Gončarov Polynomials in Partition Lattices and Exponential Families

Dedicated to Joseph Kung

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Abstract

Classical Gončarov polynomials arose in numerical analysis as a basis for the solutions of the Gončarov interpolation problem. These polynomials provide a natural algebraic tool in the enumerative theory of parking functions. By replacing the differentiation operator with a delta operator and using the theory of finite operator calculus, Lorentz, Tringali and Yan introduced the sequence of generalized Gončarov polynomials associated to a pair $(\Delta, \mathcal{Z})$ of a delta operator Δ and an interpolation grid $\mathcal{Z}$. Generalized Gončarov polynomials share many nice algebraic properties and have a connection with the theories of binomial enumeration and order statistics. In this paper we give a complete combinatorial interpretation for any sequence of generalized Gončarov polynomials. First we show that they can be realized as weight enumerators in partition lattices. Then we give a more concrete realization in exponential families and show that these polynomials enumerate various enriched structures of vector parking functions.

Keywords: Gončarov polynomials, partition lattices, exponential family
AMS Classification: 05A10, 05A15, 05A18, 06A07

1 Introduction

The classical Gončarov interpolation problem in numerical analysis was introduced by Gončarov [2, 3] and Whittaker[17]. It asks for a polynomial $f(x)$ of degree n such that the ith derivative of $f(x)$ at a given point a_i has value b_i for $i = 0, 1, 2, ..., n$. The solution is obtained by taking linear combinations of the (classical) Gončarov polynomials, or the Abel-Gončarov polynomials, which have been studied extensively by analysts; see e.g. [2, 10, 1, 4]. Gončarov polynomials also play a crucial role in Combinatorics due to their close relations to parking functions. A (classical)

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parking function is a sequence $(a_1, a_2, ..., a_n)$ of positive integers such that for every $i = 1, 2, ..., n$, there are at least i terms that are less than or equal to i. For example, the sequences $(1, 2, 3, 4)$ and $(2, 1, 4, 1)$ are both parking functions while $(2, 2, 3, 4)$ is not. The set of parking functions stays in the center of enumerative combinatorics, with many generalizations and connections to other research areas, such as hashing and linear probing in computer science, graph theory, interpolation theory, diagonal harmonics, representation theory, and cellular automaton. See the comprehensive survey [19] for more on the combinatorial theory of parking functions.

The connection between Gončarov polynomials and combinatorics was first found by Joseph Kung, who in a short note [8] of 1981 proved that classical Gončarov polynomials give the probability distribution of the order statistics of n independent uniform random variables, and its difference analog describes the order statistics of discrete, injective functions. These results were further developed in [9] to an explicit correspondence between classical Gončarov polynomials and vector parking functions. Inspired by the rich theory on delta operators and finite operator calculus, which is a unified theory on linear operators analogous to the differentiation operator D and special polynomials, Lorentz, Tringali, and the second author of the present paper introduced the generalized Gončarov polynomials [11] as a basis for the solutions to the Gončarov interpolation problem with respect to a delta operator. Many algebraic and analytic properties of classical Gončarov polynomials have been extended to the generalized version.

A natural question is to find the combinatorial interpretations for the generalized Gončarov polynomials. To answer this question we need to understand the combinatorial significance of delta operators. In the third paper of the seminal series *On the Foundations of Combinatorial Theory III*, Mullin and Rota [12] developed the basic theory of delta operators and their associated sequence of polynomials. Such sequences of polynomials are of binomial type and occur in many combinatorial problems when objects can be pieced together out of small, connected objects. Mullin and Rota's work provides a realization of binomial sequences in combinatorial problems. However, this realization is only valid for binomial sequences whose coefficients are non-negative integers, and so excludes many basic counting polynomials, for example, the falling factorial $x_{(n)} = x(x-1)\cdots(x-n+1)$. Mullin and Rota hint at a generalization of their theory to incorporate such cases. Using the language of partitions, partition types and partition categories, Ray [14] proved that every polynomial sequence of binomial type can be realized as a weighted enumerator in partition lattices.

In this paper we give a complete combinatorial interpretation of the generalized Gončarov polynomials, first in Ray's partition lattices and then in a more concrete model, the exponential families, as described by Wilf [18]. Basically, to any polynomial sequence of binomial type and any given interpolation grid $\mathcal{Z}$ there is an associated sequence of Gončarov polynomials. While the sequence of binomial type can be realized as weighted enumerators in partition lattices or in an exponential family, the associated Gončarov polynomials count those structures which also encode vector parking functions. In other words, the generalized Gončarov polynomials characterize structures in a binomial enumeration problem that are subject to certain order-statistic constraints. Our results cover the initial attempt in [11] which provides a combinatorial interpretation for some families of generalized Gončarov polynomials in a structure called *reluctant functions*.

The rest of the paper is organized as follows. In Section 2, we recall the basic theory of delta operators and binomial enumeration, as well as the concepts of generalized Gončarov polynomials and vector parking functions. In Section 3, we describe the realization of generalized Gončarov polynomials in partition lattices and weight functions. Then, in Section 4, we study a more concrete realization of Gončarov polynomials as type-enumerator in exponential families. We end the paper in Section 5 with a few closing remarks.

2 Background

2.1 Delta Operators and Binomial Enumeration

We recall the basic theory of delta operators and their associated sequence of basic polynomials as developed by Rota, Kahaner, and Odlyzko [15]. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{K}[x]$ the vector space of all polynomials in the variable x over $\mathbb{K}$. For each $a \in \mathbb{K}$, let E_a denote the shift operator $\mathbb{K}[x] \to \mathbb{K}[x] : f(x) \mapsto f(x+a)$. A linear operator $\mathbf{s} : \mathbb{K}[x] \to \mathbb{K}[x]$ is called *shift-invariant* if $\mathbf{s}E_a = E_a\mathbf{s}$ for all $a \in \mathbb{K}$, where the multiplication is the composition of operators.

Definition 1. A delta operator Δ is a shift-invariant operator satisfying $\Delta(x) = a$ for some nonzero constant a.

Definition 2. Let Δ be a delta operator. A polynomial sequence $\{p_n(x)\}_{n\geq 0}$ is called the sequence of basic polynomials, or the associated basic sequence of Δ if

(i) $p_0(x) = 1$;

(ii) Degree of $p_n(x)$ is n and $p_n(0) = 0$ for each $n \geq 1$;

(iii) $\Delta(p_n(x)) = np_{n-1}(x)$.

Every delta operator has a unique sequence of basic polynomials, which is a sequence of binomial type (or binomial sequence) that satisfies

$$p_n(u+v) = \sum_{i\geq 0} \binom{n}{i} p_i(u) p_{n-i}(v), \tag{1}$$

for all $n \geq 0$. Conversely, every polynomial sequence of binomial type is the associated basic sequence of some delta operator.

Let $\mathbf{s}$ be a shift-invariant operator, and Δ a delta operator. Then $\mathbf{s}$ can be expanded uniquely as a formal power series of Δ. If

$$\mathbf{s} = \sum_{k\geq 0} \frac{a_k}{k!} \Delta^k,$$

we say that $f(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k$ is the Δ-indicator of $\mathbf{s}$. In fact, the correspondence

$$f(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k \longleftrightarrow \sum_{k\geq 0} \frac{a_k}{k!} \Delta^k$$

is an isomorphism from the ring $\mathbb{K}[[t]]$ of formal power series in t onto the ring of shift-invariant operators. Under this isomorphism, a shift-invariant operator is invertible if and only if its Δ-indicator $f(t)$ satisfies $f(0) \neq 0$, and it is a delta operator if and only if $f(0) = 0$ and $f'(0) \neq 0$, i.e., $f(t)$ has a compositional inverse $g(t)$ satisfying $f(g(t)) = g(f(t)) = t$.

Another important result is the generating function for the sequence of basic polynomials $\{p_n(x)\}_{n\geq 0}$ associated to a delta operator Δ. Let $f(t)$ be the D-indicator of Δ, where $D = d/dx$ is the differentiation operator. Let $g(t)$ be the compositional inverse of $f(t)$. Then,

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = \exp(xg(t)). \tag{2}$$

The operator $\Lambda = g(D)$ is called the *conjugate delta operator* of Δ, and $\{p_n(x)\}_{n\geq 0}$ is the *conjugate sequence* of Λ. It is easy to see that if $p_n(x) = \sum_{k\geq 1} p_{n,k}x^k$, then $g(t) = \sum_{k\geq 1} p_{k,1}\frac{t^k}{k!}$.

Polynomial sequences of binomial type are closely related to the theory of binomial enumeration. Consider the following model. Assume $\mathcal{B}$ is a family of discrete structures. For a finite set E, let $\Pi(E)$ be the poset of all partitions π of E, ordered by refinement, and write $|\pi|$ for the number of blocks of π. Define a *k-assembly of $\mathcal{B}$-structures on E* as a partition π of the set E into $|\pi| = k$ blocks such that each block of π is endowed with a $\mathcal{B}$-structure. Let $B_k(E)$ denote the set of all such k-assemblies. For example, when $\mathcal{B}$ is a set of rooted trees, a k-assembly of $\mathcal{B}$-structures on E is a forest of k rooted trees with vertex set E. We can also take $\mathcal{B}$ to be other structures, such as permutations, complete graphs, posets, etc. Assume that the cardinality of $B_k(E)$ depends only on the cardinality of E, but not its content. In other words, there is a bijection between $B_k(E)$ and $B_k([n])$ where $[n] = \{1, 2, ..., n\}$ and $|E| = n$.

Definition 3. Let

$$b_{n,k} = \begin{cases} |B_k([n])|, & \text{if } k \leq n \\ 0, & \text{if } k > n, \end{cases}$$

where $b_{0,0} = 1$ and $b_{n,0} = 0$ for $n \geq 1$.

Theorem 1 ([12])**.** *Assume $b_{1,1} \neq 0$. If $b_n(x) = \sum_{k=1}^{n} b_{n,k}x^k$ is the enumerator for assemblies of $\mathcal{B}$-structures on $[n]$, then $(b_n(x))_{n\geq 0}$ is a sequence of polynomials of binomial type.*

Theorem 1 provides a realization of binomial sequences in combinatorial problems. If we think of x as a positive integer such that $|X| = x$ for some set X, then we can interpret $b_n(x)$ as the number of assemblies of $\mathcal{B}$-structures on $[n]$, where each block carries a label from X. From this viewpoint, it is easy to see that $(b_n(x))_{n\geq 0}$ is of binomial type.

This realization is only valid for binomial sequences whose coefficients are non-negative integers, and so excludes many polynomial sequences naturally appearing in combinatorics, for example, the falling factorials $x_{(n)}$. Mullin and Rota expanded their construction slightly by considering the *monomorphic classes*, in which different blocks receive different labels from X, and hence the counting polynomial becomes $\tilde{b}_n(x) = \sum_{k=1}^{n} b_{n,k}x_{(k)}$. Ray [14] extended Mullin-Rota's theory and developed the concept of partition categories, and he proved that any binomial sequence can be realized as a weight enumerator in partition lattices. We will use Ray's model in Section 3.

2.2 Generalized Gončarov Polynomials and Vector Parking Functions

Let $\mathcal{Z} = (z_i)_{i\geq 0}$ be a fixed sequence with values in $\mathbb{K}$, where $\mathbb{K}$ is the scalar field. For our purpose, it suffices to take $\mathbb{K}$ to be $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. We call $\mathcal{Z}$ the interpolation grid and $z_i \in \mathcal{Z}$ the

i-th interpolation node. Let $\mathcal{T} = (t_n(x;\Delta,\mathcal{Z}))_{n\geq 0}$ be the unique sequence of polynomials that satisfies

$$\varepsilon_{z_i}\Delta^i(t_n(x;\Delta,\mathcal{Z})) = n!\delta_{i,n}, \tag{3}$$

where ε_{z_i} is evaluation at z_i.

Definition 4. The polynomial sequence $\mathcal{T} = (t_n(x;\Delta,\mathcal{Z}))_{n\geq 0}$ determined by (3) is called the *sequence of generalized Gončarov polynomials* associated with the pair $(\Delta,\mathcal{Z})$ and $t_n(x;\Delta,\mathcal{Z})$ is the n-th generalized Gončarov polynomial relative to the same pair.

This sequence $\mathcal{T}$ has a number of interesting algebraic properties. One of them is a recurrence formula described as follows: Let $t_n(x) = t_n(x;\Delta,\mathcal{Z})$ and $\{p_n(x)\}_{n\geq 0}$ be the basic sequence associated to Δ. Then

$$p_n(x) = \sum_{i=0}^{n}\binom{n}{i}\, p_{n-i}(z_i)\ t_i(x). \tag{4}$$

We remark that by definition, to compute the generalized Gončarov polynomials given the basic sequence, one would find the conjugate operator Λ via (2), compute Δ by solving for the compositional inverse of the D-indicator of Λ, and then find the n-th polynomial $t_n(x)$ of the sequence by using (3). The computation required in this process can be quite involved. However, (4) gives a recursive formula which can be used as an alternative definition for $t_n(x)$, which is much more convenient in combinatorial problems. For other algebraic properties of generalized Gončarov polynomials, see [11].

Classical Gončarov polynomials have a combinatorial interpretation in enumeration. Let $\vec{u} = (u_i)_{i\geq 1}$ be a sequence of non-decreasing positive integers. A (vector) $\vec{u}$-parking function of length n is a sequence $(x_1,...,x_n)$ of positive integers whose order statistics, i.e., the nondecreasing rearrangement $(x_{(1)},x_{(2)},...,x_{(n)})$, satisfy the inequalities $x_{(i)} \leq u_i$ for all $i = 1,\ldots,n$. Vector parking functions can be described via a parking process of n cars trying to park along a line of $x \geq n$ parking spots. Cars enter one by one in order, and before parking, each driver has a preferred parking spot. Each driver goes to her preferred spot directly and parks in the first spot available from there, if there exists one. A $\vec{u}$-parking function is a sequence of drivers' preferences such that at least i cars prefer to park in the first u_i spots, for all $i = 1,\ldots,n$. When $u_i = i$, we recover the classical parking functions, which were originally introduced by Konheim and Weiss [7] and are the preference sequences such that $x = n$ and every driver can find some spot to park. In general, the n-th Gončarov polynomials associated to the pair $(D,-\mathcal{Z})$ counts the number of $\vec{z}$-parking functions of length n, where $\vec{z} = (z_0,z_1,\ldots,z_{n-1})$ is the initial segment of the grid $\mathcal{Z}$; see [9].

A concrete realization of $t_n(x;\Delta,\mathcal{Z})$ for some other delta operators Δ is found in a combinatorial object called reluctant functions whose underlying structure are families of labeled trees. In [11], it is proved for some properly defined Δ and $\mathcal{Z}$, $t_n(x;\Delta,\mathcal{Z})$ enumerates the number of reluctant functions in a certain binomial class $\mathcal{B}$ whose label sequences are $\vec{z}$-parking functions. The object of the present paper is to extend this result and prove that for any delta operator, the generalized Gončarov polynomials (up to a scaling) have a realization as a weighted enumerator in partition lattices and in any exponential family.

3 Gončarov Polynomials in Partition Lattices

In this section we give a generic, or universal realization of generalized Gončarov polynomials in weighted enumeration over partition lattices. Our result is built on Ray's solution of the realization problem for arbitrary sequences of binomial types in the context of partition categories. Here, we will simplify his notation and present his construction in terms of incidence algebra of partially ordered sets, which was the language originally used by combinatorialists, e.g., see Joni and Rota [6].

For any finite set S, let $\Pi(S)$ denote the set of all partitions of S, and write Π_n for $\Pi([n])$. Elements of $\Pi(S)$ are partially ordered by refinement: that is, define $\pi \leq \sigma$ if every block of π is contained in a block of σ. In particular, $\Pi(E)$ has a unique maximal element $\hat{1}$ that has only one block and a unique minimal element $\hat{0}$ for which every block is a singleton. Let $|\pi|$ be the number of blocks of π and $\Pi(\pi)$ be the partitions of the set that consists of blocks of π, When $\pi \leq \sigma$, the *induced partition* σ/π is the partition σ viewed as an element of $\Pi(\pi)$. Define the *class* of (π, σ) as the sequence $\lambda = (\lambda_1, \lambda_2, ...)$ of non-negative integers such that λ_i is the number of blocks of size i in the partition σ/π, for $1 \leq i \leq |\pi|$. It follows that

$$\sum_{i\geq 1} i\lambda_i = |\pi| \qquad \text{and} \qquad \sum_{i\geq 1} \lambda_i = |\sigma|.$$

Example 1. Let $E = [8]$, $\pi = \{1\}, \{2\}, \{345\}, \{67\}, \{8\}$, $\sigma = \{1345\}, \{2\}, \{678\} \in \Pi_8$. Then, $\sigma/\pi = \{(1), (345)\}, \{(2)\}, \{(67), (8)\} \in \Pi(\pi)$. The class of (π, σ) is $\lambda = (1, 2, 0, 0, ...)$, where we have $\sum_{i\geq 1} i\lambda_i = |\pi| = 5$ and $\sum_{i\geq 1} \lambda_i = |\sigma| = 3$. □

We recall the basic notation in incidence algebra. Let P be a finite poset and A a commutative ring with unity. Denote by $Int(P)$ the set of all intervals of P, i.e., the set $\{(x, y) : x \leq y\}$. The *incidence algebra* $I(P, A)$ of P over A is the A-algebra of all functions

$$f : Int(P) \to A,$$

where multiplication is defined via the convolution

$$fg(x, y) = \sum_{x\leq z\leq y} f(x, z)g(z, y).$$

The algebra $I(P, A)$ is associative with identity δ, where

$$\delta(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

An element $f \in I(P, A)$ is invertible under the multiplication if and only if $f(x, x)$ is invertible in A for every $x \in P$.

In this paper we are concerned with the case $P = \Pi_n$, the partition lattice of $[n]$, and $A = \mathbb{K}[w_2, w_3, \ldots]$, where $w_2, w_3, ...$ are independent variables. In addition, we set $w_1 = 1$.

Definition 5. Assume $\pi \leq \sigma$ in Π_n and the class of (π, σ) is $\lambda = (\lambda_1, \lambda_2, \ldots)$. Define the *zeta-type function* $w(\pi, \sigma) \in I(\Pi_n, A)$ by letting

$$w(\pi, \sigma) = w_1^{\lambda_1} w_2^{\lambda_2} \ldots w_{|\pi|}^{\lambda_{|\pi|}}. \tag{5}$$

Note that $w(\pi,\pi) = 1$ for all π. Hence w is invertible. The inverse of w is called the *Möbius-type function* and denoted by μ^w. Explicitly, $\mu^w(\pi,\pi) = 1$ and for $\pi < \sigma$,

$$\mu^w(\pi,\sigma) = -\sum_{\pi \le \tau < \sigma} \mu^w(\pi,\tau) w(\tau,\sigma).$$

When all $w_i = 1$, the zeta-type function and the Möbius-type function become the zeta function and the Möbius function of Π_n respectively.

Example 2. Consider the lattice Π_3. Then for all $\pi < \sigma$, $w(\pi,\sigma) = w_2$ except that $w(\hat{0},\hat{1}) = w_3$. Consequently, $\mu^w(\pi,\sigma) = -w_2$ if $\pi < \sigma$ except that $\mu^w(\hat{0},\hat{1}) = 3w_2^2 - w_3$. □

Define the *zeta-type enumerator* $\{a_n(x;w)\}_{n\ge 0}$ and *Möbius-type enumerator* $\{b_n(x;w)\}_{n\ge 0}$ as follows. Let $a_0(x;w) = b_0(x;w) = 1$ and for $n \ge 1$,

$$a_n(x;w) = \sum_{\pi \in \Pi_n} w(\hat{0},\pi)\ x^{|\pi|}, \tag{6}$$

$$b_n(x;w) = \sum_{\pi \in \Pi_n} \mu^w(\hat{0},\pi)\ x^{|\pi|}. \tag{7}$$

Theorem 2 ([14])**.** *1. The polynomial sequences $\{a_n(x;w)\}_{n\ge 0}$ and $\{b_n(x;w)\}_{n\ge 0}$ are of binomial type.*

2. Let Λ be the delta operator whose D-indicator is given by $g(t) = t + \sum_{i\ge 2} w_i t^i/i!$. Then $\{a_n(x;w)\}_{n\ge 0}$ is the conjugate sequence of Λ and $\{b_n(x;w)\}_{n\ge 0}$ is the basic sequence of Λ.

For $n = 0,1,\ldots,4$, the polynomials $a_n(w,x)$ and $b_n(w,x)$ are

$$\begin{aligned}
a_0(x;w) &= 1,\\
a_1(x;w) &= x,\\
a_2(x;w) &= x^2 + w_2 x,\\
a_3(x;w) &= x^3 + 3w_2x^2 + w_3x,\\
a_4(x;w) &= x^4 + 6w_2x^3 + (4w_3 + 3w_2^2)x^2 + w_4x,
\end{aligned}$$

and

$$\begin{aligned}
b_0(x;w) &= 1,\\
b_1(x;w) &= x,\\
b_2(x;w) &= x^2 - w_2 x,\\
b_3(x;w) &= x^3 - 3w_2x^2 + (3w_2^2 - w_3)x,\\
b_4(x;w) &= x^4 - 6w_2x^3 + (15w_2^2 - 4w_3)x^2 + (10w_2w_3 - w_4 - 15w_2^3)x.
\end{aligned}$$

The linear coefficient in $b_n(w;x)$ is $\mu_n^w = \mu^w(\hat{0},\hat{1})$ in Π_n. Assume Δ is the conjugate delta operator of Λ. Then $\{a_n(x;w)\}_{n\ge 0}$ is the basic sequence of Δ and $\{b_n(x;w)\}_{n\ge 0}$ is the conjugate sequence of Δ. The operator Δ can be written as $\Delta = \sum_{n\ge 1} \mu_n^w D^n/n!$. Since $w_1 = 1$, each μ_n^w is a polynomial of $w_2, w_3, \ldots$. If we take w_1 to be a variable, μ_n^w would be a polynomial in $w_1^{-1}, w_2, w_3, \ldots$.

The condition $w_1 = 1$ is equivalent to the equation $a_1(x;w) = x$. Since the weight variables $w_2, w_3, \ldots$ can take arbitrary values, Theorem 2 implies that any polynomial sequence

$\{p_n(x)\}_{n\geq 0}$ of binomial type with $p_1(x) = x$ can be realized as the zeta-type weight enumerator or the Möbius-type weight enumerator over partition lattices. Note that for any scalar $k \neq 0$, if a sequence $\{p_n(x)\}_{n\geq 0}$ is the basic sequence of Δ and the conjugate sequence of Λ, then $\{p_n/k^n\}_{n\geq 0}$ is the basic sequence of $k\Delta$ and the conjugate sequence of $g(D/k)$ where $g(t)$ is the D-indicator of Λ. Hence Theorem 2 covers all polynomial sequences of binomial type up to a scaling.

In the problem of counting assemblies of $\mathcal{B}$-structures outlined in Section 2.1, the enumerator $\sum_k b_{n,k}x^k$ in Theorem 1 is a specialization of the polynomial $a_n(x;w)$, where w_n is the number of $\mathcal{B}$-structures on a block of size n. For example, when $\mathcal{B}$ is the set of rooted trees, $w_n = n^{n-1}$ and hence $a_n(x;w) = x(x+n)^{n-1}$, the n-th Abel polynomial.

Our objective is to fit the generalized Gončarov polynomials into this model and present a combinatorial interpretation in terms of weight-enumeration in partition lattices. Following the notation of Theorem 2, let Δ be the conjugate delta operator of Λ. Given an interpolation grid $\mathcal{Z}$, we denote by $t_n(x;w,\mathcal{Z})$ the n-th generalized Gončarov polynomial relative to the pair $(\Delta,\mathcal{Z})$. We use this notation to emphasize the role of the zeta-type function $w(\pi,\sigma)$.

To get a formula for the polynomial $t_n(x;w,\mathcal{Z})$, we use the recurrence (4) in Section 2.2. Since $a_n(x;w)$ is the basic sequence of Δ, $\{t_n(x;w,\mathcal{Z})\}_{n\geq 0}$ is the unique sequence of polynomials that satisfies the recurrence

$$a_n(x;w) = \sum_{i=0}^{n}\binom{n}{i}a_{n-i}(z_i;w)\ t_i(x;w,\mathcal{Z}). \tag{8}$$

In other words,

$$t_n(x;w,\mathcal{Z}) = a_n(x;w) - \sum_{i=0}^{n-1}\binom{n}{i}a_{n-i}(z_i;w)\ t_i(x;w,\mathcal{Z}). \tag{9}$$

In particular, $t_0(x;w,\mathcal{Z}) = 1$ and $t_1(x;w,\mathcal{Z}) = a_1(x;w) - a_1(z_0;w) = x - z_0$. Here we again assume $w_1 = 1$ and hence $a_1(x;w) = x$. Since if Δ is changed to $k\Delta$, the corresponding $t_n(x;w,\mathcal{Z})$ just changes to $t_n(x;w,\mathcal{Z})/k^n$, again we cover all the cases up to a scaling.

Assume x is a positive integer and $X = \{1,2,\ldots,x\}$. Then $a_n(x;w)$ is the zeta-type weight enumerator of all the block-labeled partitions, where each block of the partition carries a label from X. In symbols,

$$a_n(x;w) = \sum_{\pi\in\Pi_n} w(\hat{0},\pi)\cdot|\{f:\mathrm{Block}(\pi)\to X\}|,$$

where Block(π) is the set of blocks of π. For a partition π with a block-labeling f, we record the labeling by the list $f_\pi = (x_1,x_2,\ldots,x_n)$, where $x_i = f(B_j)$ whenever i is in the block B_j of π.

Let $\vec{z} = (z_0,z_1,\cdots,z_{n-1})$ be the initial segment of the grid $\mathcal{Z}$. Furthermore, assume that $z_0 \leq z_1 \cdots \leq z_{n-1}$ are positive integers with $z_{n-1} < x$.

Define the set $\mathcal{PF}_\pi(\mathcal{Z})$ as the set of all block-labelings of π that are also $\vec{z}$-parking functions, i.e.,

$$\mathcal{PF}_\pi(\mathcal{Z}) = \{f:\mathrm{Block}(\pi)\to X \mid f_\pi \text{ is a } \vec{z}\text{-parking function}\}. \tag{10}$$

More precisely, $\mathcal{PF}_\pi(\mathcal{Z})$ is the set of block-labelings of π such that the order statistics of $f_\pi = (x_1,x_2,\ldots,x_n)$ satisfies $x_{(i)} \leq z_{i-1}$ for $i = 1,\ldots,n$. Let $PF_\pi(\mathcal{Z})$ be the cardinality of $\mathcal{PF}_\pi(\mathcal{Z})$.

Our main result of this section is the following theorem.

Theorem 3. *Assume $t_n(x; w, \mathcal{Z})$ is the n-th generalized Gončarov polynomial defined by* (8) *with a positive increasing integer sequence $\mathcal{Z} = (z_0, z_1, ...)$. Let x be an integer larger than z_{n-1}. Then,*

$$t_n(0; \omega, -\mathcal{Z}) = t_n(x; \omega, x - \mathcal{Z}) = \sum_{\pi \in \Pi_n} w(\hat{0}, \pi) \cdot PF_\pi(\mathcal{Z}), \tag{11}$$

where $x - \mathcal{Z} = (x - z_0, x - z_1, x - z_2, \dots)$ and $-\mathcal{Z} = (-z_0, -z_1, -z_2, \dots)$.

The first equality follows from [11, Prop.3.5] that was proved by verifying the defining equation (3), and the second equality follows from the recurrence (8) and Lemma 4 proved next. Note that all three parts of (11) are polynomials of $z_0, z_1, \dots, z_{n-1}$, hence (11) is a polynomial identity.

Lemma 4. *For every $n \geq 0$, it holds that*

$$a_n(x; w) = \sum_{i=0}^{n} \binom{n}{i} a_{n-i}(x - z_i; w) \sum_{\pi \in \Pi_i} w(\hat{0}, \pi) \cdot PF_\pi(\mathcal{Z}) \tag{12}$$

Proof. Again we assume that x and z_i are positive integers and $z_0 < z_1 < \cdots < z_{n-1} < x$. For a finite set E and P, let $\mathcal{S}(E, P)$ be the set of pairs (π, f) where π is a partition of the set E and f is a function from Block(π) to P. Then the left-hand side of (12) counts the set $\mathcal{S}([n], X)$ by the zeta-type weight function $w(\hat{0}, \pi)$. Note that if π has blocks $B_1, B_2, \dots, B_k$, then

$$w(\hat{0}, \pi) = \prod_{j=1}^{k} w_{|B_j|}.$$

For a pair $(\pi, f) \in \mathcal{S}([n], X)$ with $f_\pi = (x_1, x_2, \dots, x_n)$, let $\mathbf{inc}(f_\pi) = (x_{(1)}, x_{(2)}, \dots, x_{(n)})$ be the non-decreasing rearrangement of the terms of f_π. Set

$$i(f) = \max\{k : x_{(j)} \leq z_{j-1} \ \forall j \leq k\}.$$

Thus, the maximality of $i = i(f)$ means that

$$x_{(1)} \leq z_0, \ x_{(2)} \leq z_1, \dots, x_{(i)} \leq z_{i-1}$$

and

$$z_i < x_{(i+1)} \leq x_{(i+2)} \leq \cdots \leq x_{(n)} \leq x.$$

In the case that $x_{(j)} > z_{j-1}$ for all j, we have $i(f) = 0$.

Assume $(x_{r_1}, ..., x_{r_i})$ is the subsequence of f_π from which the non-decreasing sequence $(x_{(1)}, x_{(2)}, ..., x_{(i)})$ is obtained. Let $R_1 = \{r_1, r_2, \dots, r_i\} \subseteq [n]$. Then it is easy to see that R_1 must be a union of some blocks of π, while $R_2 = [n] \setminus R_1$ is the union of the remaining blocks of π. Let π_1 be the restriction of π on R_1 and π_2 the restriction of π on R_2. Thus π is a disjoint union of π_1 and π_2. Furthermore, let f_i be the restriction of f on R_i. Then f_1 is a map from the blocks of π_1 to $\{1, \dots, z_i\}$ that is also a $\vec{z}$-parking function, and f_2 is a map from blocks of π_2 to the set $X \setminus [z_i] = \{z_i + 1, \dots, x\}$.

Let $\mathcal{S}^P(E, X)$ be the subset of $\mathcal{S}(E, X)$ such that for each pair (π, f), the sequence f_π is a $\vec{z}$-parking function. Then the above argument defines a decomposition of $(\pi, f) \in \mathcal{S}([n], X)$

into pairs $(\pi_1, f_1) \in \mathcal{S}^P(R_1, X)$ and $(\pi_2, f_2) \in \mathcal{S}(R_2, X \setminus [z_i])$. Conversely, any pairs of (π_1, f_1) and (π_2, f_2) described above can be reassembled into a partition π of $[n]$ with labels in X. In other words, the set $\mathcal{S}([n], X)$ can be written as a disjoint union of Cartesian products as

$$\mathcal{S}([n], X) = \biguplus_{i; R_1 \subseteq [n]: |R_1| = i} \mathcal{S}^P(R_1, X) \times \mathcal{S}(R_2, X \setminus [z_i]). \tag{13}$$

In addition, if π is the disjoint union of π_1 and π_2, then

$$w(\hat{0}, \pi) = w(\hat{0}, \pi_1) w(\hat{0}, \pi_2).$$

Putting the above results together, we have

$$\begin{aligned} a_n(x; w) &= \sum_{(\pi, f) \in \mathcal{S}([n], X)} w(\hat{0}, \pi) \\ &= \sum_{i=0}^{n} \sum_{R_1 : |R_1| = i} \left(\sum_{(\pi_1, f_1) \in \mathcal{S}^p(R_1, X)} w(\hat{0}, \pi_1) \cdot \sum_{(\pi_2, f_2) \in \mathcal{S}(R_2, X \setminus [z_i])} w(\hat{0}, \pi_2) \right) \\ &= \sum_{i=0}^{n} \binom{n}{i} a_{n-i}(x - z_i; w) \sum_{(\pi_1, f_1) \in \mathcal{S}^p(R_1, X)} w(\hat{0}, \pi_1) \\ &= \sum_{i=0}^{n} \binom{n}{i} a_{n-i}(x - z_i; w) \sum_{\pi \in \Pi_i} w(\hat{0}, \pi) PF_\pi(\mathcal{Z}). \end{aligned}$$

The last equation follows from the definition of $PF_\pi(\mathcal{Z})$. □

Example 3. From the recurrence (8) we get

$$t_2(x; w, \mathcal{Z}) = x^2 + (w_2 - 2z_1)x + (2z_0 z_1 - z_0^2 - w_2 z_0).$$

Hence $t_2(0; w, -\mathcal{Z}) = 2z_0 z_1 - z_0^2 + w_2 z_0$. On the other hand, there are two partitions in Π_2. For $\pi = \{12\}$, clearly $w(\hat{0}, \{12\}) = w_2$ and $PF_{\{12\}}(\mathcal{Z}) = z_0$. For $\pi = \{1\}\{2\}$, $w(\hat{0}, \pi) = 1$ and $PF_\pi(\mathcal{Z})$ is the number of pairs of positive integers (x, y) such that $\min(x, y) \le z_0$ and $\max(x, y) \le z_1$. It is easy to check that there are $2z_0 z_1 - z_0^2$ such pairs. □

Since $\{a_n(x; w)\}$ gives a generic form of the sequence of polynomials of binomial type, $\{t_n(x; w, \mathcal{Z})\}$ is the generic form of the generalized Gončarov polynomials. In particular, from Theorem 3 we see that when $w_2 = w_3 = \cdots = 0$, $t_n(0; w, -\mathcal{Z})$ gives the number of $\vec{z}$-parking functions of length n.

4 Gončarov Polynomials in Exponential Families

In this section, we explore a more concrete realization of generalized Gončarov polynomials in exponential families, which are picturesque models that deal with counting structures that are built out of connected pieces and can be applied to many combinatorial problems.

4.1 Exponential Families

Exponential families are combinatorial models based on the partition lattices where the enumeration are captured by the exponential generating functions. The description of exponential families and their relation to the incidence algebra of Π_n can be found in standard textbooks, e.g., [16, Section 5.1]. Here we adopt Wilf's description of exponential families [18] in the context of 'playing cards' and 'hands'.

Suppose that there is given an abstract set P of 'pictures', which typically are the connected structures. A *card* $\mathcal{C}(S,p)$ is a pair consisting of a finite label set S of positive integers and a picture $p \in P$. The weight of $\mathcal{C}$ is $|S|$. If $S = [n]$, the card is called *standard*. A *hand* H is a set of cards whose label sets form a partition of $[n]$ for some n. The weight of a hand is the sum of the weights of the cards in the hand. The *n-th deck* $\mathcal{D}_n$ is the set of all standard cards of weights n. We require that $\mathcal{D}_n$ is always finite. An *exponential family* $\mathcal{F}$ is the collection of decks $\mathcal{D}_1, \mathcal{D}_2, \dots$.

In an exponential family, let $d_i = |\mathcal{D}_i|$ and $h_{n,k}$ be the number of hands H of weight n that consist of k cards. Let $h_0(x) = 1$ and for $n \geq 1$,

$$h_n(x) = \sum_{k=1}^{n} h_{n,k} x^k. \tag{14}$$

Then the main counting theorem, *the exponential formula*, states that these polynomials satisfy the generating relation:

$$e^{xD(t)} = \sum_{n \geq 0} h_n(x) \frac{t^n}{n!}, \tag{15}$$

where $D(t) = \sum_{k \geq 1} d_k t^k / k!$. In other words, if $d_1 = h_{1,1} \neq 0$, $\{h_n(x)\}_{n \geq 0}$ is a sequence of binomial type that is conjugate to the delta operator $\sum_{k \geq 1} d_k D^k / k!$.

Example 4. *Set Partitions*: Here, a card is a label set $[n]$ with a 'picture' of n dots. Each deck $\mathcal{D}_n$ consists of the single card of weight n, and a hand is just a partition of the set $[n]$. Thus, $h_{n,k}$ is the number of partitions of the set $[n]$ into k classes, which is $S(n,k)$, the Stirling number of the second kind. □

Example 5. *Permutations and their Cycles*: Each card is a cyclic permutation on a label set S. The deck $\mathcal{D}_n$ consists of all distinct cyclic permutations on $[n]$ so $d_n = (n-1)!$ and a hand is a permutation of $[n]$ consisting of k cycles. Thus, $h_{n,k}$ is the number of permutations on $[n]$ that have k cycles, that is, the signless Stirling number of the first kind $c(n,k)$. □

Note that we can interpret x^k in $h_n(x)$ as the number of maps from the set of cards in a hand to the set $X = \{1, \dots, x\}$ for some positive integer x. Hence $h_n(x)$ counts the number of hands of weight n in which each card is labeled by an element of X. This set-up gives a natural combinatorial interpretation for binomial polynomial sequences whose coefficients are positive integers.

In *Foundation III* [12] Mullin and Rota introduced a structure called *reluctant functions*, which can be used to give a combinatorial interpretation for some generalized Gončarov polynomials; see [11]. We remark that reluctant functions is a special case of exponential families, in which the 'pictures' are certain sets of trees. We will show that by taking the type enumerator an exponential family actually provides a combinatorial model for all generalized Gončarov polynomials.

4.2 Type Enumerator in Exponential Families

In a given exponential family $\mathcal{F}$, we have seen that

$$h_n(x) = \sum_{k=1}^{n} h_{n,k} x^k = \sum_{H} |\{f: \text{ cards in } H \to X\}| \tag{16}$$

where H ranges over all hands of weight n. For a hand H consisting of cards $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$ of weights $t_1, t_2, \dots, t_k$, define the type of H as

$$\text{type}(H) = y_{t_1} y_{t_2} \cdots y_{t_k},$$

where $y_1, y_2, \dots,$ are free variables.

Let

$$h_n(x; \mathbf{y}) = \sum_{H: \text{ weight } n} \text{type}(H) |\{f: \text{ cards in } H \to X\}|. \tag{17}$$

Then we have the following form of the exponential formula.

Proposition 5. *The sequence of type enumerators $\{h_n(x;\mathbf{y})\}_{n\geq 0}$, viewed as a polynomial in x, is a sequence of polynomials of binomial type satisfying the equation*

$$\sum_{n\geq 0} h_n(x;\mathbf{y}) \frac{t^n}{n!} = \exp\left(x \sum_{k\geq 1} d_k y_k \frac{t^k}{k!} \right). \tag{18}$$

Proof. We compare the formula of $h_n(x;\mathbf{y})$ with that of $h_n(x)$. Note for $n \geq 1$, $h_n(x)$ can be computed by

$$h_n(x) = \sum_{k\geq 1} \sum_{H=\{\mathcal{C}_1,\dots,\mathcal{C}_k\}} d_{t_1} d_{t_2} \cdots d_{t_k} x^k, \tag{19}$$

where $\{\mathcal{C}_1, \cdots, \mathcal{C}_k\}$ is a hand of weight n and t_i is the weight of card $\mathcal{C}_i$, while

$$h_n(x;\mathbf{y}) = \sum_{k\geq 1} \sum_{H=\{\mathcal{C}_1,\dots,\mathcal{C}_k\}} d_{t_1} d_{t_2} \cdots d_{t_k} y_{t_1} y_{t_2} \cdots y_{t_k} x^k. \tag{20}$$

The exponential formula for $h_n(x)$ then implies Proposition 5. □

Remark. Comparing to the generic form $a_n(x;w)$ in the previous section, we see that $h_n(x;\mathbf{y})$ corresponds to the case where the variables in the zeta-type function are determined by $w_n = d_n y_n$. As far as $d_1 \neq 0$, we can obtain arbitrary polynomial sequences of binomial type by taking suitable values for the y_i-variables.

4.3 Sequence of Generalized Goncarov Polynomials

Let $Z = (z_i)_{i\geq 0}$ be an interpolation grid. For the binomial sequence $\{h_n(x;\mathbf{y})\}_{n\geq 0}$ defined in an exponential family $\mathcal{F}$, we can consider the associated generalized Gončarov polynomials given by (4) with $p_n(x)$ replaced by $h_n(x;\mathbf{y})$. Denote this Gončarov polynomial by $t_n(x;\mathbf{y},\mathcal{F},\mathcal{Z})$ to emphasize that it has variables y_i and is defined in $\mathcal{F}$. Explicitly, $t_n(x;\mathbf{y},\mathcal{F},\mathcal{Z})$ is obtained by the recurrence

$$t_n(x;\mathbf{y},\mathcal{F},\mathcal{Z}) = h_n(x;y) - \sum_{i=0}^{n-1} \binom{n}{i} h_{n-i}(z_i;y) t_i(x;\mathbf{y},\mathcal{F},\mathcal{Z}). \tag{21}$$

Suppose $X = \{1,2,...,x\}$ and assume that $z_0 \le z_1 \le \cdots \le z_{n-1}$ are integers in X. Let $\vec{z} = (z_0, z_1, \ldots, z_{n-1})$. For a hand $H = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ of weight n with a function f from $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ to X, denote by f_H the list $(x_1, x_2, \ldots, x_n)$, where $x_i = f(\mathcal{C}_j)$ if i is in the label set of $\mathcal{C}_j$. Let

$$\mathcal{PF}_H(\mathcal{Z}) = \{f : \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\} \to X \mid f_H \text{ is a } \vec{z}\text{-parking function}\},$$

and $PF_H(\mathcal{Z})$ the cardinality of $\mathcal{PF}_H(\mathcal{Z})$. Then we have the following analog of Theorem 3.

Theorem 6. *For $n \ge 0$,*

$$t_n(0; \mathbf{y}, \mathcal{F}, -\mathcal{Z}) = t_n(x; \mathbf{y}, \mathcal{F}, x - \mathcal{Z}) = \sum_{H:\ \textit{of weight } n} \text{type}(H) \cdot PF_H(\mathcal{Z}). \tag{22}$$

Here by convention, the third term of (22) *equals* 1 *when* $n = 0$.

Theorem 6 follows from (21) and the following recurrence relation

$$h_n(x; \mathbf{y}) = \sum_{i=0}^{n} \binom{n}{i} h_{n-i}(x - z_i; \mathbf{y}) \sum_{H:\ \text{of weight } i} \text{type}(H) \cdot PF_H(\mathcal{Z}), \tag{23}$$

whose proof is similar to that of Lemma 4. In an exponential family $\mathcal{F}$, let $\mathcal{A}(S, X)$ be the set of pairs (H, f) such that H is a hand whose label sets form a partition of S and f is a function from the cards in H to X. Then the basic ingredients of the proof are that

(1) type(H) is a multiplicative function only depending on the weights of cards in H, and

(2) The set $\mathcal{A}([n], X)$ can be decomposed into a disjoint union of Cartesian products of the form

$$\mathcal{A}^P(R, X) \times \mathcal{A}([n] \setminus R, X \setminus [z_i]),$$

where $\mathcal{A}^P(R, X) = \{(H, f) \in \mathcal{A}(R, X) :\ f_H \text{ is a } \vec{z}\text{-parking function}\}$, and the disjoint union is taken over all the subsets R of $[n]$.

We skip the details of the proof of Eq. (23).

We illustrate the above results and some connections to combinatorics in the exponential families given in Examples 4 and 5. There are many other exponential families in which the type enumerator and associated Gončarov polynomials have interesting combinatorial significance.

1. Let $\mathcal{F}_1$ be the exponential family of set partitions described in Example 4. In this family, $d_i = 1$ for all i and $h_n(x) = \sum_{k=0}^{n} S(n,k) x^k$. In the type enumerator, if we substitute $y_1 = 1$ and $y_i = w_i$ for $i \ge 2$, then $h_n(x; \mathbf{y})$ is exactly the same as the generic sequence $a_n(x; w)$ in (6), and consequently $t_n(x; \mathbf{y}, \mathcal{F}_1, \mathcal{Z})$ is the same as the generic Gončarov polynomial $t_n(x; w, \mathcal{Z})$ defined by (9). In particular, if all $y_i = 1$, $t_n(0; \mathbf{y}, \mathcal{F}_1, -\mathcal{Z})$ gives a formula for the number of $\vec{z}$-parking functions with the additional structure that cars arrive in disjoint groups, and drivers in the same group always prefer the same parking spot.

   When $y_i = 1$ and $z_i = 1 + i$ for all i, the first few terms of the Gončarov polynomials

$t_n(x) = t_n(x; \mathbf{y}, \mathcal{F}_1, -\mathcal{Z})$ are

$$\begin{aligned} t_0(x) &= 1 \\ t_1(x) &= x+1 \\ t_2(x) &= x^2+5x+4 \\ t_3(x) &= x^3+12x^2+40x+29 \\ t_4(x) &= x^4+22x^3+163x^2+453x+311 \end{aligned}$$

In particular, for $x = 0$ we get the sequence $1, 1, 4, 29, 311, \ldots$. This is sequence A030019 in the On-Line Encyclopedia of Integer Sequences (OEIS) [13], where it is interpreted as the number of labeled spanning trees in the complete hypergraph on n vertices (all hyper-edges having cardinality 2 or greater). It would be interesting to find a direct bijection between the hyper-trees and the parking-function interpretation.

2. Let $\mathcal{F}_2$ be the exponential family of the permutations and their cycles, as described in Example 5. Here $d_n = (n-1)!$ and $h_n(x) = \sum_{k=0}^{n} c(n,k)x^k = x^{(n)}$, where the $c(n,k)$ is the signless Stirling numbers of the first kind and $x^{(n)}$ is the rising factorial $x(x+1)\cdots(x+n-1)$. When $y_1 = 1$, the Gončarov polynomial $t_n(x; \mathbf{y}, \mathcal{F}_2, \mathcal{Z})$ can be obtained from the generic form $t_n(x; w, \mathcal{Z})$ by replacing w_n with $(n-1)!y_n$ for $n \geq 2$. When all $y_i = 1$, i.e. $y = \mathbf{1}$, $t_n(0; \mathbf{1}, \mathcal{F}_2, -\mathcal{Z})$ gives a formula for the number of $\vec{z}$-parking functions with the addition requirement that cars are formed in disjoint cycles, and drivers in the same cycle prefer the same parking spot.
In addition, when $y = \mathbf{1}$, and $\mathcal{Z}$ is the arithmetic progression $z_i = a + bi$, the Goncarov polynomial is
$$t_n(x; \mathbf{1}, \mathcal{F}_2; -\mathcal{Z}) = (x+a)(x+a+nb+1)^{(n-1)}. \tag{24}$$
Another combinatorial interpretation of $t_n(0; \mathbf{1}, \mathcal{F}_2, -\mathcal{Z})$ is given in [11, Section 6.7], where it shows that $t_n(0; \mathbf{1}, \mathcal{F}, -\mathcal{Z})$ is $n!$ times the number of lattice paths from $(0,0)$ to $(x-1, n)$ with strict right boundary $\mathcal{Z}$. For example, when $z_i = a + bi$ for some positive integers a and b, $\frac{1}{n!}t_n(0; \mathbf{1}, \mathcal{F}_2, -\mathcal{Z})$ is the number of lattice paths from $(0,0)$ to $(x-1, n)$ which stay strictly to the left of the points $(a+ib, i)$ for $i = 0, 1, \ldots, n$. In particular for $a = 1$ and $b = k$, it counts the number of labeled lattice paths from the origin to (kn, n) that never pass below the line $x = yk$. In that case (24) gives $\frac{1}{1+kn}\binom{(k+1)n}{n}$, the n-th k-Fuss-Catalan number.

Remark. We can also consider the injective functions in the definition of $h_n(x)$ and $h_n(x; y)$ in (16) and (17), where the term x^k is replaced by the lower factorial $x_{(k)} = x(x-1)\cdots(x-k+1)$. In other words, cards of a hand are labeled by X with the additional property that different cards get different labels. Some examples are given in [11, Section 6] and called *monomorphic classes*. A result analogous to Theorem 6 still holds for the momomorphic classes of an exponential family.

As a final result we point out an explicit formula to compute the constant coefficient of the generalized Gončarov polynomial whenever we know the basic sequence $\{p_n(x)\}_{n\geq 0}$. It is proved in [11] and only depends on the recurrence (4) and the fact that $p_n(0) = 0$ for $n > 0$. The proof does not need an explicit formula for the delta operator Δ and hence the result is easier to use when we need to compute the value of $t_n(0;\mathbf{y},\mathcal{F},-\mathcal{Z})$ in a given exponential family.

Let $\{p_n(x)\}_{n\ge 0}$ be a sequence of binomial type and $\mathcal{Z}=(z_0,z_1,\dots)$ be a given grid. Assume $\{t_n(0;-\mathcal{Z})\}_{n\ge 0}$ is defined by the recurrence relation

$$t_n(0;-\mathcal{Z}) = -\sum_{i=0}^{n-1}\binom{n}{i}p_{n-i}(-z_i)t_i(0),$$

for $n\ge 1$ and $t_0(0;-\mathcal{Z})=1$. Then for $n\ge 1$, $t_n(0;-\mathcal{Z})$ can be expressed as a summation over ordered partitions.

Given a finite set S with n elements, an *ordered partition* of S is an ordered list $(B_1,\dots,B_k)$ of disjoint nonempty subsets of S such that $B_1\cup\cdots\cup B_k=S$. If $\rho=(B_1,\dots,B_k)$ is an ordered partition of S, then we set $|\rho|=k$. For each $i=1,2,\dots,k$, we let $b_i=b_i(\rho)=|B_i|$, and $s_i:=s_i(\rho):=\sum_{j=1}^{i}b_j$. In particular, set $s_0(p)=0$. Let $\mathcal{R}_n$ be the set of all ordered partitions of the set $[n]$.

Theorem 7 ([11])**.** *For $n\ge 1$,*

$$\begin{aligned} t_n(0;-\mathcal{Z}) &= \sum_{\rho\in\mathcal{R}_n}(-1)^{|\rho|}\prod_{i=0}^{k-1}p_{b_{i+1}}(-z_{s_i}) \\ &= \sum_{\rho\in\mathcal{R}_n}(-1)^{|\rho|}p_{b_1}(-z_0),\cdots p_{b_k}(-z_{s_{k-1}}). \end{aligned} \tag{25}$$

The following list is the formulas for the first several Gončarov polynomials.

$$\begin{aligned} t_0(0;-\mathcal{Z}) &= 1 \\ t_1(0;-\mathcal{Z}) &= -p_1(-z_0) \\ t_2(0;-\mathcal{Z}) &= 2p_1(-z_0)p_1(-z_1)-p_2(-z_0) \\ t_3(0;-\mathcal{Z}) &= -p_3(-z_0)+3p_2(-z_0)p_1(-z_2)+3p_1(-z_0)p_2(-z_1)-6p_1(-z_0)p_1(-z_1)p_1(-z_2). \end{aligned}$$

4.4 Degenerate Cases

In an exponential family, the polynomial $h_n(x)$ or $h_n(x;\mathbf{y})$ may not always have degree n, e.g., when $d_1=h_{1,1}=0$. We say that such polynomial sequences and the corresponding exponential families are *degenerate*. For a degenerate sequence of polynomials, there is no delta operator for which the sequence is the basic or the conjugate sequence. Nevertheless, the exponential formulas (15) and (18) are still true. Hence the sequences $\{h_n(x)\}_{n\ge 0}$ and $\{h_n(x;\mathbf{y})\}_{n\ge 0}$ still satisfy the binomial-type identity (1).

Without a delta operator, we cannot define the generalized Gončarov interpolation problems. However, we can still introduce the generalized Gončarov polynomials via the recurrence (4). Furthermore, we will prove in Theorem 8 that the shift invariance of Gončarov polynomials can also be derived from (4). Therefore, Theorems 6 and 7 still hold true for the degenerate exponential families since all the proofs follow from the binomial-type identity (1) and the recurrence (4).

Theorem 8. *Assume $\{p_n(x)\}_{n\ge 0}$ is a polynomial sequence of binomial type with $p_0(x)=1$, but the degree of $a_n(x)$ is not necessary n. Let $t_n(x;\mathcal{Z})$ be defined by the recurrence relation*

$$t_n(x;\mathcal{Z}) = p_n(x)-\sum_{i=0}^{n-1}\binom{n}{i}p_{n-i}(z_i)t_i(x;\mathcal{Z}). \tag{26}$$

For any scalar η and the interpolation grid $\mathcal{Z} = \{z_0, z_1, z_2, \dots\}$, let $\mathcal{Z} + \eta$ be the sequence $(z_0 + \eta, z_1 + \eta, z_2 + \eta, \cdots)$. Then we have

$$t_n(x + \eta; \mathcal{Z} + \eta) = t_n(x; \mathcal{Z}) \tag{27}$$

for all $n \geq 0$.

Proof. We prove Theorem 8 by induction on n. The initial case $n = 0$ is trivial since $t_0(x; \mathcal{Z}) = 1$ for all x and any grid $\mathcal{Z}$. Assume Eq. (27) is true for all indices less than n. We compute $t_n(x + \eta; \mathcal{Z} + \eta)$. By definition

$$t_n(x + \eta; \mathcal{Z} + \eta) = p_n(x + \eta) - \sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i + \eta) t_i(x + \eta; \mathcal{Z} + \eta). \tag{28}$$

By the inductive hypothesis $t_i(x + \eta; \mathcal{Z} + \eta) = t_i(x; \mathcal{Z})$ for $i < n$ and the binomial identity of $p_n(x)$, the right-hand side of (28) can be written as

$$\begin{aligned} & \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(\eta) - \sum_{i=0}^{n-1} \binom{n}{i} \left(\sum_{j=0}^{n-i} \binom{n-i}{j} p_{n-i-j}(z_i) p_j(\eta) \right) t_i(x; \mathcal{Z}) \\ = & \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(\eta) - \sum_{\substack{i+j \leq n \\ \text{except } (i,j)=(n,0)}} \binom{n}{i} \binom{n-i}{j} p_j(\eta) p_{n-i-j}(z_i) t_i(x; \mathcal{Z}) \end{aligned} \tag{29}$$

Since

$$\binom{n}{i}\binom{n-i}{j} = \frac{n!}{i!j!(n-i-j)!} = \binom{n}{j}\binom{n-j}{i},$$

then (29) can be expressed as

$$\begin{aligned} & \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(\eta) - \sum_{\substack{i+j \leq n \\ \text{except } (i,j)=(n,0)}} \binom{n}{j} \binom{n-j}{i} p_j(\eta) p_{n-i-j}(z_i) t_i(x; \mathcal{Z}) \\ = & \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(\eta) - \sum_{j=1}^{n} \binom{n}{j} p_j(\eta) \sum_{i=0}^{n-j} \binom{n-j}{i} p_{n-i-j}(z_i) t_i(x; \mathcal{Z}) \\ & - \sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i) t_i(x; \mathcal{Z}). \end{aligned} \tag{30}$$

The last summation in (30) corresponds to the terms with $j = 0$. Note that

$$\sum_{i=0}^{n-j} \binom{n-j}{i} p_{n-i-j}(z_i) t_i(x; \mathcal{Z}) = p_{n-j}(x).$$

Hence For. (30) is equal to

$$\begin{aligned} & \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(\eta) - \sum_{j=1}^{n} \binom{n}{j} p_j(\eta) p_{n-j}(x) - \sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i) t_i(x; \mathcal{Z}) \\ = & \; p_n(x) - \sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i) t_i(x; \mathcal{Z}) \\ = & \; t_n(x; \mathcal{Z}). \end{aligned}$$

This finishes the proof. □

The next example shows a degenerate exponential family.

Example 6. *2-Regular simple graphs.* In this exponential family a card is an undirected cycle on a label set $[m]$ (where $m \geq 3$). The deck $\mathcal{D}_n$ consists of all undirected circular arrangements of n letters so $d_n = \frac{1}{2}(n-1)!$ for $n \geq 3$ and $d_1 = d_2 = 0$. A hand is then a undirected simple graph on the vertex set $[n]$, which is 2-regular, that is, every vertex has degree 2. Thus, $h_{n,k}$ is the number of undirected 2-regular simple graphs on n vertices consisting of k cycles. Denote by $\mathcal{F}_3$ this exponential family.

For $\mathcal{F}_3$, the type enumerators are $h_0(x,\mathbf{y}) = 1$, $h_1(x;\mathbf{y}) = h_2(x;\mathbf{y}) = 0$, $h_3(x;\mathbf{y}) = y_3 x$, $h_4(x;\mathbf{y}) = 2y_4 x$, $h_5(x,\mathbf{y}) = 12 y_5 x$, and $h_6(x;\mathbf{y}) = 60 y_6 x + 10 y_3^2 x^2$, etc. Although the degree of $h_n(x;\mathbf{y})$ is not n, the exponential formula still holds:

$$\sum_{k=0}^{n} h_n(x;\mathbf{y}) \frac{t^k}{k!} = \exp\left(x \sum_{k\geq 3} y_k \frac{t^k}{2k} \right).$$

We compute by the recurrence (21) that

$$\begin{aligned}
t_0(x;\mathbf{y},\mathcal{F}_3,\mathcal{Z}) &= 1 \\
t_1(x;\mathbf{y},\mathcal{F}_3,\mathcal{Z}) &= t_2(x;y,\mathcal{F}_3,\mathcal{Z}) = 0 \\
t_3(x;\mathbf{y},\mathcal{F}_3,\mathcal{Z}) &= y_3(x-z_0), \\
t_4(x;\mathbf{y},\mathcal{F}_3,\mathcal{Z}) &= 3y_3(x-z_0), \\
t_5(x;\mathbf{y},\mathcal{F}_3,\mathcal{Z}) &= 12y_5(x-z_0), \\
t_6(x;\mathbf{y},\mathcal{F}_3,\mathcal{Z}) &= 10y_3^2x^2 + 60y_6x - 20y_3^2z_3x - 60y_6z_0 - 10y_3^2z_0^2 + 20y_3^2z_0z_3.
\end{aligned}$$

The equation

$$t_n(0;\mathbf{y},\mathcal{F}_3,-\mathcal{Z}) = \sum_{H:\text{ of weight } n} \text{type}(H)\cdot PF_H(\mathcal{Z})$$

is still true. For example, for $n = 6$, $t_6(0;y,\mathcal{F}_3,-\mathcal{Z}) = 60y_6z_0 + 20y_3^2z_0z_3 - 10y_3^2z_0^2$. The term $60y_6z_0$ comes from the $5!/2 = 60$ 6-cycles, and the terms $10y_3^2(2z_0z_3 - z_0^2)$ comes from the 10 hands each with two 3-cycles. □

5 Closing Remarks

In this paper we present the combinatorial interpretation of an arbitrary sequence of Gončarov polynomials associated with a polynomial sequence of binomial type. There are many other combinatorial problems that provide a formal framework of coalgebras, bialgebras, or Hopf algebras [6]. In those problems the counting sequences satisfy an identity that is analogous to the binomial-type identity (1), with the binomial coefficients $\binom{n}{i}$ replaced by some other section coefficients. For example, the theory of binomial enumeration proposed by Mullin and Rota [12] was generalized to an abstract context and applied to dissecting schemes by Henle [5]. It would be an interesting project to investigate the role of generalized Gončarov polynomials in these other dissecting schemes and discrete structures. As suggested by Henle, this research may lead to connections to rook polynomials, order invariants of posets, Tutte invariants of combinatorial geometries, cycle indices and symmetric functions, and many others.

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