix A, and $\mathbf{t}_i$ the i-th column vector of A. Denote by $\{t_{\mathbf{n}}(\mathbf{x}; Z)\}_{\mathbf{n} \in \mathbb{N}^d}$ the sequence of delta Gončarov polynomials associated with Δ and Z. Then $\{t_{\mathbf{n}}(\mathbf{x}; Z)\}_{\mathbf{n} \in \mathbb{N}^d}$ is the basic sequence of the system of delta operators $(\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_d)$, with $\mathfrak{s}_i = \mathfrak{d}_i \mathcal{E}_{\mathbf{t}_i}$ for all i.*

This can be seen by writing down the orthogonality conditions for the system $(\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_d)$, or from the proof of Theorem 4.2.

5. Closed formulas for bivariate delta Abel polynomials

In the rest of the paper we will show how to derive compact formulas for a family of multivariate delta Abel polynomials using the properties of delta operators.

The systems of delta operators considered here are *separable*, in the sense that $\mathfrak{d}_i = D_i \mathfrak{L}_i$ for each i, where $\mathfrak{L}_i$ is an invertible shift-invariant operator acting only on x_i, i.e., the indicator of $\mathfrak{L}_i$ is a formal power series in the variable x_i only. The classical case where $\mathfrak{d}_i = D_i$ is a special example in this family.

We shall give a closed formula for the multivariate delta Abel polynomials, which are delta Gončarov polynomials whose defining grid is a linear transformation of $\mathbb{N}^d$ by a $d \times d$ matrix. From Theorem 4.2, these are exactly the delta Gončarov polynomials that are of binomial type.

For simplicity and clarity, in this section we describe the approach and the results in the bivariate case, where the variables are x and y. The general case is dealt with in a similar manner in the next section.

In [15], it was conjectured and then verified that, for a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{23}$$

the classical bivariate Abel polynomials on the grid

$$Z = \{(x_{i,j}, y_{i,j})\}_{(i,j) \in \mathbb{N}^2}, \quad \text{where } (x_{i,j}, y_{i,j}) = A(i,j)^t \text{ for all } i, j \in \mathbb{N}, \tag{24}$$

are given by

$$[(x - x_{0,n})(y - y_{m,0}) - x_{0,n} y_{m,0}] (x - x_{m,n})^{m-1} (y - y_{m,n})^{n-1}. \tag{25}$$

We will derive this formula using the techniques of delta operators. For that we will need the concept of bivariate Pincherle derivatives and their properties, see [16, 21]. These operators were also called "Lie derivatives" in [3, 7].

Let $\mathfrak{L}$ be a shift-invariant operator on $\mathbb{R}[x, y]$. Then the Pincherle derivatives of $\mathfrak{L}$ relative x and y are, respectively, the linear operators defined by taking, for every $p(x, y) \in \mathbb{R}[x, y]$,

$$\mathfrak{L}'_x(p(x, y)) = \mathfrak{L}(xp(x, y)) - x\mathfrak{L}(p(x, y))$$

and

$$\mathfrak{L}'_y(p(x, y)) = \mathfrak{L}(yp(x, y)) - y\mathfrak{L}(p(x, y)).$$

If $\mathfrak{L}$ is shift-invariant, then so are $\mathfrak{L}'_x$ and $\mathfrak{L}'_y$. In addition, the indicator function of $\mathfrak{L}_x$ (resp. $\mathfrak{L}_y$) is $\partial f(x, y)/\partial x$ (resp. $\partial f(x, y)/\partial y$), where $f(x, y)$ is the indicator of $\mathfrak{L}$. Consequently, we have:

1. $(D_x)'_x = \mathfrak{I}$, the identity operator, and $(D_x)'_y = 0$;

2. $(\mathfrak{L}\mathfrak{S})'_x = (\mathfrak{L})'_x \mathfrak{S} + \mathfrak{L}(\mathfrak{S})'_x$ for any shift-invariant operator $\mathfrak{S}$;

3. $(\mathfrak{L}D_x)'_x = \mathfrak{L} + (\mathfrak{L})'_x D_x$;

4. $(\mathcal{E}_{(a,b)})'_x = a\mathcal{E}_{(a,b)}$, $(\mathcal{E}_{(a,b)})'_y = b\mathcal{E}_{(a,b)}$.

Two formulas from [21] we will be using are based on the fact that any univariate delta operator can be written as $\mathfrak{d} = D\mathfrak{L}$, where $\mathfrak{L}$ is an invertible shift-invariant operator. Let $\mathfrak{d} = D\mathfrak{L}$ be a delta operator with basic sequence $\{p_n\}_{n=0}^\infty$. Then

$$p_n(x) = \mathfrak{d}'\mathfrak{L}^{-n-1}(x^n) \tag{26}$$

and

$$p_n(x) = x\mathfrak{L}^{-n}(x^{n-1}). \tag{27}$$

The crucial step enabling the use of delta operators is Proposition 4.3, which allows us to solve a Gončarov interpolation problem on the type of linear grid used here by determining a basic sequence.

Theorem 5.1. *Let $(\mathfrak{d}_x, \mathfrak{d}_y)$ with $\mathfrak{d}_x = D_x\mathfrak{L}_x$, $\mathfrak{d}_y = D_y\mathfrak{L}_y$ be a separable pair of delta operators, and let Z be a linear transformation of $\mathbb{N}^2$ by a 2×2 matrix A as in Eqs. (23) and (24). Denote by $(t_{m,n}((x,y);Z))_{(m,n)\in\mathbb{N}^2}$ the sequence of Abel polynomials associated with the pair $(\mathfrak{d}_x, \mathfrak{d}_y)$ and the grid Z, and fix $m, n \in \mathbb{N}$. The following holds:*

(i) *Let p_m and q_n be the (univariate) basic sequences of $\mathfrak{d}_x$ and $\mathfrak{d}_y$ respectively. Then $t_{m,n}((x,y);Z)$ is given by*

$$\begin{vmatrix} x - x_{0,n} & y_{m,0} \\ x_{0,n} & y - y_{m,0} \end{vmatrix} \frac{p_m(x - x_{m,n})}{x - x_{m,n}} \cdot \frac{q_n(y - y_{m,n})}{y - y_{m,n}},$$

where $x_{i,j} = a_{1,1}i + a_{1,2}j$ and $y_{i,j} = a_{2,1}i + a_{2,2}j$ for all $0 \le i \le m$ and $0 \le j \le n$.

(ii) *Let $\mathfrak{s}_1 = D_x\mathfrak{L}_x\mathcal{E}_{(a_{1,1},a_{2,1})}$ and $\mathfrak{s}_2 = D_y\mathfrak{L}_y\mathcal{E}_{(a_{1,2},a_{2,2})}$. Then*

$$t_{m,n}((x,y);Z) = J(\mathfrak{s}_1, \mathfrak{s}_2)\mathcal{E}_{a_{1,1},a_{2,1}}^{-m-1}\mathcal{E}_{a_{1,2},a_{2,2}}^{-n-1}\mathfrak{L}_x^{-m-1}\mathfrak{L}_y^{-n-1}(x^m y^n), \tag{28}$$

where

$$J(\mathfrak{s}_1, \mathfrak{s}_2) = \begin{vmatrix} (\mathfrak{s}_1)'_x & (\mathfrak{s}_1)'_y \\ (\mathfrak{s}_2)'_x & (\mathfrak{s}_2)'_y \end{vmatrix}. \tag{29}$$

(iii) *$t_{m,n}((x,y);Z) = T_{m,n}(x^m y^n)$, where $T_{m,n}$ is the operator*

$$\begin{vmatrix} (\mathfrak{d}_x)'_x + a_{1,1}\mathfrak{d}_x & a_{2,1}\mathfrak{d}_x \\ a_{1,2}\mathfrak{d}_y & (\mathfrak{d}_y)'_y + a_{2,2}\mathfrak{d}_y \end{vmatrix}\mathcal{E}_{(a_{1,1},a_{2,1})}^{-m}\mathcal{E}_{(a_{1,2},a_{2,2})}^{-n}\mathfrak{L}_x^{-m-1}\mathfrak{L}_y^{-n-1}. \tag{30}$$

Proof. The proof consists of first showing that (ii) and (iii) are equivalent. Then we prove that (iii) yields the basic sequence of the pair $(\mathfrak{s}_1, \mathfrak{s}_2)$, which, by Proposition 4.3, is the same as the bivariate Abel polynomial associated with the pair $(\mathfrak{d}_x, \mathfrak{d}_y)$ and the grid Z. Lastly, we use (26) and (27) to derive (i).

Step 1. (ii) and (iii) are equivalent. We calculate

$$(\mathfrak{s}_1)'_x = \mathcal{E}_{(a_{1,1},a_{2,1})}[(\mathfrak{d}_x)'_x + a_{1,1}\mathfrak{d}_x];$$
$$(\mathfrak{s}_1)'_y = a_{2,1}\mathcal{E}_{(a_{1,1},a_{2,1})}\mathfrak{d}_x;$$
$$(\mathfrak{s}_2)'_x = a_{1,2}\mathcal{E}_{(a_{1,2},a_{2,2})}\mathfrak{d}_y;$$
$$(\mathfrak{s}_2)'_y = \mathcal{E}_{(a_{1,2},a_{2,2})}[(\mathfrak{d}_y)'_y + a_{2,2}\mathfrak{d}_y].$$

Substituting these values into (28) and (29), we obtain (iii).

Step 2. (iii) yields the basic sequence of the pair $(\mathfrak{s}_1, \mathfrak{s}_2)$. Let, for $m, n \geq 0$,

$$s_{m,n}(x,y) = T_{m,n}(x^m y^n).$$

By induction, it is enough to show that $s_{m,n}$ satisfies the following:

(a) $s_{0,0}(x,y) = 1$;

(b) $s_{m,n}(0,0) = 0$ if $m \geq 1$ or $n \geq 1$;

(c) $\mathfrak{s}_1(s_{m,n}) = m s_{m-1,n}$, $\mathfrak{s}_2(s_{m,n}) = n s_{m,n-1}$, and $\mathfrak{s}_1(s_{0,n}) = \mathfrak{s}_2(s_{m,0}) = 0$ for all $m, n \geq 1$.

Proof of (a). All the operators that appear in Eq. (30) commute with each other. Accordingly, we compute that $\mathfrak{d}_x(1) = \mathfrak{d}_y(1) = 0$, while $(\mathfrak{d}_x)'_x = \mathfrak{L}_x + D_x(\mathfrak{L}_x)'_x$ and $(\mathfrak{d}_y)'_y = \mathfrak{L}_y + D_y(\mathfrak{L}_y)'_y$. It follows that

$$s_{0,0}(x,y) = \mathfrak{L}_x^{-1}\mathfrak{L}_x\mathfrak{L}_y^{-1}\mathfrak{L}_y(1) = 1.$$

Proof of (c). Notice that $\mathfrak{s}_1 = D_x\mathfrak{L}_x\mathcal{E}_{(a_{1,1},a_{2,1})}$ commutes with all the operators involved in the definition of $s_{m,n}$ (but not with x^m or y^n). Therefore, we find that, for all $m, n \in \mathbf{N}$ such that $m + n \geq 1$, $\mathfrak{s}_1(s_{m,n}(x,y))$ is equal to

$$\begin{vmatrix} (\mathfrak{d}_x)'_x + a_{1,1}\mathfrak{d}_x & a_{2,1}\mathfrak{d}_x \\ a_{1,2}\mathfrak{d}_y & (\mathfrak{d}_y)'_y + a_{2,2}\mathfrak{d}_y \end{vmatrix} \mathcal{E}_{(a_{1,1},a_{2,1})}^{-m+1}\mathcal{E}_{a_{1,2},a_{2,2}}^{-n}\mathfrak{L}_x^{-m}\mathfrak{L}_y^{-n}(mx^{m-1}y^n),$$

that is, $\mathfrak{s}_1(s_{m,n}(x,y)) = mT_{m-1,n}(x^m y^n) = m s_{m-1,n}(x,y)$. In a similar way, we see that $\mathfrak{s}_2(s_{m,n}) = n s_{m,n-1}$ for all $m, n \in \mathbb{N}$ with $m + n \geq 1$.

Proof of (b). For all $m, n \geq 0$, we have from the definition of $s_{m,n}$ and the commutativity of shift-invariant operators that

$$s_{m,n}(x,y) = \mathcal{E}_{(a_{1,1},a_{2,1})}^{-m}\mathcal{E}_{(a_{1,2},a_{2,2})}^{-n}(\mathcal{H}_{m,n}(x,y)) \tag{31}$$

where the polynomial $\mathcal{H}_{m,n}(x,y)$ is given by the 2×2 determinant

$$\begin{vmatrix} (\mathfrak{d}_x)'_x\mathfrak{L}_x^{-m-1}(x^m) + a_{1,1}D_x\mathfrak{L}_x^{-m}(x^m) & a_{2,1}\mathfrak{L}_x^{-m}D_x(x^m) \\ a_{1,2}\mathfrak{L}_y^{-n}D_y(y^n) & (\mathfrak{d}_y)'_y\mathfrak{L}_y^{-n-1}(y^n) + a_{2,2}D_y\mathfrak{L}_y^{-n}(y^n) \end{vmatrix}.$$

Now we combine the previous equations with the univariate formulas (26) and (27), which we use in the form

$$(\mathfrak{d}_x)'_x\mathfrak{L}^{-m-1}(x^m) = x\mathfrak{L}^{-m}(x^{m-1}).$$

If $m \geq 1$ and $n \geq 1$, we get

$$\mathcal{H}_{m,n}(x,y) = \begin{vmatrix} x\mathfrak{L}_x^{-m}(x^{m-1}) + ma_{1,1}\mathfrak{L}_x^{-m}(x^{m-1}) & ma_{2,1}\mathfrak{L}_x^{-m}(x^{m-1}) \\ na_{1,2}\mathfrak{L}_y^{-n}(y^{n-1}) & y\mathfrak{L}_y^{-n}(y^{n-1}) + na_{2,2}\mathfrak{L}_y^{-n}(y^{n-1}) \end{vmatrix},$$

which ultimately leads to

$$s_{m,n}(x,y) = \mathcal{E}_{(a_{1,1},a_{2,1})}^{-m}\mathcal{E}_{(a_{1,2},a_{2,2})}^{-n} \begin{vmatrix} x + ma_{1,1} & ma_{2,1} \\ na_{1,2} & y + na_{2,2} \end{vmatrix} \mathfrak{L}_x^{-m}\mathfrak{L}_y^{-n}(x^{m-1}y^{n-1}).$$

In these computations, the monomial x^m (respectively, y^n) must stay to the right of any operator acting on x^m (respectively, on y^n).

Applying the shifts, and taking into account that $\mathcal{E}_{(a_{1,1},a_{2,1})}^{-m} = \mathcal{E}_{(-ma_{1,1},-ma_{2,1})}$ and $\mathcal{E}_{(a_{1,2},a_{2,2})}^{-n} = \mathcal{E}_{(-na_{1,2},-na_{2,2})}$, we obtain that $s_{m,n}(x,y)$ equals

$$\begin{vmatrix} x - na_{1,2} & ma_{2,1} \\ na_{1,2} & y - ma_{2,1} \end{vmatrix} \mathfrak{L}_x^{-m}\mathfrak{L}_y^{-n}((x - ma_{1,1} - na_{1,2})^{m-1}(y - ma_{1,1} - na_{2,2})^{n-1}),$$

which simplifies to give

$$s_{m,n}(x,y) = \begin{vmatrix} x - x_{0,n} & y_{m,0} \\ x_{0,n} & y - y_{m,0} \end{vmatrix} \mathfrak{L}_x^{-m}\mathfrak{L}_y^{-n}((x - x_{m,n})^{m-1}(y - y_{m,n})^{n-1}). \qquad (32)$$

Evaluating at $(x,y) = (0,0)$, we get $s_{m,n}(0,0) = 0$.

On the other hand, if one of m and n is zero, say $m \geq 1$ and $n = 0$, then, starting from Eq. (31), using (26) and (27) only on x, and recalling that $(\mathfrak{d}_y)'_y = \mathfrak{L}_y + D_y(\mathfrak{L}_y)'_y$, we have

$$\begin{aligned} s_{m,0}(x,y) &= \mathcal{E}_{(a_{1,1},a_{2,1})}^{-m} \begin{vmatrix} x\mathfrak{L}_x^{-m}(x^{m-1}) + a_{1,1}m\mathfrak{L}_x^{-m}(x^{m-1}) & a_{2,1}m\mathfrak{L}_x^{-m}(x^{m-1}) \\ 0 & 1 \end{vmatrix} \\ &= \mathcal{E}_{(a_{1,1},a_{2,1})}^{-m}[x + ma_{1,1}]\mathfrak{L}_x^{-m}(x^{m-1}) \\ &= x\mathfrak{L}_x^{-m}((x - ma_{1,1})^{m-1}). \end{aligned} \qquad (33)$$

Evaluating at $(0,0)$ yields $s_{m,0}(0,0) = 0$. Similarly for $m = 0$ and $n \geq 1$. This confirms (b).

Step 3. We prove (i). We have from Eq. (27) that

$$\mathfrak{L}_x^{-m}(x^{m-1}) = \frac{p_m(x)}{x} \quad \text{and} \quad \mathfrak{L}_y^{-n}(y^{n-1}) = \frac{q_n(y)}{y}.$$

These, together with Eqs. (32) and (33), yields (i). $\qquad \square$

By way of example, we now consider some special cases of the general formulas implied by Theorem 5.1.

Example 1. If A is the identity matrix, then the basic sequence of the pair $(\mathfrak{d}_x, \mathfrak{d}_y)$ is $\{p_m(x)q_n(y)\}_{(m,n)\in\mathbb{N}^2}$, and the delta Gončarov polynomials are

$$t_{m,n}((x,y);Z) = \frac{xp_m(x-m)}{x-m} \cdot \frac{yq_n(y-n)}{y-n}.$$

Example 2. For the classical Abel polynomials, where $\mathfrak{d}_x = D_x$ and $\mathfrak{d}_y = D_y$, we have $p_m(x) = x^m$ and $q_n(y) = y^n$. Hence Theorem 5.1(i) provides the same formula (25) obtained in [15].

Example 3. Forward differences are given by the delta operator $\mathfrak{d} = \mathcal{E}_1 - \mathfrak{I} = D\mathfrak{L}$, where $\mathfrak{L}f(x) = \int_x^{x+1} f(t)dt$ is the Bernoulli operator. The n-th basic polynomial for this operator is the lower factorial function

$$p_n(x) = (x)_{(n)} = x(x-1)\cdots(x-n+1).$$

So, if the delta operators $\mathfrak{d}_x$ and $\mathfrak{d}_y$ are forward difference operators relative to x and y respectively, then the delta Abel polynomial $t_{m,n}((x,y);Z)$ is equal to

$$[(x - x_{0,n})(y - y_{m,0}) - x_{0,n}y_{m,0}] \cdot (x - x_{m,n} - 1)_{(m-1)}(y - y_{m,n} - 1)_{(n-1)}.$$

Similarly, backward differences are given by the delta operator $\mathfrak{d} = \mathfrak{I} - \mathcal{E}_{-1} = D\mathfrak{L}$, with $\mathfrak{L}f(x) = \int_{x-1}^{x} f(t)dt$. The n-th basic polynomial associated to this operator is the upper factorial function

$$p_n(x) = (x)^{(n)} = x(x+1)\cdots(x+n-1).$$

So, if the delta operators $\mathfrak{d}_x$ and $\mathfrak{d}_y$ are, respectively, backward difference operators relative to x and y, then the delta Abel polynomial $t_{m,n}((x,y);Z)$ equals

$$[(x - x_{0,n})(y - y_{m,0}) - x_{0,n}y_{m,0}] \cdot (x - x_{m,n} + 1)^{(m-1)}(y - y_{m,n} + 1)^{(n-1)}.$$

6. Closed formulas for multivariate delta Abel polynomials

In this section we give closed formulas for those multivariate delta Abel polynomials whose system of delta operators $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ is separable, which means that, for each $i = 1, \ldots, d$,

$$\mathfrak{d}_i = D_i \sum_{j=0}^{\infty} a_j^{(i)} D_i^j = D_i \mathfrak{L}_i, \tag{34}$$

for some real coefficients $a_0^{(i)}, a_1^{(i)}, \ldots$ with $a_0^{(i)} \neq 0$.

Let $\{p_n^{(i)}\}_{n \in \mathbb{N}}$ be the basic sequence of $\mathfrak{d}_i$. Note that $p_n^{(i)}$ is only a function of the variable x_i and that the delta operator $\mathfrak{d}_i$ contains only partial derivatives with respect to x_i. The grid Z is a linear transformation of $\mathbb{N}^d$ by a certain $d \times d$ matrix $A = (a_{i,j})$.

Theorem 6.1. *Let the system of delta operators $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$, the matrix A, and the grid Z be defined as above. Let B be the $d \times d$ diagonal matrix with*

$$B_{i,i} = x_i - z_{\mathbf{n},i} \qquad (1 \leq i \leq d), \tag{35}$$

and C the $d \times d$ matrix whose (i,j)-entry is given by

$$C_{i,j} = z_{n_i \mathbf{e_i},j}, \tag{36}$$

where, for a vector $\mathbf{k} \in \mathbb{N}^d$, $z_{\mathbf{k},j}$ is the j-th entry of the point $z_{\mathbf{k}}$. Moreover, let F be the $d \times d$ diagonal matrix of operators with

$$F_{i,i} = (\mathfrak{d}_i)'_{x_i} \qquad (1 \leq i \leq d), \tag{37}$$

and G the $d \times d$ matrix of operators whose (i,j)-entry is $G_{i,j} = a_{i,j}\mathfrak{d}_i$.

Denote by $(t_{\mathbf{n}}(\mathbf{x};Z))_{\mathbf{n} \in \mathbb{N}^d}$ the sequence of delta Abel polynomials associated with $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_d)$ and the grid Z, and let $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbf{N}^d$ be fixed. The following statements hold:

(i) *We have*

$$t_{\mathbf{n}}(\mathbf{x}; Z) = \det(B + C) \prod_{i=1}^{d} \frac{p_{n_i}^{(i)}(x_i - z_{\mathbf{n},i})}{x_i - z_{\mathbf{n},i}}. \tag{38}$$

(ii) *Let* $\mathfrak{s}_i = \mathfrak{d}_i \mathcal{E}_{\mathbf{t}_i}$, *where* $\mathbf{t}_i$ *is the i-th column vector of the matrix A. Then*

$$t_{\mathbf{n}}(\mathbf{x}; Z) = J((\mathfrak{s})'_{\mathbf{x}}) \left(\prod_{i=1}^{d} \mathfrak{L}_i^{-n_i-1} \right) \left(\prod_{i=1}^{d} \mathcal{E}_{\mathbf{t}_i}^{-n_i-1} \right) \mathbf{x}^{\mathbf{n}}, \tag{39}$$

where $J((\mathfrak{s})'_{\mathbf{x}})$ is the Jacobian matrix whose (i,j)-entry is $(\mathfrak{s}_i)'_{x_j}$.

(iii) *We have*

$$t_{\mathbf{n}}(\mathbf{x}; Z) = \det(F + G) \left(\prod_{i=1}^{d} \mathfrak{L}_i^{-n_i-1} \right) \left(\prod_{i=1}^{d} \mathcal{E}_{\mathbf{t}_i}^{-n_i} \right) \mathbf{x}^{\mathbf{n}}. \tag{40}$$

Proof. We only sketch the proof here since the computations are all similar to those in the bivariate case.

Step 1. We prove that (39) and (40) are equivalent by calculating that

$$(\mathfrak{s}_i)'_{x_j} = \begin{cases} (\mathfrak{d}_i)'_{x_i} \mathcal{E}_{\mathbf{t}_i} + a_{i,i} \mathfrak{d}_i \mathcal{E}_{\mathbf{t}_i} & \text{if } i = j \\ a_{j,i} \mathfrak{d}_i \mathcal{E}_{\mathbf{t}_i} & \text{if } i \neq j \end{cases}.$$

Substituting into (39) we obtain (40).

Step 2. We check that Eq. (40) yields the basic sequence of the system $(\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_d)$, since then this is also the desired delta Gončarov polynomials, by Proposition 4.3. For that, denote by $s_{\mathbf{n}}(\mathbf{x})$ the right-hand side of (40). By induction, it suffices to prove that:

(a) $s_{\mathbf{0}}(\mathbf{x}) = 1$;

(b) $s_{\mathbf{n}}(0, 0) = 0$ if $\mathbf{n} \neq \mathbf{0}$;

(c) $\mathfrak{s}_i(s_{\mathbf{n}}) = n_i s_{\mathbf{n}-\mathbf{e}_i}$ for all $i = 1, \ldots, d$.

Point (a) is immediate, when considering that $\mathfrak{d}_i(1) = D_i(1) = 0$ and

$$(\mathfrak{d}_i)'_{x_i} \mathfrak{L}_i^{-1}(1) = (\mathfrak{L}_i + D_i(\mathfrak{L}_i)'_{x_i}) \mathfrak{L}_i^{-1}(1) = 1.$$

As for (b), we use again Equations (26) and (27) to get $(\mathfrak{d}_i)'_{x_i} \mathfrak{L}_i^{-n_i-1} x_i^{n_i} = x_i \mathfrak{L}_i^{-n_i} x_i^{n_i-1}$. Hence if $n_i \geq 1$ for all i, $s_{\mathbf{n}}(\mathbf{x})$ can be expressed as

$$s_{\mathbf{n}}(\mathbf{x}; Z) = \left(\prod_{i=1}^{d} \mathcal{E}_{\mathbf{t}_i}^{-n_i} \right) \det(K(\mathbf{x})) \prod_{i=1}^{d} \mathfrak{L}_i^{-n_i}(x_i^{n_i-1}),$$

where $K(\mathbf{x})$ is the $d \times d$ matrix with entries

$$k_{i,j} = \begin{cases} x_i + n_i a_{i,i} & \text{if } i = j \\ n_i a_{j,i} & \text{if } i \neq j \end{cases}.$$

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Applying the shift operators first, we have

$$s_{\mathbf{n}}(\mathbf{x}) = \det(M(\mathbf{x})) \prod_{i=1}^{d} \left(\mathfrak{L}_i^{-n_i}((x_i - z_{\mathbf{n},i})^{n_i-1}) \right), \tag{41}$$

where $M(\mathbf{x})$ is the $d \times d$ matrix with entries

$$m_{i,j} = \begin{cases} x_i + n_i a_{i,i} - \sum_{j=1}^{d} n_j a_{i,j} & \text{if } i = j \\ n_i a_{j,i} & \text{if } i \neq j \end{cases}.$$

When $\mathbf{x} = \mathbf{0}$, $\det(M) = 0$ since every column of $M(\mathbf{0})$ has sum zero. Hence $s_{\mathbf{n}}(\mathbf{0}) = 0$ whenever $\mathbf{n}$ has no zero entry.

For the case that some $n_i = 0$ but $\mathbf{n} \neq \mathbf{0}$, the computation reduces to an equivalent one at a lower dimension and with an index vector whose entries are all positive. Since the above argument applies to all dimensions $d \geq 1$, we conclude $s_{\mathbf{n}}(\mathbf{0}) = 0$ for all $\mathbf{n} \neq \mathbf{0}$.

As for (c), note that $\mathfrak{s}_i = D_i \mathfrak{L}_i \mathcal{E}_{\mathbf{t}_i}$ commutes with all operators in the definition of $s_{\mathbf{n}}$. Hence we can compute $\mathfrak{s}_i(s_{\mathbf{n}})$ by applying $\mathfrak{s}_i$ to $\mathbf{x}^{\mathbf{n}}$ first, then applying the other operators in the formula of $s_{\mathbf{n}}$. This yields exactly the formula of $n_i s_{\mathbf{n}-\mathbf{e}_i}$.

Step 3. We derive Eq. (38) from (40). Notice that in terms of $\mathbf{n}$ and the matrix $A = (a_{i,j})$, Eqs. (35) and (36) can be re-written as

$$B_{i,i} = x_i - \sum_{j=1}^{d} n_j a_{i,j}, \qquad C_{i,j} = n_i a_{j,i}.$$

Hence the matrix M in (41) is exactly $B+C$. Formula (38) is then obtained by using the equation

$$\mathfrak{L}_i^{-n_i}((x - z_{\mathbf{n},i})^{n_i-1}) = \frac{p_{n_i}^{(i)}(x_i - z_{\mathbf{n},i})}{x_i - z_{\mathbf{n},i}}.$$

This completes our proof. $\qquad \square$

REMARK 3. Given a grid $Z \subseteq \mathbb{R}^d$ and a vector $\mathbf{v} \in \mathbb{R}^d$, let $Z + \mathbf{v} = \{z_{\mathbf{k}} + \mathbf{v} : z_{\mathbf{k}} \in Z\}$. It is easy to verify that the delta Gončarov polynomials are translation invariant, i.e., they satisfy $t_{\mathbf{n}}(\mathbf{x} + \mathbf{v}; Z + \mathbf{v}) = t_{\mathbf{n}}(\mathbf{x}; Z)$.

As a consequence, if Z' is an affine transformation of $\mathbb{N}^d$, i.e., $Z' = \{z'_{\mathbf{k}} = \mathbf{v} + A\mathbf{k} : \mathbf{k} \in \mathbb{N}^d\}$ for some $d \times d$ matrix A, then Theorem 6.1 yields a closed formula for the delta Gončarov polynomials associated with the grid Z' by the equation $t_{\mathbf{n}}(\mathbf{x}; Z') = t_{\mathbf{n}}(\mathbf{x} - \mathbf{v}; Z)$, where $Z = \{z_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d, z_{\mathbf{k}} = A\mathbf{k}\}$.

REMARK 4. In [15, Theorem (4.3)] a formula was given for the classical trivariate Abel polynomials. It was noted that the formula there was not valid for all linear matrices and suggested the question of finding a closed general formula for the classical Abel polynomials in three variables.

Since in the classical case, the system of delta operators is separable, Eq. (38) gives a complete answer for this question. In dimension 3 let the variables be x, y, z, then (38) specializes to

$$t_{m,n,p}((x,y,z); Z) = \begin{vmatrix} x - x_{0,n,p} & y_{m,0,0} & z_{m,0,0} \\ x_{0,n,0} & y - y_{m,0,p} & z_{0,n,0} \\ x_{0,0,p} & y_{0,0,p} & z - z_{m,n,0} \end{vmatrix} \cdot$$

$$\cdot \frac{p_m(x - x_{m,n,p})}{x - x_{m,n,p}} \cdot \frac{q_n(y - y_{m,n,p})}{y - y_{m,n,p}} \cdot \frac{r_p(z - z_{m,n,p})}{z - z_{m,n,p}}, \tag{42}$$

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where $\{p_m\}$, $\{q_n\}$ and $\{r_p\}$ are the basic sequences of the univariate operators $\mathfrak{d}_x$, $\mathfrak{d}_y$ and $\mathfrak{d}_z$ respectively. When $(\mathfrak{d}_x, \mathfrak{d}_y, \mathfrak{d}_z) = (D_x, D_y, D_z)$, using $p_m(x) = x^m$, $q_n(y) = y^n$ and $r_p(z) = z^p$, we obtain from (42) a complete formula for the classical trivariate Abel polynomials.

Acknowledgments

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