

Exact Formulas for Moments of Sums of Classical Parking Functions

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Abstract

For given positive integers a and b , an $[a, b]$ -parking function of length n is a sequence (x_1, x_2, \dots, x_n) of positive integers whose order statistics $x_{(i)}$ (the sequence obtained by rearranging the original sequence in non-decreasing order) satisfy the inequalities $x_{(i)} \leq a + (i - 1)b$ for all i . Using elementary methods, we derive explicit formulas for higher moments of sums and reversed sums of parking functions of length n . These formulas are finite double sums with reasonably simple terms. They are obtained by solving a recursion based on a combinatorial decomposition which decomposes a sequence of positive integers into a “maximum” parking function and a subsequence all of whose terms have “high” values. Moments of reversed sums of ordinary parking functions (the case when $a = b = 1$) have a surprising connection with the enumeration of “sparsely-edged” graphs. From this connection, we obtain exact formulas and derive, using routine methods, asymptotic formulas (due originally to E. M. Wright) for the number of connected labeled graphs on N vertices and $N + k$ edges. Our method yields a new formula for the Wright constants in terms of a sequence which satisfies a linear recursion.

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1 Introduction

An *ordinary parking function* of length n is a sequence (x_1, x_2, \dots, x_n) of positive integers whose order statistics (the sequence $(x_{(1)}, x_{(2)}, \dots, x_{(n)})$ obtained by rearranging the original sequence in non-decreasing order) satisfy $x_{(i)} \leq i$. Ordinary parking functions arose in the theory of linear probes [7] and they have been extensively studied (see [5]). In particular, computer scientists are interested in the moments of their sums. See [1, 2, 4, 6].

In the paper [9], we studied \mathbf{u} -parking function, where \mathbf{u} is a sequence (u_1, u_2, u_3, \dots) of non-decreasing positive integers. These are sequences of positive integers whose order statistics satisfy $x_{(i)} \leq u_i$. Using Gončarov polynomials, we obtain Appell relations for generating functions of moments of sums of \mathbf{u} -parking functions and show how, in principle, one can obtain explicit formulas for the moments in terms of Gončarov polynomials. When specialized to the case $u_i = i$ of ordinary parking functions, the formulas for the mean and second moment are different from (but equivalent to) the known formulas obtained earlier in [1, 2, 6]. In this paper, we use a variation of our method in [9, 10] to obtain higher-moment formulas similar to the known formulas for the first and second moment.

Higher moments of sums of ordinary parking functions have been studied by Knuth [6] and Flajolet, Poblete and Viola [1] as part of an cost analysis of linear probing and hashing. In particular, Knuth obtained exact formulas for mean and second moment of reversed sums of ordinary parking functions. Using singularity analysis and a functional equation for the “tree function”, Flajolet, Poblete and Viola obtained asymptotic formulas and the limit distribution for higher moments. They also show, in principle, how one might obtain exact formulas for any specific higher moment by a “pumping” process. Further work on the limit distribution can be found in [4].

Our approach is different from these earlier work. We begin with an combinatorial decomposition for sequences of positive integers. The special case of this decomposition for ordinary parking function appeared in the paper by Konheim and Weiss [7] which founded the subject. This decomposition yields a linear recursion for the higher moments. We solve this recursion and obtain exact formulas which are double sums with terms which are as simple and explicit as one might reasonably expect. From the exact formulas, one could, with sufficient patience, obtain asymptotic formulas to as high an order as one wishes. Our approach is deliberately elementary and explicit, with the emphasis on how we discovered the formulas. We expect that once the formulas are known, it would be possible, as is common in mathematics, to verify them by other methods.

In addition, we shall work with $[a, b]$ -parking functions, which are \mathbf{u} -parking functions where u_i equals $a + (i - 1)b$. We call this the “classical” case. There are at least three reasons for doing this. One is that such parking functions have come up naturally, for example, in labeling regions in extended Shi arrangements, and in enumeration of multicolored graphs. See [15, 18]. Another is that although explicit computations are slightly more complicated than for ordinary parking functions, the proofs are the same. Finally, “parametrizing” the problem clarifies how things work, and, in particular, the effect of the first term a and the difference b on the moments.

As is well-known [9, 12, 15], there are

$$a(a + nb)^{n-1}$$

$[a, b]$ -parking functions. Let

$$S_n = X_1 + X_2 + \dots + X_n$$

be the sum of a random $[a, b]$ -parking function (X_1, X_2, \dots, X_n) of length n chosen with uniform distribution from the sample space of all $[a, b]$ -parking functions. Let $E_k(n; a, b)$ be the expected value of the k th (falling) factorial moment of S_n , that is,

$$E_k(n; a, b) = \frac{1}{a(a + nb)^{n-1}} \sum_{(x_1, x_2, \dots, x_n)} (x_1 + x_2 + \dots + x_n)_k,$$

where $(N)_k$ is the falling factorial $N(N - 1) \dots (N - k + 1)$ and the sum ranges over all $[a, b]$ -parking functions (x_1, x_2, \dots, x_n) .

In [10], we prove the following formula for expected sums of $[a, b]$ -parking functions:

$$E_1(n; a, b) = \frac{n(a + nb + 1)}{2} - \frac{1}{2} \sum_{j=1}^n \binom{n}{j} \frac{j!b^j}{(a + nb)^{j-1}} \quad (1.1)$$

There are similar formulas for the higher moments.

Theorem 1.1. *The k th factorial moment $E_k(n; a, b)$ of the sum of a random $[a, b]$ -parking function of length n equals*

$$U_k(n) + \sum_{r=1}^k \left[\binom{k}{r} U_{k-r}(n) \left(\sum_{j=1}^n \binom{n}{j} \frac{j!b^j}{(a + nb)^{j-1}} \tau_r(j) \right) \right], \quad (1.2)$$

where $U_r(x)$ is a polynomial of degree $2r$ derived from the r th factorial moments of the sum of a random sequence with terms from a discrete interval and $\tau_r(x)$ is a polynomial of degree not exceeding $3r - 3$. The polynomials $\tau_r(x)$ do not depend on k and their coefficients are polynomials in a and b .

The polynomials $U_r(x)$ will be defined in Section 2. A method to calculate the polynomials $\tau_r(x)$ recursively will be given implicitly in the proof of Theorem 1.1.

The proof of Theorem 1.1 is somewhat complicated. In Section 2, we describe the combinatorial decomposition underlying the proof. From this decomposition, we derive, in Section 3, linear recursions for factorial moments of sums of $[a, b]$ -parking functions. These recursions are special cases of recursions for \mathbf{u} -parking functions derived in [9]. We also recast the linear recursions as matrix equations. To solve the matrix equations, we need results about Stirling numbers, two-way sums, and matrix relations (Section 4). These three threads are woven together in Section 5 to prove Theorem 1.1. The proof does not yield the exact degree (almost certainly $3r - 3$) of $\tau_r(x)$. In Section 6, we show that for infinitely many r the exact degree of $\tau_r(x)$ is $3r - 3$ by deriving a recursion for the coefficient t_r of x^{3r-3} in $\tau_r(x)$. In Section 7, we use Theorem 1.1 to obtain formulas for the moments of reversed sums of parking functions. These formulas are used in the last section to derive asymptotic formulas for the number of “sparsely-edged” graphs and a new way to calculate the Wright constants occurring in these formulas.

In this paper, we shall work with vectors of dimension $N + 1$ and $(N + 1) \times (N + 1)$ matrices, where the indices run from 0 to N . If $f(x)$ is a function, then

$$\overrightarrow{f(i)}$$

is the $(N + 1)$ -dimensional column vector with i th coordinate equal to $f(i)$ and

$$\overrightarrow{f(i)^*}$$

is the $(N + 1)$ -dimensional column vector $(0, f(1), f(2), \dots, f(N))^T$, with the zeroth coordinate equal to 0 and the i th coordinate equal to $f(i)$ for $i \geq 1$. Furthermore, we denote by

$$\mathcal{D}(f(i))$$

the diagonal matrix whose ii th entry is $f(i)$. The matrix \mathcal{P} is the lower triangular matrix

$$\left[(-1)^{n-i} \binom{n}{i} a(a + ib)^{n-1} \right]_{0 \leq n, i \leq N}.$$

The matrix \mathcal{Q} is lower triangular matrix

$$\left[\binom{i}{j} \frac{j!b^j}{(a + ib)^{j-1}} \right]_{0 \leq i, j \leq N}.$$

2 The underlying combinatorial decomposition

We shall use the following special case of a combinatorial decomposition proved in [9].

Theorem 2.1. *There is a bijection between the set $[1, x]^n$ of all length- n integer sequences with terms in the set $\{1, 2, \dots, x\}$ and the disjoint union of Cartesian products*

$$\bigcup_{\substack{\{i_1, i_2, \dots, i_m\} \\ 0 \leq m \leq n}} \text{Park}(i_1, i_2, \dots, i_m) \times [a + mb + 1, x]^{n-m},$$

where $\text{Park}(i_1, i_2, \dots, i_m)$ is the set of subsequences indexed by the set $\{i_1, i_2, \dots, i_m\}$ which are $[a, b]$ -parking functions of length m , and $[a + mb + 1, x]^{n-m}$ is the set of subsequences indexed by the complement of $\{i_1, i_2, \dots, i_m\}$ which form length- $(n - m)$ integer sequences with terms in $\{a + mb + 1, a + mb + 2, \dots, x\}$.

To use Theorem 2.1, we need formulas for factorial moments of a sum $X_1 + X_2 + \dots + X_n$ of independent uniformly distributed random variables X_i taking values in the discrete interval $\{A, A + 1, \dots, B\}$, where A and B are positive integers and $A < B$. Recall that the probability generating function of a uniformly distributed random variable X_i on $\{A, A + 1, \dots, B\}$ is

$$\frac{1}{B + 1 - A} (t^A + t^{A+1} + \dots + t^B),$$

and hence, the (exponential) factorial moment generating function of the sum $X_1 + X_2 + \dots + X_n$ is

$$\frac{1}{(B + 1 - A)^n} ((1 + t)^A + (1 + t)^{A+1} + \dots + (1 + t)^B)^n.$$

In particular, the k th factorial moment $U_k(n; A, B)$ can be obtained by differentiating the generating function k times and setting $t = 0$. For example, $U_0(n; A, B) = 1$ and

$$U_1(n; A, B) = \frac{1}{2}n(A + B). \quad (2.1)$$

To calculate $U_2(n; A, B)$, we differentiate twice and set $t = 0$ to obtain

$$\begin{aligned} U_2(n; A, B) &= n(n-1) \left[\frac{A + (A+1) + \dots + B}{B+1-A} \right]^2 + n \left[\frac{A(A-1) + (A+1)A + \dots + B(B-1)}{B+1-A} \right] \\ &= \frac{1}{4}n(n-1)(A+B)^2 + \frac{1}{3}n[B(B-1) + A(A+B-2)]. \end{aligned} \quad (2.2)$$

Similarly,

$$\begin{aligned} U_3(n; A, B) &= \frac{1}{8}n(n-1)(n-2)(A+B)^3 + \frac{1}{2}n(n-1)(A+B)[B(B-1) + A(A+B-2)] \\ &\quad + \frac{1}{4}n[B(B-1)(B-2) + A[(B-1)(B-2) + (A-1)(A+B-4)]]. \end{aligned} \quad (2.3)$$

For arbitrary k , consider the function $p(t; A, B)$ defined by

$$p(t; A, B) = \frac{1}{B+1-A} ((1+t)^A + (1+t)^{A+1} + \dots + (1+t)^B). \quad (2.4)$$

Then

$$\begin{aligned} p^{(s)}(0; A, B) &= \left. \frac{d^s}{dt^s} p(t) \right|_{t=0} \\ &= \frac{(B+1)_{s+1} - (A)_{s+1}}{(B+1-A)(s+1)}. \end{aligned}$$

In particular, $p^{(s)}(0; A, B)$ is a polynomial in the variables A and B of degree s . For example, $p^{(0)}(0; A, B) = 1$ and $p^{(1)}(0; A, B) = (A + B)/2$. If $k \geq 1$, then the k th factorial moment $U_k(n; A, B)$ is a linear combination of terms of the form

$$(n)_r (p^{(0)}(0; A, B))^{n-r} (p^{(1)}(0; A, B))^{a_1} \cdots (p^{(k)}(0; A, B))^{a_k},$$

where $1 \leq r \leq k$, $a_i \geq 0$, $a_1 + \cdots + a_k = r$, $a_1 + 2a_2 + \cdots + ka_k = k$. Hence, $U_k(n; A, B)$ is a polynomial in the variables n , A and B , with “leading” term

$$(n)_k (p^{(1)}(0; A, B))^k = n(n-1) \cdots (n-k+1)(A+B)^k / 2^k$$

when expanded as a linear combination of falling factorials in n .

From the factorial moments $U_k(n; A, B)$, we define the polynomials $U_k(n)$ in the single variable n by

$$U_k(n) := U_k(n; a + nb + 1, 0). \quad (2.5)$$

In particular, $U_0(n) = 1$,

$$U_1(n) = \frac{1}{2}n(a + nb + 1) \quad (2.6)$$

$$U_2(n) = \frac{1}{4}n(n-1)(a + nb + 1)^2 + \frac{1}{3}n(a + nb + 1)(a + nb - 1). \quad (2.7)$$

$$U_3(n) = \frac{1}{8}n(n-1)(n-2)(a + nb + 1)^3 + \frac{1}{2}n(n-1)(a + nb + 1)^2(a + nb - 1) \quad (2.8)$$

$$+ \frac{1}{4}n(a + nb + 1)[2 + (a + nb)(a + nb - 3)]. \quad (2.9)$$

In this notation, we can rewrite Eq. (1.1) as

$$E_1(n; a, b) = U_1(n) + \sum_{j=1}^n \binom{n}{j} \frac{j!b^j}{(a + nb)^{j-1}} \tau_1(j),$$

where $\tau_1(j) = -\frac{1}{2}$.

Since the exponential generating function for $U_k(n; A, B)$ is the n th power of a fixed function, the moments $U_k(n; A, B)$ satisfy the following identity of binomial type

$$U_k(x + y; A, B) = \sum_{j=0}^k \binom{k}{j} U_j(x; A, B) U_{k-j}(y; A, B)$$

as polynomials in the variables x and y . In particular, we have the following identity.

Lemma 2.2.

$$U_k(n; a + ib + 1, 0) = \sum_{j=0}^k \binom{k}{j} U_{k-j}(n - i; a + ib + 1, 0) U_j(i)$$

3 The linear recursion

In this section, we begin the proof of Theorem 1.1. For brevity's sake, we shall use the notation $T_0(n) = 1$, and for $n \geq 1$,

$$T_r(n) = \sum_{j=1}^n \binom{n}{j} \frac{j!b^j}{(a + nb)^{j-1}} \tau_r(j). \quad (3.1)$$

The proof of Theorem 1.1 is by induction on k . The case $k = 0$ is obviously true. We assume that $k \geq 1$ and that $E_r(n; a, b)$ is given by the formula in Theorem 1.1 for $0 \leq r \leq k - 1$.

Using the combinatorial decomposition given in Theorem 2.1, we can calculate the k th factorial moment of a sum of a random sequence in $[1, x]^n$ in two different ways (see Theorem 7.1 in [9]). This yields the following linear recursion for $E_k(n; a, b)$:

$$x^n U_k(n; 1, x) = \sum_{i=0}^n \binom{n}{i} (x - a - bi)^{n-i} a (a + bi)^{i-1} \cdot \left[\sum_{s=0}^k \binom{k}{s} E_s(i; a, b) U_{k-s}(n - i; a + ib + 1, x) \right].$$

This is an identity holding for all sufficiently large positive integers x and hence, it holds as a polynomial identity in the variable x . Setting $x = 0$, we obtain the linear recursion

$$\begin{aligned} & \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a (a + bi)^{n-1} E_k(i; a, b) \\ &= - \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a (a + bi)^{n-1} \left(\sum_{s=0}^{k-1} \binom{k}{s} E_s(i; a, b) U_{k-s}(n - i; a + ib + 1, 0) \right). \end{aligned} \quad (3.2)$$

For example, when $k = 2$, this linear recursion is

$$\begin{aligned} & \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a (a + ib)^{n-1} E_2(i; a, b) \\ &= - \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a (a + ib)^{n-1} [E_1(i; a, b)(n - i)(a + ib + 1) \\ & \quad + \frac{1}{4}(n - i)(n - i - 1)(a + ib + 1)^2 + \frac{1}{3}(n - i)(a + ib + 1)(a + ib - 1)]. \end{aligned} \quad (3.3)$$

The left hand side of the recursion (3.2) is the n th coordinate of the vector

$$\overrightarrow{\mathcal{P}E_k(i; a, b)}.$$

We shall put the right hand side into a similar form. Using the induction hypothesis that for $0 \leq s \leq k - 1$, the s th factorial moments $E_s(i; a, b)$ are given by formula (1.2), we write the inner sum on the right hand side of (3.2) as

$$\sum_{s=0}^{k-1} \binom{k}{s} U_{k-s}(n - i; a + ib + 1, 0) \left(\sum_{r=0}^s \binom{s}{r} U_{s-r}(i) T_r(i) \right).$$

Changing the order of summation, this can be rewritten as

$$\sum_{r=0}^{k-1} \left(\sum_{s=r}^{k-1} \binom{k}{s} \binom{s}{r} U_{k-s}(n - i; a + ib + 1, 0) U_{s-r}(i) \right) T_r(i).$$

The coefficient of $T_r(i)$ can be further simplified. Changing the index of summation from s to $l = s - r$ and using the binomial coefficient identity

$$\binom{k}{l+r} \binom{l+r}{r} = \binom{k}{r} \binom{k-r}{l},$$

the coefficient of $T_r(i)$ equals

$$\binom{k}{r} \sum_{l=0}^{(k-r)-1} \binom{k-r}{l} U_{(k-r)-l}(n - i; a + ib + 1, 0) U_l(i).$$

By Lemma 2.2, this equals

$$-\binom{k}{r}[U_{k-r}(i) - U_{k-r}(n; a + ib + 1, 0)].$$

Thus, the right hand side of (3.2) consists of two sums.

The first sum is

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a(a+ib)^{n-1} \left(\sum_{r=0}^{k-1} \binom{k}{r} U_{k-r}(i) T_r(i) \right).$$

Since the inner sum depends only on i , the first sum is the n th coordinate of the vector

$$\overrightarrow{\mathcal{P} \sum_{r=0}^{k-1} \binom{k}{r} U_{k-r}(i) T_r(i)}.$$

The second sum,

$$-\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a(a+ib)^{n-1} \left(\sum_{r=0}^{k-1} \binom{k}{r} U_{k-r}(n; a + ib + 1, 0) T_r(i) \right), \quad (3.4)$$

is more complicated. To handle it, observe that $U_{k-r}(n; a + ib + 1, 0)$ is a linear combination of products

$$n(n-1) \dots (n-m+1) f(a+ib)$$

of a falling factorial in n and a polynomial in $a+ib$, where $1 \leq m \leq k-r$ and the degree of the polynomial $f(x)$ does not exceed $k-r$. Hence, the second sum is a linear combination of sums of the form

$$n^\alpha \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a(a+bi)^{n-1+\beta} T_r(i).$$

When $r=0$, such a sum is the n th coordinate of the vector

$$\mathcal{D}(i^\alpha) \overrightarrow{\mathcal{P}(a+ib)^\beta}.$$

When $r > 0$, such a sum is the n th coordinate of the vector

$$\mathcal{D}(i^\alpha) \mathcal{P} \mathcal{D}((a+ib)^\beta) \overrightarrow{\mathcal{Q}_{T_r}(i)^*}.$$

In all cases, $0 < \alpha \leq k-r$, $0 \leq \beta \leq k-r$, and $0 \leq r \leq k-1$.

Putting all the previous calculations together, we conclude that the linear recursion can be written as the matrix equation

$$\overrightarrow{\mathcal{P} E_k(i; a, b)} = \overrightarrow{\mathcal{P} \sum_{r=0}^{k-1} \binom{k}{r} U_{k-r}(i) T_r(i)} + C + D, \quad (3.5)$$

where C is the linear combination of all vectors of the form $\mathcal{D}(i^\alpha) \overrightarrow{\mathcal{P}(a+ib)^\beta}$ and D is the linear combination of all vectors of the form $\mathcal{D}(i^\alpha) \mathcal{P} \mathcal{D}((a+ib)^\beta) \overrightarrow{\mathcal{Q}_{T_r}(i)^*}$, where $r \geq 1$. In both C and D , the coefficients are constants.

For example, in the case $k=2$, we can use the formula for $E_1(i; a, b)$ given in Eq. (1.1) and obtain the following explicit matrix equation

$$\begin{aligned} \overrightarrow{\mathcal{P} E_2(i; a, b)} &= \overrightarrow{\mathcal{P} U_2(i) + 2U_1(i) T_1(i)} - \frac{1}{4} \mathcal{D}(i^2) \overrightarrow{\mathcal{P}(a+ib+1)^2} - \mathcal{D}(i) \overrightarrow{\mathcal{P}[\frac{1}{12}(a+ib)^2 - \frac{1}{2}(a+ib) - \frac{7}{12}]} \\ &\quad + \frac{1}{2} \mathcal{D}(i) \mathcal{P} \mathcal{D}(a+ib+1) \overrightarrow{\mathcal{Q}_1^*} \end{aligned}$$

We shall solve the matrix equation by applying \mathcal{P}^{-1} to both sides. Before we can do this, we need tools to simplify the linear combinations C and D . These tools will be developed in Section 4.

4 Technical Lemmas

4.1 Stirling numbers

Stirling numbers play an unexpected role in our calculations. This is because of the following lemma, which is stated in a redundant, but more useful, form.

Lemma 4.1.

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a(a+ib)^m i^\ell = n! \sum_{k=\max(0, n-\ell)}^m \binom{m}{k} a^{m-k+1} b^k S(k+\ell, n),$$

where $S(r, n)$ is a Stirling number of the second kind and equals the number of partitions of an r -element set into n non-empty blocks. In particular, if $0 \leq m \leq n-1$, then

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a(a+ib)^m = 0, \quad (4.1)$$

and

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a(a+ib)^m = n! a b^n \sum_{k=0}^{m-n} \binom{m}{n+k} a^{m-n-k} b^k S(n+k, n). \quad (4.2)$$

Proof. Expand $(a+bi)^m$ with the binomial theorem, use the identity of Stirling ([17]; see, for example, [16], page 34),

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} i^r = n! S(r, n),$$

and observe that $S(m, n) = 0$ if $m < n$. □

Simple counting arguments yield the following result.

Lemma 4.2.

$$S(n+k, n) = \sum_{r=k+1}^{2k} \pi_r \binom{n+k}{r}, \quad (4.3)$$

where π_r is the number of ways of partitioning a set of size r into $r-k$ parts, with each part containing at least two elements. In particular, $S(n+k, n)$ is a polynomial in n of degree $2k$ and the coefficient of n^{2k} is $\pi_{2k}/(2k)!$, which equals $1/(k!2^k)$.

For example,

$$\begin{aligned} S(n, n) &= 1, \\ S(n+1, n) &= \binom{n+1}{2}, \\ S(n+2, n) &= \binom{n+2}{3} + 3 \binom{n+2}{4}, \\ S(n+3, n) &= \binom{n+3}{4} + 10 \binom{n+3}{5} + 15 \binom{n+3}{6}, \end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a(a+ib)^n &= ab^n n!, \\ \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} a(a+ib)^{n+1} &= ab^n n! \left[\frac{1}{2}bn^2 + \left(\frac{1}{2}b+a\right)n + a \right].\end{aligned}$$

Lemmas 4.1 and 4.2 yield information about how the matrix \mathcal{P} acts on vectors.

Lemma 4.3. *Let $f(x)$ be a polynomial of degree δ . Then*

$$\mathcal{P} \overrightarrow{f(i)} = \overrightarrow{i!g(i)},$$

where $g(x)$ is a polynomial of degree $2(\delta - 1)$. In particular,

$$\mathcal{D}(i^\alpha) \overrightarrow{\mathcal{P}f(a+ib)} = \overrightarrow{ab^i i!g(i)},$$

where $g(x) = x^\alpha h(x)$ is a polynomial of degree $\alpha + 2(\delta - 1)$ with coefficients polynomials in a and b .

4.2 Two-way Sums

We shall need the following elementary lemmas.

Lemma 4.4. *Let $p(x)$ be a polynomial of degree α , $q(x)$ a polynomial of degree β , and c be a non-negative integer. Then the sum*

$$\sum_{i=c}^n p(i)q(n-i)$$

is a polynomial in n of degree $\alpha + \beta + 1$.

Proof. Expanding the product $p(x)q(y)$, it suffices to consider the sums

$$T_{\alpha,\beta}(n) = \sum_{i=c}^n (n-i)^\alpha i^\beta.$$

However,

$$T_{\alpha,\beta}(n) = \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} n^{\alpha-j} \left(\sum_{i=c}^n i^{\beta+j} \right).$$

Since the sum

$$\sum_{i=c}^n i^{\beta+j}$$

is a polynomial in n of degree $\beta + j + 1$, the lemma follows. □

Another elementary result about two-way sums is the following lemma.

Lemma 4.5. *The leading coefficient $L(\alpha, \beta)$, that is, the coefficient of $n^{\alpha+\beta+1}$, of the two-way sum*

$$\sum_{i=c}^n (n-i)^\alpha i^\beta$$

as a polynomial in n equals

$$\int_0^1 (1-x)^\alpha x^\beta dx.$$

Proof. The leading coefficient is the limit

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=c}^n (n-i)^\alpha i^\beta}{n^{\alpha+\beta+1}}.$$

This equals the beta-function integral $\int_0^1 (1-x)^\alpha x^\beta dx$. □

Using Lemma 4.5 and integration by parts, we have, for $\alpha \geq 1$,

$$\begin{aligned} L(\alpha, \beta) &= \frac{\alpha}{\beta+1} \int_0^1 (1-x)^{\alpha-1} x^{\beta+1} dx. \\ &= \frac{\alpha}{\beta+1} L(\alpha-1, \beta+1). \end{aligned}$$

Since $L(0, \beta) = 1/(\beta+1)$, this yields

$$L(\alpha, \beta) = \frac{\alpha!}{(\beta+1)(\beta+2)\dots(\beta+\alpha+1)} \quad (4.4)$$

when α is an non-negative integer.

4.3 Matrix relations

In this subsection, we prove several relations among the matrices \mathcal{P} , \mathcal{Q} , and $\mathcal{D}((a+ib)^\alpha)$.

Lemma 4.6.

$$\mathcal{P}\mathcal{D}((a+ib)^\alpha)\mathcal{Q} = \mathcal{N}\mathcal{K}(\alpha),$$

where \mathcal{N} is the diagonal matrix $\mathcal{D}(ab^i i!)$ and $\mathcal{K}(\alpha)$ is the lower triangular matrix with n th entry equal to

$$\sum_{t=0}^{\alpha} \binom{n-j+\alpha}{n-j+t} (a+jb)^{\alpha-t} b^t S(n-j+t, n-j).$$

if $n \geq j$ and zero otherwise.

Proof. The n th entry of the product $\mathcal{P}\mathcal{D}((a+ib)^\alpha)\mathcal{Q}$ equals

$$\sum_{i=j}^n (-1)^{n-i} \binom{n}{i} \binom{i}{j} j! b^j a (a+ib)^{n-j+\alpha}.$$

Changing indices from i to $i-j$ and regrouping terms, this can be simplified to

$$\frac{ab^j n!}{(n-j)!} \sum_{t=0}^{n-j} (-1)^{n-j-t} \binom{n-j}{t} (A+tb)^{n-j+\alpha},$$

where $A = a+jb$. Applying Lemma 4.1, we obtain

$$\binom{n}{j} ab^j j! (n-j)! b^{n-j} \sum_{t=0}^{\alpha} \binom{n-j+\alpha}{n-j+t} A^{\alpha-t} b^t S(n-j+t, n-j)$$

which equals

$$n! ab^n \sum_{t=0}^{\alpha} \binom{n-j+\alpha}{n-j+t} A^{\alpha-t} b^t S(n-j+t, n-j).$$

□

For example, the n th entry of $\mathcal{PD}((a+ib)^2)\mathcal{Q}$ is $ab^n n!$ times

$$b^2 S(n-j+2, n-j) + (n-j+2)(a+jb)bS(n-j+1, n-j) + \binom{n-j+2}{2}(a+jb)^2.$$

By Lemma 4.2, $S(n-j+2, n-j)$ is a polynomial of degree 4 in $n-j$ and $S(n-j+1, n-j)$ is a polynomial of degree 2 in $n-j$. Hence, the n th entry of $\mathcal{PD}((a+ib)^2)\mathcal{Q}$ is a polynomial of total degree 4 in the variables j and $n-j$. This degree-counting argument works in general. Noting that the coefficients of the polynomial are polynomials in a and b , we have the following corollary.

Corollary 4.7. *Let $f(x)$ be a polynomial of degree α . Then, the n th entry of $\mathcal{PD}(f(a+ib))\mathcal{Q}$ is $ab^n n!$ times a polynomial in the variables j and $n-j$ of total degree 2α if $n \geq j$ and zero otherwise. Furthermore, the coefficients of the polynomial are polynomials in a and b .*

An important special case of Lemma 4.6 is when $\alpha = 0$.

Corollary 4.8.

$$\mathcal{P}\mathcal{Q} = \mathcal{N}\mathcal{L}$$

where \mathcal{N} is the diagonal matrix whose i th entry is $ab^i i!$ and \mathcal{L} is the lower triangular matrix with all i th entry equal to 1 if $i \geq j$ and zero otherwise. In particular,

$$\mathcal{P}^{-1} = \mathcal{Q}\mathcal{L}^{-1}\mathcal{N}^{-1}.$$

The action of $\mathcal{L}^{-1}\mathcal{N}^{-1}$ on a vector can be explicitly described. Multiplying a column vector \vec{a}_i on the left by \mathcal{N}^{-1} divides the i th coordinate a_i by $ab^i i!$. The matrix \mathcal{L} is the summation matrix and sends \vec{a}_i to the vector whose i th coordinate is $a_0 + a_1 + \dots + a_i$. Hence, the inverse \mathcal{L}^{-1} is the backward difference matrix, with all diagonal entries 1, all sub-diagonal entries -1 , and all other entries zero. Multiplying the vector \vec{a}_i on the left by \mathcal{L}^{-1} results in the vector $a_i - a_{i-1}$, obtained by taking the backward difference of the coordinates a_i , with the convention that $a_{-1} = 0$.

Lemma 4.9. *Let $f(x)$ be a polynomial of degree α and $g(x)$ be a polynomial of degree ϵ . Then,*

$$\mathcal{PD}(f(a+ib))\mathcal{Q}\overrightarrow{g(i)^*} = \overrightarrow{!ab^i h(i)^*}$$

where $h(x)$ is a polynomial of degree $2\alpha + \epsilon + 1$ with coefficients polynomials in a and b .

Proof. By Lemma 4.6, for any $\delta \geq 0$, the vector

$$\mathcal{PD}((a+ib)^\delta)\mathcal{Q}\overrightarrow{g(i)^*} = \mathcal{N}\mathcal{K}(\delta)\overrightarrow{g(i)^*}.$$

Hence, the zeroth coordinate of the left hand vector is zero. Now let $K_{nj}(\delta)$ be n th entry of matrix $\mathcal{K}(\delta)$. Then, if $n \geq 1$, the n th coordinate of the above vector equals

$$ab^n n! \sum_{j=1}^n K_{nj}(\delta)g(j). \quad (4.5)$$

By Corollary 4.7, $K_{nj}(\delta)$ is a polynomial in the variables j and $n-j$ of total degree 2δ . Hence the sum in (4.5) is a two-way sum with summand having total degree $2\delta + \epsilon$ and, by Lemma 4.4, it is a polynomial in n of degree $2\delta + \epsilon + 1$. \square

5 Finishing the proof of Theorem 1.1

We continue the proof of Theorem 1.1. The aim is to put the expression $\mathcal{P}^{-1}(C + D)$ into a simple form.

The vectors in the linear combination C has the form

$$\mathcal{D}(i^\alpha)\overrightarrow{\mathcal{P}f(a+ib)},$$

where $0 < \alpha \leq k$ and the polynomial $f(x)$ has degree at most k . By Lemma. 4.3, this vector equals

$$\overrightarrow{ab^i i! i^\alpha g(i)},$$

where $g(x)$ is a polynomial of degree at most $2k - 2$.

From Cor. 4.8, the inverse matrix \mathcal{P}^{-1} equals $\mathcal{Q}\mathcal{L}^{-1}\mathcal{N}^{-1}$. Hence, applying \mathcal{P}^{-1} removes the factor of $ab^i i!$, takes the difference of the vector $\overrightarrow{i^\alpha g(i)}$, and applies \mathcal{Q} . Since the zero coordinate of the vector $\overrightarrow{ab^i i! i^\alpha g(i)}$ is 0, this results in the vector

$$\overrightarrow{\mathcal{Q}h_1(i)^*},$$

where $h_1(x)$ is the difference polynomial $x^\alpha g(x) - (x-1)^\alpha g(x-1)$ and has degree at most $3k - 3$.

Vectors in the linear combination D has the form

$$\mathcal{D}(i^\alpha)\overrightarrow{\mathcal{P}\mathcal{D}(f(a+ib))\mathcal{Q}\tau_r(i)^*},$$

where $1 \leq \alpha \leq k - r$ and the polynomial $f(x)$ has degree β at most $k - r$. By Lemma 6.4, this vector equals

$$\overrightarrow{ab^i i! i^\alpha h(i)^*}.$$

Using the induction hypothesis that $\tau_r(x)$ has degree $3r - 3$, it follows from Lemma 6.4 that the degree of $x^\alpha h(x)$ is at most

$$\alpha + 2\beta + (3r - 3) + 1.$$

This expression is maximized by taking $\alpha = \beta = k - r$ and the maximum equals $3k - 2$. As in the earlier case, applying \mathcal{P}^{-1} yields a vector of the form $\overrightarrow{\mathcal{Q}h_2(i)^*}$, where $h_2(x)$ is a polynomial of degree at most $3k - 3$.

Summarizing, we conclude that

$$\mathcal{P}^{-1}(C) = \overrightarrow{\mathcal{Q}h_1(i)^*} \quad \text{and} \quad \mathcal{P}^{-1}(D) = \overrightarrow{\mathcal{Q}h_2(i)^*},$$

where $h_1(x)$ and $h_2(x)$ are polynomials of degree at most $3k - 3$. We can now conclude the proof of Theorem 1.1 by setting $\tau_k(x) = h_1(x) + h_2(x)$.

For example, when $k = 2$,

$$\begin{aligned} h_1(x) &= -\frac{b}{2}x^3 + \left(\frac{b}{4} - \frac{3a}{4}\right)x^2 + \left(\frac{a}{12} - \frac{b}{12} - 1\right)x + 1, \\ h_2(x) &= \frac{2b}{3}x^3 + \left(\frac{3a}{4} - \frac{b}{4}\right)x^2 + \left(\frac{b}{12} - \frac{a}{4} + 1\right)x - \frac{1}{2}. \end{aligned}$$

and

$$\tau_2(x) = \frac{b}{6}x^3 - \frac{a}{6}x + \frac{1}{2}.$$

This yields the case $k = 2$ of Theorem 1.1.

Theorem 5.1. *The second factorial moment of the sum of a random $[a, b]$ -parking function of length n equals*

$$\begin{aligned} & \frac{1}{4}n(n-1)(a+nb+1)^2 + \frac{1}{3}n(a+nb+1)(a+nb-1) \\ & - \frac{n(a+nb+1)}{2} \sum_{j=1}^n \binom{n}{j} \frac{j!b^j}{(a+nb)^{j-1}} + \sum_{j=1}^n \binom{n}{j} \frac{j!b^j}{(a+nb)^{j-1}} \left(\frac{b}{6}j^3 - \frac{a}{6}j + \frac{1}{2} \right). \end{aligned}$$

One can, with some patience, work out $\tau_k(x)$ explicitly for any given k . For example,

$$\begin{aligned}\tau_3(x) = & -\frac{7}{960}b^2x^6 + \left(-\frac{69}{320}b^2 + \frac{3}{160}ab\right)x^5 + \left(-\frac{7}{16}ab - \frac{43}{192}b^2\right)x^4 + \left(-\frac{5}{192}b^2 - \frac{11}{32}ab - \frac{1}{2}b - \frac{1}{3}a^2\right)x^3 \\ & + \left(\frac{1}{16}ab - \frac{3}{160}b^2 - \frac{1}{4}a^2\right)x^2 + \left(1 + \frac{1}{12}a^2 + \frac{1}{2}a - \frac{1}{20}ab - \frac{1}{120}b^2\right)x - 1.\end{aligned}$$

6 Leading coefficients

In this section, we derive a linear recursion for the degree- $(3k-3)$ coefficients of $\tau_k(x)$. This recursion implies that for infinitely many k , the degree of $\tau_k(x)$ is exactly $3k-3$.

The highest possible power of x in $h_1(x)$ originates from the leading term

$$n(n-1)\dots(n-k+1)(a+ib+1)^k/2^k$$

in $U_k(n; a+ib+1, 0)$. The leading monomial in the variables n and i in this term is $n^k(a+ib)^k/2^k$. By Lemma 4.1, the n th coordinate of $(1/2^k)\mathcal{D}(i^k)\overrightarrow{\mathcal{P}(a+ib)^k}$ is a linear combination of Stirling numbers, with the ‘‘leading’’ term $(ab^{k+n}n!/2^k)S(n+k-1, n)$. By Lemma 4.5, the highest power of n in the expansion of $S(n+k-1, n)$ has exponent $2k-2$ and coefficient $1/2^{k-1}(k-1)!$. Hence, taking into account the effect of \mathcal{N}^{-1} , and \mathcal{L}^{-1} , as well as the negative sign in the recursion, the leading term of $h_1(x)$ has exponent $3k-3$ and coefficient

$$-\frac{(3k-2)b^{k-1}}{2^{2k-1}(k-1)!}. \quad (6.1)$$

We note that the factor of $3k-2$ comes from taking the backward difference of x^{3k-2} .

Let $b^{r-1}t_r$ be the coefficient of x^{3r-3} in $\tau_r(x)$. By Eq. (1.1), $t_1 = -\frac{1}{2}$. The highest possible power of x in $h_2(x)$ comes from the vectors

$$\mathcal{D}(i^{k-r})\mathcal{P}\mathcal{D}((a+ib)^{k-r})\overrightarrow{\mathcal{Q}\tau_r(i)^*}.$$

By Lemma 4.6, the n th entry of the matrix $\mathcal{P}\mathcal{D}((a+ib)^{k-r})\mathcal{Q}$ is a polynomial in the variables $n-j$ and j of total degree $2(k-r)$. It follows from Lemma 4.2 that when expanded as a linear combination of monomials in $n-j$ and j , the part of the n th entry of total degree $2(k-r)$ equals $ab^n n!$ times

$$b^{k-r} \sum_{s=0}^{k-r} \frac{1}{2^{k-r-s}(k-r-s)!s!} (n-j)^{2(k-r)-s} j^s = \frac{b^{k-r}}{2^{k-r}(k-r)!} (n^2-j^2)^{k-r}. \quad (6.2)$$

For example, when $k-r=2$, the part with total degree 4 is $ab^n n!$ times

$$b^2 \left[\frac{1}{8}(n-j)^4 + \frac{1}{2}(n-j)^3 j + \frac{1}{2}(n-j)^2 j^2 \right] = \frac{b^2}{8} (n^2-j^2)^2.$$

Thus, for $n \geq 1$, the n th coordinate of $\mathcal{P}\mathcal{D}((a+ib)^{k-r})\overrightarrow{\mathcal{Q}\tau_r(i)^*}$ equals $ab^n n! \sum_{j=1}^n \mathcal{K}(k-r)\tau_r(j)$, where the part of highest power of n is $ab^n n!$ times

$$\frac{b^{k-1}t_r}{2^{k-r}(k-r)!} \sum_{j=1}^n (n^2-j^2)^{k-r} j^{3r-3}. \quad (6.3)$$

By Lemma 4.4, the highest power of n in (6.3) is n^{2k+r-2} and the coefficient of the highest power is

$$\frac{b^{k-1}t_r}{2^{k-r}(k-r)!} \int_0^1 (1-x^2)^{k-r} x^{3r-3} dx = \frac{b^{k-1}t_r}{2^{k-r}(k-r)!} \frac{1}{2} L\left(k-r, \frac{3r-4}{2}\right). \quad (6.4)$$

Using Eq. (4.4), the above formula equals

$$\frac{b^{k-1}t_r}{(3r-2)(3r)(3r+2)\dots(2k+r-2)}. \quad (6.5)$$

Hence, taking into account the effects of $\mathcal{D}(i^{k-r})$, \mathcal{N}^{-1} , and \mathcal{L}^{-1} , we derive from Eq. (3.4) that the coefficient of x^{3k-3} of $h_2(x)$ equals

$$-(3k-2)b^{k-1}\sum_{r=1}^{k-1}\binom{k}{r}\frac{t_r}{2^{k-r}(3r-2)(3r)(3r+2)\dots(2k+r-2)}. \quad (6.6)$$

Since $\tau_k(x) = h_1(x) + h_2(x)$, we obtain the following recursion for the coefficients t_r :

$$t_k = -(3k-2)\left(\frac{1}{2^{2k-1}(k-1)!} + \sum_{r=1}^{k-1}\binom{k}{r}\frac{t_r}{2^{k-r}\prod_{j=1}^{k-r+1}(3r+2j-4)}\right).$$

Rewriting the recursion, we have the following result.

Theorem 6.1. *Let $b^{k-1}t_k$ be the coefficient of x^{3k-3} in $\tau_k(x)$. Then $t_1 = -\frac{1}{2}$, and $\{t_k\}$ satisfies the recursion*

$$\frac{1}{2^{2k-1}(k-1)!} + \sum_{r=1}^k\binom{k}{r}\frac{t_r}{2^{k-r}\prod_{j=1}^{k-r+1}(3r+2j-4)} = 0. \quad (6.7)$$

From the starting point $t_1 = -\frac{1}{2}$, the recursion yields $t_2 = \frac{1}{6}$, $t_3 = -\frac{7}{960}$,

$$t_4 = \frac{10}{2^7 \cdot 3^3 \cdot 5 \cdot 7}, \quad t_5 = \frac{13}{2^{12} \cdot 3^4 \cdot 7 \cdot 11},$$

and

$$t_6 = \frac{31}{2^9 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13}.$$

We remark that since $t_5 > 0$, the sequence t_k does not alternate in sign. However, Theorem (6.1) implies that there are infinitely many indices k for which $t_k \neq 0$, and hence, the degree of $\tau_k(x)$ is exactly $3k-3$ for infinitely many k . We conjecture that the degree of $\tau_k(x)$ is always $3k-3$.

7 Reversed Sums

The *reversed sum* R_n for a \mathbf{u} -parking functions is defined by

$$R_n = u_1 + u_2 + \dots + u_n - S_n.$$

The reversed sum is needed in many applications. For example, for ordinary parking functions, the statistics of reversed sums match the statistics of inversions of labeled trees [15].

In the classical case when $u_i = a + (i-1)b$,

$$R_n = an + b\binom{n}{2} - S_n.$$

We will use the notation

$$R_n = cn - S_n, \quad (7.1)$$

where $c = a + b(n - 1)/2$ (and is a function of n). Let $F_k(n; a, b)$ be the k th falling factorial moment of R_n . From Eq. (7.1) and the binomial theorem for falling factorials, we have

$$F_k(n; a, b) = \sum_{r=0}^k (-1)^r \binom{k}{r} (cn)_{k-r} E_r(n; a, b).$$

Using the notation given in Eq. (3.1) and Theorem 1.1, and changing order of summation, we obtain

$$\begin{aligned} F_k(n; a, b) &= \sum_{r=0}^k (-1)^r \binom{k}{r} (cn)_{k-r} \left[\sum_{m=0}^r \binom{r}{m} U_{r-m}(n) T_m(n) \right] \\ &= \sum_{m=0}^k \binom{k}{m} T_m(n) \left[\sum_{r=m}^k (-1)^r \binom{k-m}{r-m} \right] (cn)_{k-r} U_{r-t}(n). \end{aligned}$$

Changing the index from r to $s = r - t$, the inner sum becomes

$$\sum_{s=0}^{k-t} (-1)^s \binom{k-t}{s} (cn)_{k-t-s} U_s(n), \quad (7.2)$$

which is a polynomial in n . Let

$$W_k(t) = \sum_{s=0}^k (-1)^s \binom{k}{s} (cn)_{k-s} U_s(n). \quad (7.3)$$

For example, $W_0(n) = 1$, $W_1(n) = \frac{1}{2}n(a - b - 1)$,

$$W_2(n) = \frac{1}{4}n(n-1)(a-b-3)^2 + \frac{1}{12}n(b^2n^2 + (2ab - 12b)n + 4a^2 - 6ab + 3b^2 - 24a + 12b - 4).$$

Theorem 7.1. *The k th factorial moment $F_k(n; a, b)$ of the reversed sum of a random $[a, b]$ -parking function of length n equals*

$$W_k(n) + \sum_{r=1}^k \left[(-1)^r \binom{k}{r} W_{k-r}(n) \left(\sum_{j=1}^n \binom{n}{j} \frac{j!b^j}{(a+nb)^{j-1}} \tau_r(j) \right) \right] \quad (7.4)$$

where $\tau_r(x)$ is the polynomial of degree not exceeding $3r - 3$ in Theorem 1.1 and $W_k(x)$ is a polynomial of degree $3k/2$ if k is even, and $(3k - 1)/2$ if k is odd.

To prove Theorem 7.1, we just need to determine the degree of $W_k(t)$. The proof is contained in the following three lemmas (Lemma 7.2, 7.3 and 7.4) in which we also determine the leading coefficient of $W_k(t)$. This information will be needed in the next section.

Since

$$(-1)^s U_s(n) = \frac{d^s}{dt^s} (p(-t; a + nb + 1, 0))^n \Big|_{t=0}$$

where $p(t, A, B)$ is as defined in Eq. (2.4), and

$$(cn)_{k-s} = \frac{d^{k-s}}{dt^{k-s}} (1+t)^{cn} \Big|_{t=0},$$

we have, by Leibniz's rule of differentiation,

$$W_k(n) = \frac{d^k}{dt^k} (1+t)^{cn} (p(-t; a + nb + 1, 0))^n \Big|_{t=0}.$$

Let A , B , and C be positive integers such that $A < B < C$. Let

$$q(t; A, B, C) = \frac{(1+t)^C}{B+1-A} ((1-t)^A + (1-t)^{A+1} + \dots + (1-t)^B),$$

$$\begin{aligned} K_s(x, y) &= \left. \frac{d^s}{dt^s} (1+t)^x (1-t)^y \right|_{t=0} \\ &= \sum_{m=0}^s (-1)^m \binom{s}{m} (x)_{s-m} (y)_m, \end{aligned}$$

and

$$W_k(n; A, B, C) = \left. \frac{d^k}{dt^k} (q(t; A, B, C))^n \right|_{t=0}. \quad (7.5)$$

Note that

$$W_k(n) = W_k(n; a + nb + 1, 0, a + b(n-1)/2).$$

Lemma 7.2. *For s a non-negative integer,*

$$q^{(s)}(0; A, B, C) = -\frac{K_{s+1}(C, B+1) - K_{s+1}(C, A)}{(s+1)(B+1-A)}. \quad (7.6)$$

In particular, $q^{(s)}(0; A, B, C)$ is a polynomial in the variables A , B , and C of total degree s .

Proof. By Leibniz's rule,

$$\begin{aligned} q^{(s)}(0; A, B, C) &= \sum_{r=0}^s \binom{s}{r} (C)_{s-r} \left[(-1)^r \frac{(A)_r + \dots + (B)_r}{B+1-A} \right] \\ &= \frac{1}{B+1-A} \sum_{r=0}^s (-1)^r \binom{s}{r} (C)_{s-r} \left[\frac{(B+1)_{r+1} - (A)_{r+1}}{r+1} \right]. \end{aligned}$$

Using

$$\begin{aligned} &\sum_{r=0}^s (-1)^r \binom{s}{r} (C)_{s-r} \frac{(x)_{r+1}}{r+1} \\ &= -\frac{1}{s+1} \sum_{t=1}^{s+1} (-1)^t \binom{s+1}{t} (C)_{s+1-t} (x)_t \\ &= -\frac{K_{s+1}(C, x) - (C)_{s+1}}{s+1}, \end{aligned}$$

we obtain

$$q^{(s)}(0; A, B, C) = -\frac{K_{s+1}(C, B+1) - K_{s+1}(C, A)}{(B+1-A)(s+1)}.$$

□

For $k \geq 1$, the function $W_k(n; A, B, C)$ is a linear combination of terms of the form

$$(n)_r (q^{(0)}(0; A, B, C))^{n-r} (q^{(1)}(0; A, B, C))^{a_1} \dots (q^{(k)}(0; A, B, C))^{a_k},$$

where $1 \leq r \leq k$, $a_i \geq 0$, and $a_1 + a_2 + \dots + a_k = r$, $a_1 + 2a_2 + \dots + ka_k = k$. Hence, by Lemma 7.2, $W_k(n; A, B, C)$ is a polynomial in the variables n , A , B , and C of total degree $2k$, with the variable n having degree at most k .

Next, define $q_s(n)$ by the formula

$$q_s(n) = q^{(s)}(0; a + nb + 1, 0, a + b(n - 1)/2).$$

For example, $q_0(n) = 1$, $q_1(n) = \frac{1}{2}a - b - 1$,

$$q_2(n) = \frac{1}{12}[b^2n^2 + (2ab - 12b)n + 4a^2 - 6ab + 3b^2 - 24a + 12b - 4].$$

Lemma 7.3. *The degree of $q_s(n)$ is s if s is even, and less than or equal to $s - 1$ if s is odd.*

Proof. By Lemma 7.2,

$$q_s(n) = \frac{K_{s+1}(a + b(n - 1)/2, 1) - K_{s+1}(a + b(n - 1)/2, a + nb + 1)}{(s + 1)(a + nb)} \quad (7.7)$$

When s is even, the leading term in the numerator is $2(bn/2)^{s+1}$ and hence, the leading term in $q_s(n)$ is

$$\frac{b^s}{2^s(s + 1)}n^s$$

and $q_s(n)$ has degree s .

When s is odd, the coefficients of n^{s+1} in the two terms cancel in the numerator, and hence, the numerator has degree at most s . Therefore, $q_s(n)$ has degree at most $s - 1$ in this case. \square

Lemma 7.4. *The polynomial $W_k(n)$ has degree $3k/2$ if k is even, and degree $(3k - 1)/2$ if k is odd. When k is even, the leading coefficient is*

$$(k - 1)!!b^k/12^{k/2},$$

where $m!! = m(m - 2) \cdots 3 \cdot 1$ for an odd positive integer m and $(-1)!!$ is defined to be 1.

Proof. A typical term in $W_k(n)$ is of the form

$$\binom{n}{r} (q_0(n))^{n-r} (q_1(n))^{a_1} (q_2(n))^{a_2} \cdots (q_k(n))^{a_k}$$

where $1 \leq r \leq k$, $a_1 \geq 0$, $a_1 + a_2 + \dots + a_k = r$ and $a_1 + 2a_2 + \dots + ka_k = k$. By Lemma 7.3, the degree of such a term is at most

$$r + 2a_2 + 2a_3 + 4a_4 + 4a_5 + \cdots + 2\lfloor k/2 \rfloor a_k.$$

The maximum degree occurs exactly when $a_3 = a_4 = \dots = a_k = 0$ and $a_2 = \lfloor \frac{k}{2} \rfloor$. In other words, if k is even, the maximum degree $3k/2$ is achieved at $r = a_2 = k/2$ and $a_1 = a_3 = \dots = a_k = 0$; if k is odd, the maximum degree $(3k - 1)/2$ is achieved at $a_1 = 1$, $a_2 = (k - 1)/2$, $r = (k + 1)/2$, and $a_3 = \dots = a_k = 0$.

For $k = 2m$, the leading term is contained in

$$\binom{n}{m} (q_0(n))^{n-m} (q_2(n))^m.$$

One way to compute the leading coefficient is to consider terms of the form

$$Q_{i_1}(t)Q_{i_2}(t) \cdots Q_{i_{n-m}}(t)Q_{j_1}^{(2)}(t)Q_{j_2}^{(2)}(t) \cdots Q_{j_m}^{(2)}(t)$$

in

$$\frac{d^k}{dt^k} Q_1(t)Q_2(t) \cdots Q_n(t).$$

When one differentiates the product $Q_1(t)Q_2(t)\cdots Q_n(t)$ k times using the product rule, such a term is obtained by choosing m indices j_1, j_2, \dots, j_m and differentiating $Q_j(t)$ twice if j is one of the chosen indices. Hence, there are

$$\binom{n}{m} \binom{2m}{2, 2, \dots, 2}.$$

such terms. Setting $Q_i(t) = q(t; A, B, C)$ for all i , we conclude that the coefficient of

$$q(0; A, B, C)^{n-m} q^{(2)}(0; A, B, C)^m$$

in $W_k(n; A, B, C)$ is $\binom{n}{m} (2m-1)!!$, which equals $\binom{n}{k/2} (k-1)!!$. Since $q_0(n) = 1$ and the leading term in $q_2(n)$ is $b^2 n^2 / 12$, we conclude that the leading term of $W_k(n)$ is

$$(k-1)!! b^k n^{3k/2} / (12)^{k/2}.$$

□

8 The Wright Constants

We end this paper with an application to graphical enumeration. Let $c(n+1, k)$ be the number of labeled connected graphs on $n+1$ vertices with exactly $n+k$ edges. For example, $c(n+1, 0) = (n+1)^{n-1}$ by Cayley's formula. Let $C_n(q) = \sum_k c(n+1, k) q^k$ be the enumerator by the number of "excess" edges for connected labelled graphs. Kreweras [8] proved that

$$C_n(q) = \sum_{(a_1, \dots, a_n)} (1+q)^{\binom{n+1}{2} - \sum a_i}, \quad (8.1)$$

where the sum ranges over ordinary parking functions of length n . Hence

$$C(n+1, k) = \sum_j p_j \binom{j}{k}$$

where p_j is the number of ordinary parking functions that have reversed sum $\binom{n+1}{2} - \sum a_i = j$. Comparing with the definition of $F_k(n; a, b)$, we have

Theorem 8.1.

$$c(n+1, k) = \frac{(n+1)^{n-1}}{k!} F_k(n; 1, 1). \quad (8.2)$$

An alternative proof of Theorem 8.1 can be obtained by combining results in [13] and [18]. In [13], Spencer proved that

$$\frac{c(n+1, k)}{c(n+1, 0)} = E \left[\binom{M}{k} \right],$$

where M is a certain random variable defined on all labeled trees on $n+1$ vertices with uniform distribution. In [18] it is proved that M is equivalent to the reversed sum under a combinatorial bijection from the set of labeled trees to the set of ordinary parking functions. Combining these, we obtain Theorem 8.1.

In his 1977 paper [19], Wright found the following asymptotic formula: for fixed k and n tending to infinity,

$$c(n+1, k) = \rho_{k-1} (n+1)^{n-1+(3k/2)} \{1 + O(n^{-1/2})\}, \quad (8.3)$$

where the *Wright constants* ρ_k are defined by a second order recursion. (The notation ρ_{k-1} is Wright's; ρ_{k-1} equals c_k in Spencer's notation [13].)

With the explicit formulas for moments of reversed sums given in Theorem 7.1, we will derive Wright's formula using routine methods in asymptotic analysis (see, for example, [3]). This derivation yields a formula for the Wright constants in terms of the leading coefficients t_k of the polynomial $\tau_k(x)$, which satisfy a linear recursion.

We begin by noting that

$$\sum_{j=1}^n \frac{(n)_j}{(1+n)^{j-1}} j^k = (1+n) \sum_{j=2}^{n+1} \frac{(1+n)_j}{(1+n)^j} j^k (1 + O(j^{-1})). \quad (8.4)$$

From the inequality

$$\begin{aligned} \frac{(n+1)_k}{(n+1)^k} &= \prod_{j=1}^{k-1} \left(1 - \frac{j}{n+1}\right) \\ &\leq \prod_{j=1}^{k-1} \exp\left(-\frac{j}{n+1}\right) = \exp\left(-\frac{1}{n+1} \binom{k}{2}\right) \end{aligned}$$

(which gives an upper bound on the initial terms), and the asymptotic formula, for $k = o(n^{2/3})$,

$$\prod_{j=1}^{k-1} \left(1 - \frac{j}{n+1}\right) = \exp(-k^2/2n) \left[1 + O\left(\frac{k^3}{n^2}\right)\right],$$

we obtain, for large n ,

$$\begin{aligned} \sum_{j=2}^{n+1} \frac{(n+1)_j}{(n+1)^j} j^k &\sim \int_2^{n+1} j^k \exp(-j^2/2(n+1)) dj \\ &\sim (n+1)^{(k+1)/2} \int_0^\infty j^k \exp(-j^2/2) dj \\ &= (n+1)^{(k+1)/2} 2^{(k-1)/2} \Gamma\left(\frac{k+1}{2}\right), \end{aligned}$$

where $\Gamma(z)$ is the Gamma function. From this and Eq. (8.4), it follows that

$$T_r(n) \sim t_r 2^{(3r-4)/2} \Gamma\left(\frac{3r-2}{2}\right) (n+1)^{3r/2}, \quad (r \geq 1),$$

where t_r is the coefficient of x^{3r-3} in $\tau_r(x)$. Combining this with Lemma 7.4 and Theorem 7.1, and observing that the dominant term $(n+1)^{3k/2}$ in $F_k(n; 1, 1)$ can only occur in the product of $W_{k-r}(n)$ and $T_r(n)$, where $k-r$ is a positive even integer, we conclude that when k is even,

$$F_k(n; 1, 1) \sim \left[\frac{(k-1)!!}{12^{k/2}} + \sum_{0 < r \leq k, r \text{ even}} \binom{k}{r} \frac{(k-r-1)!! 2^{(3r-4)/2} \Gamma\left(\frac{3r-2}{2}\right) t_r}{12^{(k-r)/2}} \right] (n+1)^{3k/2}$$

and when k is odd,

$$F_k(n; 1, 1) \sim \left[\sum_{0 < r < k, r \text{ even}} \binom{k}{r} \frac{(k-r-1)!! 2^{(3r-4)/2} \Gamma\left(\frac{3r-2}{2}\right) t_r}{12^{(k-r)/2}} \right] (n+1)^{3k/2}.$$

The corresponding formulas for $c(n+1, k)$ follow immediately. Similar computation shows that the second term in the asymptotic expansion of $c(n+1, k)$ is $O((n+1)^{n+\frac{3k}{2}-\frac{1}{2}})$.

Our approach yields another formula for the Wright constants.

Theorem 8.2. *If k is even and $k = 2l$, then*

$$k! \rho_{k-1} = (2l-1)!! (12)^{-l} + \sum_{s=1}^l \binom{2l}{2s} \frac{t_{2s} 2^{3s-2} \Gamma(3s-1) (2l-2s-1)!!}{(12)^{l-s}},$$

and if k is odd and $k = 2l+1$, then

$$k! \rho_{k-1} = - \sum_{s=0}^l \binom{2l+1}{2s+1} \frac{t_{2s+1} 2^{3s-\frac{1}{2}} \Gamma(3s+\frac{1}{2}) (2l-2s-1)!!}{(12)^{l-s}},$$

where t_r satisfy the linear recursion (6.7).

Using the formulas $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and the values for t_r calculated in Section 6, we obtain $\rho_0 = \sqrt{2\pi}/4$, $\rho_1 = 5/24$, and $\rho_2 = 5\sqrt{2\pi}/2^8$. These values were first obtained by Wright [19] using a different method. Spencer observed in [13] that $k! \rho_{k-1}$ is also the k th power moment $E[L^k]$ of the Brownian excursion area L , which was studied numerically by Louchard [11]. Hence Theorem 8.2 also gives a formula for these moments.

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