Gončarov Polynomials and Parking Functions

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Proposed running head: Gončarov Polynomials

Abstract

Let $\mathbf{u}$ be a sequence of non-decreasing positive integers. A $\mathbf{u}$-parking function of length n is a sequence $(x_1, x_2, \ldots, x_n)$ whose order statistics (the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ obtained by rearranging the original sequence in non-decreasing order) satisfy $x_{(i)} \leq u_i$. The Gončarov polynomials $g_n(x; a_0, a_1, \ldots, a_{n-1})$ are polynomials defined by the biorthogonality relation:

$$\varepsilon(a_i)D^i g_n(x; a_0, a_1, \ldots, a_{n-1}) = n!\delta_{in},$$

where $\varepsilon(a)$ is evaluation at a and D is the differentiation operator. In this paper we show that Gončarov polynomials form a natural basis of polynomials for working with $\mathbf{u}$-parking functions. For example, the number of $\mathbf{u}$-parking functions of length n is $(-1)^n g_n(0; u_1, u_2, \ldots, u_n)$. Various properties of Gončarov polynomials are discussed. In particular, Gončarov polynomials satisfy a linear recursion obtained by expanding x^n as a linear combination of Gončarov polynomials, which leads to a decomposition of an arbitrary sequence of positive integers into two subsequences: a "maximum" $\mathbf{u}$-parking function and a subsequence consisting of terms of higher values. Many counting results for parking functions can be derived from this decomposition. We give, as examples, formulas for sum enumerators, and a linear recursion and Appell relation for factorial moments of sums of $\mathbf{u}$-parking functions.

Key Words: Gončarov polynomials, parking functions, sum enumerators

[1] Supported by NSF grant DMS-0070574 and a Sloan fellowship. Part of this research was carried out at the Institute for Advanced Study and was supported by NSF grant DMS-9729992.

1 Introduction

We shall think of finite sequences $(x_1, x_2, \ldots, x_n)$ as sequences and functions with domain $\{1, 2, \ldots, n\}$. If $(x_1, x_2, \ldots, x_n)$ is a sequence of real numbers, then the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ of *order statistics* is obtained by rearranging the original sequence in non-decreasing order. Let $\mathbf{u}$ be a non-decreasing sequence $(u_1, u_2, u_3, \ldots)$ of positive integers. A $\mathbf{u}$-*parking function* of length n is a length-n sequence $(x_1, x_2, \ldots, x_n)$ of positive integers whose sequence of order statistics satisfies $x_{(i)} \leq u_i$.

We shall call $(1, 2, 3, \ldots)$-parking functions *ordinary* parking functions. Ordinary parking functions originated in the theory of hashing and searching in computer science. They derive their name from a somewhat whimsical interpretation involving parking n cars in n spaces along a one-way street (see [11] or [9], p. 545). There is a similar interpretation for $\mathbf{u}$-parking functions when the numbers u_i are distinct: one wishes to park n cars in a longer one-way street with at least u_n spaces, but only n spaces, at positions $u_1, u_2, \ldots, u_n$, are still empty.

Ordinary parking functions have been extensively studied. In particular, it is known that the number of ordinary parking functions of length n is

$$(n+1)^{n-1},$$

a formula which is closely related to Cayley's formula for the number of labeled trees. This relation with trees had motivated much work in this area, particularly in finding bijections between ordinary parking functions and labeled trees. See [6] for an extensive list of references. Less obvious, perhaps, is the observation that the formula is (up to a sign) an evaluation of an Abel polynomial. It is this observation which led us to Gončarov polynomials.

Gončarov polynomials (see [1, 2, 7, 13]) arose in the following special case of Hermite interpolation in numerical analysis.

Gončarov Interpolation. Given two sequences of real or complex numbers $a_0, a_1, \ldots, a_n$ and $d_0, d_1, \ldots, d_n$, find a polynomial $p(x)$ of degree n such that for each $i, 0 \leq i \leq n$, the ith derivative $p^{(i)}(x)$ evaluated at a_i equals d_i.

The natural basis of polynomials for this interpolation problem is the sequence of Gončarov polynomials defined in Section 3. A special case of this is *Abel interpolation*, where the point a_i is the integer i. The Gončarov polynomials for this case are the Abel polynomials.

The appearance of Abel polynomials in both the enumeration of parking functions and Abel interpolation was one of the motivations behind this paper. We shall show that the Gončarov polynomials are the natural basis of polynomials for working with parking functions, even in the ordinary case, and the enumerative theory of ordinary parking functions can be generalized to $\mathbf{u}$-parking functions using Gončarov polynomials.

The approach in this paper is to apply results about Gončarov polynomials to parking functions. We start with a discussion of a general theory of biorthogonal polynomials in Section 2 and specialize this theory to Gončarov polynomials in Section 3. In Section 4, we present a combinatorial description of the coefficients of Gončarov polynomials in terms of rankings on ordered partitions. The key idea connecting Gončorav polynomials to parking functions is a decomposition of an arbitrary sequence of positive integers into two subsequences, a "maximum" $\mathbf{u}$-parking functions of length m and a subsequence all of whose terms are strictly larger than u_m. This is given in Section 5. An immediate application yields formulas for the number of parking functions. In Section 6, we use this decomposition to obtain linear recursions and generating function identities for sum enumerators of $\mathbf{u}$-parking functions. Finally, in Section 7, we derive a linear recursion for moments of sums of **u**-parking functions. Such moments have applications in the analysis of probing algorithms [3, 10] and the enumeration of sparsely-edged graphs [23, 28]. In two later papers [14, 15], we shall obtain simple exact formulas for these moments using the linear recursion.

2 Sequences of biorthogonal polynomials

We shall need several results about Gončarov polynomials in this paper. Many of these results are special cases of a general algebraic, that is to say, non-analytic, theory of sequences of polynomials biorthogonal to a sequence of linear functionals. Although this theory must be well-known (for some examples, see [1] or [2]), we have not been able to find an explicit description in the literature.

Consider the vector space $\mathcal{P}$ of all polynomials in the variable x over a field F of characteristic zero. Let $D : \mathcal{P} \to \mathcal{P}$ be the differentiation operator. For a scalar a in the field F, let

$$\varepsilon(a) : \mathcal{P} \to F,\ p(x) \mapsto p(a)$$

be the linear functional which evaluates $p(x)$ at a.

Let $\varphi_s(D), s = 0, 1, 2, \ldots$ be a sequence of linear operators on $\mathcal{P}$ of the form

$$\varphi_s(D) = D^s \sum_{r=0}^{\infty} b_{sr} D^r,$$

where the coefficients b_{s0} are assumed to be non-zero. Note that, although $\varphi_s(D)$ are infinite formal sums, they become finite sums when applied to a specific polynomial. Then there exists a unique sequence $p_n(x), n = 0, 1, 2, \ldots$ of polynomials such that $p_n(x)$ has degree n and

$$\varepsilon(0)\varphi_s(D)p_n(x) = n!\delta_{sn}, \tag{2.1}$$

where δ_{sn} is the Kronecker delta. To see this, let

$$p_n(x) = \sum_{k=0}^{n} c_{nk} x^k.$$

Then, for a given index n, the orthogonality relations are equivalent to the following upper triangular system of linear equations in the unknowns $c_{n,0}, c_{n,1}, c_{n,2}, \ldots, c_{n,n}$:

$$\begin{aligned}
b_{00}c_{n0} + b_{01}c_{n1} + 2!b_{02}c_{n2} + 3!b_{03}c_{n3} + \ldots + \quad n!b_{0n}c_{nn} &= 0 \\
b_{10}c_{n1} + 2!b_{11}c_{n2} + 3!b_{12}c_{n3} + \ldots + n!b_{1,n-1}c_{nn} &= 0 \\
2!b_{20}c_{n2} + 3!b_{21}c_{n3} + \ldots + n!b_{2,n-2}c_{nn} &= 0 \\
\ldots& \\
n!b_{n0}c_{nn} &= n!.
\end{aligned}$$

This system of linear equations can be solved uniquely for every index n. Hence, the polynomials $p_n(x)$ exist and they are uniquely determined by the orthogonality relations (2.1). Note also that $p_n(x)$ depends only on the operators $\varphi_0(D), \varphi_1(D), \ldots, \varphi_{n-1}(D)$. When solving this system, we need only divide by the diagonal entries b_{s0}. Hence, if we put on the extra assumption that the entries b_{s0} all equal 1, then $p_n(x)$ is monic and the coefficients of $p_n(x)$ are polynomials in the entries b_{sr}.

The polynomial sequence $p_n(x)$ is said to be *biorthogonal* to the sequence $\varphi_s(D)$ of operators, or, as some would prefer, the sequence $\varepsilon(0)\varphi_s(D)$ of linear functionals. Using Cramer's rule to solve the linear system and Laplace's expansion to group the results, we obtain the following *determinantal formula*:

$$p_n(x) = \frac{n!}{b_{00}b_{10}\cdots b_{n0}} \begin{vmatrix} b_{00} & b_{01} & b_{02} & \dots & b_{0,n-1} & b_{0n} \\ 0 & b_{10} & b_{11} & \dots & b_{1,n-2} & b_{1,n-1} \\ 0 & 0 & b_{20} & \dots & b_{2,n-3} & b_{2,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{n-1,0} & b_{n-1,1} \\ 1 & x & x^2/2! & \dots & x^{n-1}/(n-1)! & x^n/n! \end{vmatrix}. \tag{2.2}$$

Another important consequence of the fact that the initial segment $\varphi_s(D)$, $s = 0, 1, 2, \dots, n$ gives a non-singular upper triangular system of linear equations is that if $p(x)$ is a degree-n polynomial, then the conditions

$$\varepsilon(0)\varphi_i(D)p(x) = 0 \quad \text{for} \quad 0 \le i \le n$$

imply that $p(x)$ is identically zero. In particular, if $p(x)$ has degree n, then

$$p(x) = \sum_{i=0}^{n} \frac{\varepsilon(0)\varphi_i(D)p(x)}{i!} p_i(x). \tag{2.3}$$

This gives an *expansion formula*. Furthermore, the unique solution to the interpolation problem, *given numbers* $d_0, d_1, \dots, d_n$*, find a degree-*n *polynomial* $p(x)$ *such that for* $i = 0, 1, \dots, n$,

$$\varepsilon(0)\varphi_i(D)p(x) = d_i,$$

is given by the formula

$$p(x) = \sum_{i=0}^{n} \frac{d_i p_i(x)}{i!}. \tag{2.4}$$

Since

$$\varepsilon(0)\varphi_i(D)x^n = n! b_{i,n-i},$$

a special case of equation (2.3) or equation (2.4) is

$$x^n = \sum_{i=0}^{n} \frac{n! b_{i,n-i} p_i(x)}{i!}. \tag{2.5}$$

Equation (2.5) gives a *linear recursion* for $p_n(x)$. These linear recursions are perhaps the most efficient way to calculate the sequence $p_n(x)$ explicitly on a computer. Multiplying these equations by $t^n/n!$, summing over all non-negative integers n, and rearranging the right-hand side into products, we obtain the following formal power series equation (which is an instance of what one might call an *Appell relation*):

$$e^{xt} = \sum_{n=0}^{\infty} \frac{p_n(x)\varphi_n(t)}{n!}, \tag{2.6}$$

where $\varphi_n(t) = t^s \sum_{r=0}^{\infty} b_{sr} t^r$.

Another way to prove the Appell relation (2.6) is to observe that when one applies $\varepsilon(0)\varphi_s(D)$ to both sides, one obtains the same result. Observe also that when restricted to the subspace $\mathcal{P}_m$ of all polynomials of degree less than or equal to m in $\mathcal{P}$, the operators D^s are expressible as linear combinations of the operators $\varepsilon(0)\varphi_t(D), t = 0, 1, 2, \ldots, m$. Hence, one also obtains the same result when D^s is applied to both sides of the Appell relation, that is, the coefficient of x^s are the same on both sides.

We end with a matrix version of the linear recursion. We can rewrite the first $n+1$ instances of equation (2.5) as the matrix equation

$$\overrightarrow{x^i} = \mathcal{B}\,\overrightarrow{p_i(x)},$$

where

$$\overrightarrow{x^i} = [1, x, x^2, \ldots, x^n]^T,$$

$$\overrightarrow{p_i(x)} = [p_0(x), p_1(x), p_2(x), \ldots, p_n(x)]^T,$$

and $\mathcal{B}$ is the $(n+1) \times (n+1)$ lower triangular matrix

$$\left[\binom{i}{j}(i-j)!b_{j,i-j}\right]_{0 \le i,j \le n}.$$

We use the convention that the binomial coefficient $\binom{i}{j}$ is zero if $j > i$. For example, when $n = 3$, we have

$$\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_{01} & 1 & 0 & 0 \\ 2b_{02} & 2b_{11} & 1 & 0 \\ 6b_{03} & 6b_{12} & 3b_{21} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ p_1(x) \\ p_2(x) \\ p_3(x) \end{bmatrix}$$

However, we also have

$$\overrightarrow{p_i(x)} = \mathcal{C}\,\overrightarrow{x^i},$$

where $\mathcal{C}$ is the $(n+1) \times (n+1)$ lower triangular *coefficient matrix*

$$[c_{ij}]_{0 \le i,j \le n}$$

whose entries c_{ij} are coefficients of the polynomials $p_i(x)$. We use the convention that c_{ij} is zero when $j > i$. Hence, we conclude that the two lower triangular matrices $\mathcal{B}$ and $\mathcal{C}$ are inverses of each other. In particular,

$$\overrightarrow{p_i(x)} = \mathcal{B}^{-1}\,\overrightarrow{x^i}. \tag{2.7}$$

This gives a determinantal formula for $p_n(x)$ which is row and column reducible to equation (2.2).

Summarizing, we have shown that the biorthogonality relations, the linear recursions, the Appell relation, and the matrix form of the linear recursions all define the same sequence $p_n(x)$ of polynomials.

Sequences of polynomials of binomial type are special cases of sequences of biorthogonal polynomials. We shall use a description of polynomials of binomial type given in the classic paper of Mullin and Rota [17]. Recall that a sequence $p_n(x)$ of polynomials is of binomial type if and only if

$$\sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} = e^{xf(t)}, \tag{2.8}$$

for some formal power series $f(t)$ such that $f(0) = 0$ and $Df(0) \neq 0$. These conditions are equivalent to the condition that $f(t)$ have a compositional inverse in the ring of formal power series. Let $g(t)$ be the compositional inverse of $f(t)$. Then, substituting $g(t)$ for t in equation (2.8), we obtain the Appell relation

$$e^{xt} = \sum_{n=0}^{\infty} p_n(x) \frac{[g(t)]^n}{n!}.$$

From this, we conclude that sequences of polynomials of binomial type are precisely sequences of polynomials biorthogonal to operator sequences of the form

$$\varphi_s(D) = [g(D)]^s,$$

where $g(t)$ is a formal power series with $g(0) = 0$ and $Dg(0) \neq 0$.

3 Algebraic properties of Gončarov polynomials

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of numbers or variables called *nodes.* The sequence of *Gončarov polynomials*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}), \ n = 0, 1, 2, \ldots$$

is the sequence of polynomials biorthogonal to the operators

$$E^{a_s} D^s,$$

where for any number or variable a, the operator E^a is the shift by a, that is,

$$E^a p(x) = p(x + a).$$

Because $\varepsilon(0)E^a = \varepsilon(a)$, the sequence of Gončarov polynomials $g_n(x; a_0, a_1, \ldots, a_{n-1})$ are defined by the orthogonality relations

$$\varepsilon(a_s) D^s g_n(x; a_0, a_1, \ldots, a_{n-1}) = n! \delta_{sn}.$$

Since

$$E^a = \sum_{r=0}^{\infty} \frac{a^r D^r}{r!} = e^{aD},$$

the sequence of Gončarov polynomials is biorthogonal to the sequence

$$D^s \sum_{r=0}^{\infty} \frac{a_s^r D^r}{r!}.$$

As indicated by the notation, $g_n(x; a_0, a_1, \ldots, a_{n-1})$ depends only on the nodes $a_0, a_1, \ldots, a_{n-1}$. Indeed, from equation (2.2), we have the *determinantal formula,*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = n! \begin{vmatrix} 1 & a_0 & \frac{a_0^2}{2!} & \frac{a_0^3}{3!} & \cdots & \frac{a_0^{n-1}}{(n-1)!} & \frac{a_0^n}{n!} \\ 0 & 1 & a_1 & \frac{a_1^2}{2!} & \cdots & \frac{a_1^{n-2}}{(n-2)!} & \frac{a_1^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & a_2 & \cdots & \frac{a_2^{n-3}}{(n-3)!} & \frac{a_2^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_{n-1} \\ 1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \end{vmatrix}.$$

From equations (2.5) and (2.6), we have the *linear recursion*

$$x^n = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x; a_0, a_1, \ldots, a_{i-1})$$

and the *Appell relation*

$$e^{xt} = \sum_{n=0}^{\infty} g_n(x; a_0, a_1, \ldots, a_{n-1}) \frac{t^n e^{a_n t}}{n!}.$$

Finally, from equation (2.3), we have the *expansion formula.* If $p(x)$ is a polynomial of degree n, then

$$p(x) = \sum_{i=0}^{n} \frac{\varepsilon(a_i) D^i p(x)}{i!} g_i(x; a_0, a_1, \ldots, a_{i-1}).$$

We turn now to properties specific to the sequence of Gončarov polynomials. The Gončarov polynomials can be equivalently defined by the *differential relations*

$$D g_n(x; a_0, a_1, \ldots, a_{n-1}) = n g_{n-1}(x; a_1, a_2, \ldots, a_{n-1}),$$

with initial conditions

$$g_n(a_0; a_0, a_1, \ldots, a_{n-1}) = \delta_{0n}.$$

(To see this, check that the orthogonality relations are satisfied.) Integrating the differential relations, we obtain the *integral relation*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = n \int_{a_0}^{x} g_{n-1}(t; a_1, a_2, \ldots, a_{n-1}) dt.$$

Iterating this, we obtain the *integral formula*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = n! \int_{a_0}^{x} dt_1 \int_{a_1}^{t_1} dt_2 \cdots \int_{a_{n-1}}^{t_{n-1}} dt_n.$$

The integral relation makes it clear (by induction) that $g_n(x; a_0, a_1, \ldots, a_{n-1})$ is a homogeneous polynomial with integer coefficients in the variables $x, a_0, a_1, \ldots, a_{n-1}$ of total degree n. It also gives a quick way to calculate Gončarov polynomials of low degree by hand. For example,

$$\begin{aligned}
g_0(x) &= 1, \\
g_1(x; a_0) &= x - a_0, \\
g_2(x; a_0, a_1) &= x^2 - 2a_1 x + 2a_0 a_1 - a_0^2, \\
g_3(x; a_0, a_1, a_2) &= x^3 - 3a_2 x^2 + (6a_1 a_2 - 3a_1^2)x - a_0^3 + 3a_0^2 a_2 - 6a_0 a_1 a_2 + 3a_0 a_1^2.
\end{aligned}$$

Using a change of variable, the integral relation and induction, or, observing that the differential operator is "shift-invariant" or commutes with shifts, one obtains the following useful *shift formula:*

$$g_n(x + \xi; a_0 + \xi, a_1 + \xi, \ldots, a_{n-1} + \xi) = g_n(x; a_0, a_1, \ldots, a_{n-1}).$$

The integral formula also suggests a formula which shows the effect of shifting or perturbing a single node. Using the identity

$$\int_{a_m}^{t} F(t) dt = \int_{a_m}^{a_m + b_m} F(t) dt + \int_{a_m + b_m}^{t} F(t) dt$$

at the mth integral in the integral formula, we obtain the *perturbation formula:*

$$
\begin{aligned}
&g_n(x; a_0, \ldots, a_{m-1}, a_m + b_m, a_{m+1}, \ldots, a_{n-1}) = g_n(x; a_0, \ldots a_{m-1}, a_m, a_{m+1}, \ldots, a_{n-1}) \\
-\ &\binom{n}{m} g_{n-m}(a_m + b_m; a_m, a_{m+1}, \ldots, a_{n-1}) g_m(x; a_0, a_1, \ldots, a_{m-1}).
\end{aligned}
$$

Applying the perturbation formula repeatedly, we can perturb any subset of nodes. For example, the following formula allows us to perturb an initial segment of length $n - m + 1$:

$$
\begin{aligned}
& g_n(x; a_0 + b_0, a_1 + b_1, \ldots, a_{n-m} + b_{n-m}, a_{n-m+1}, \ldots, a_{n-1}) \\
= \ & g_n(x; a_0, a_1, \ldots, a_{n-m}, a_{n-m+1}, \ldots, a_{n-1}) \\
& - \sum_{i=0}^{n-m} \binom{n}{i} g_{n-i}(a_i + b_i; a_i, a_{i+1}, \ldots, a_{n-1}) g_i(x; a_0 + b_0, a_1 + b_1, \ldots, a_{i-1} + b_{i-1}).
\end{aligned}
$$

In general, perturbation formulas can also be obtained by expanding the unperturbed polynomial $g_n(x; a_0, a_1, \ldots, a_{n-1})$ as a series in suitably perturbed Gončarov polynomials.

In general, there are no nice closed-form expressions for Gončarov polynomials. But such expressions exist for two special cases studied in analysis. The first is the case when all the nodes a_i equals a. In this case,

$$g_n(x; a, a, \ldots, a) = (x - a)^n$$

and Gončarov interpolation is just expansion as a power series at $x = a$. For this case, the linear recursion specializes to the binomial identity

$$x^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} (x - a)^i,$$

The second case (which includes the first as a special case) is when $a_0, a_1, a_2, \ldots$ form an arithmetic progression. This is the case of *Abel polynomials* and we have

$$g_n(x; y, y + b, y + 2b, \ldots, y + (n - 1)b) = (x - y)(x - y - nb)^{n-1}. \tag{3.1}$$

In particular,

$$g_n(x; 0, 1, 2, \ldots, n - 1) = x(x - n)^{n-1}.$$

The linear recursion is

$$x^n = \sum_{i=0}^{n} \binom{n}{i} (y + ib)^{n-i} (x - y)(x - y - ib)^{i-1}.$$

Substituting $x + y$ for x in the second identity, we obtain *Abel's binomial theorem*, (see, e.g., [20]),

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} (y + ib)^{n-i} x(x - ib)^{i-1}.$$

With the substitution $x + y + nb$ for x, $y + nb$ for y, and $-b$ for b, we obtain *Hurwitz's versions* of Abel's binomial theorem:

$$(x + y + nb)^n = \sum_{i=0}^{n} \binom{n}{i} (y + (n - i)b)^{n-i} x(x + ib)^{i-1},$$

or, changing indices from i to $n-i$,

$$(x+y+nb)^n = \sum_{i=0}^{n} \binom{n}{i} (y+ib)^i x(x+(n-i)b)^{n-i-1}. \tag{3.2}$$

4 Coefficients of Gončarov polynomials

The main result in this section is a combinatorial interpretation of the coefficients of Gončarov polynomials. We first show that it suffices to consider only the constant terms.

Expanding $g_n(x+y; a_0, \dots, a_{n-1})$ as a Taylor expansion in x and using the differential relations, we obtain

$$g_n(x+y; a_0, a_1, \dots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(y; a_i, a_{i+1}, \dots, a_{n-1}) x^i. \tag{4.1}$$

This is a shifted or parametrized analogue of a Sheffer relation, but *not* an actual Sheffer relation unless all the nodes a_i are equal. Thus, the Gončarov polynomials may be viewed as a "shifted" Sheffer sequence for the operator D (see [18]). The beginnings of a theory of "shifted" or "decentralized" umbral calculus has been developed in [21].

Setting $y=0$ in equation (4.1), we obtain

$$g_n(x; a_0, a_1, \dots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(0; a_i, a_{i+1}, \dots, a_{n-1}) x^i. \tag{4.2}$$

Thus, coefficients of Gončarov polynomials are constant terms of (shifted) Gončarov polynomials. In particular, we have the following special case of equation (2.7).

(4.1) Lemma. Let $\mathcal{A}$ be the lower triangular matrix

$$\left[\binom{i}{j} a_j^{i-j} \right]_{0 \le i,j \le n}.$$

Then, its inverse $\mathcal{A}^{-1}$ is the lower triangular coefficient matrix

$$\left[\binom{i}{j} g_{i-j}(0; a_j, a_{j+1}, \dots, a_{i-1}) \right]_{0 \le i,j \le n}.$$

In particular,

$$\mathcal{A}^{-1} \overrightarrow{x^i} = \overrightarrow{g_i(x; a_0, a_1, \dots, a_{n-1})}.$$

We shall now give a combinatorial interpretation of the constant terms of Gončarov polynomials. This interpretation is obtained by considering the number f_n of monomials in the constant term $g_n(0; a_0, a_1, \dots, a_{n-1})$, counted with multiplicity. The sequence f_n starts $1, 1, 3, 13, 75, \dots$. Using, say, the integral relation, it is easy to show that the numbers f_n satisfy the recurrence

$$f_n = \sum_{i=1}^{n} \binom{n}{i} f_{n-i} \qquad (n \ge 1),$$

and have exponential generating function

$$\sum_{n=0}^{\infty} \frac{f_n t^n}{n!} = \frac{1}{2-e^t}.$$

From this, we see (from [22], say) that f_n is the number of *preferential arrangements*, or ordered partitions of the set with n elements. These observations suggest that there is an interpretation of the constant term $g_n(0; a_0, a_1, \ldots, a_{n-1})$ in terms of objects related to ordered partitions.

From an ordered partition $B_1, B_2, \ldots, B_m$ of a set $\{x_1, x_2, \ldots, x_n\}$ with n elements, one can associate a *ranking* $\rho : \{x_1, x_2, \ldots, x_n\} \to \{0, 1, 2, \ldots, n-1\}$ as follows: if an element x_i is in the jth block B_j, then defined

$$\rho(x_i) = \sum_{l<j} |B_l|.$$

In particular, $\rho(x_i) = 0$ whenever x_i is in the first block B_1. We define the order $|\rho|$ to be the size of the image of ρ, which is also the number of blocks in the ordered partition associated with ρ. For example, from the ordered partition $\{2,4\}$, $\{5\}$, $\{1,3\}$ of $\{1,2,3,4,5\}$, one obtains the ranking defined by $\rho(2) = \rho(4) = 0$, $\rho(5) = 2$, and $\rho(1) = \rho(3) = 3$. Rankings are characterized by the property: for every element x_i, there are exactly $\rho(x_i)$ elements x_j such that $\rho(x_j) < \rho(x_i)$.

(4.2) Theorem.

$$g_n(0; a_0, a_1, \ldots, a_{n-1}) = \sum_{\rho} (-1)^{|\rho|} \prod_{j=0}^{n-1} a_{\rho(j)},$$

where the sum ranges over all rankings ρ of $\{1, 2, \ldots, n\}$.

Proof. The theorem holds when $n = 0$. When $n > 0$, the constant terms of Gončarov polynomials satisfy the recursion

$$g_n(0; a_0, a_1, \ldots, a_{n-1}) = -\sum_{i=0}^{n-1} \binom{n}{i} a_i^{n-i} g_i(0; a_0, a_1, \ldots, a_{i-1})$$

obtained by setting $x = 0$ in the linear recursion. We shall show that the sum on the right hand side of the equation in Theorem 4.2 satisfies the same recursion. Let $\mathcal{R}[n]$ be the set of all rankings on $\{1, 2, \ldots, n\}$. Divide $\mathcal{R}[n]$ into groups $\mathcal{R}[n, i]$ according to the maximum value i taken by the ranking, so that

$$\mathcal{R}[n, i] = \{\rho : \max\{\rho(1), \rho(2), \ldots, \rho(n)\} = i\}.$$

If ρ is in $\mathcal{R}[n, i]$, then the inverse image $\rho^{-1}(i)$ must contain exactly $n-i$ numbers. Thus, there is a bijection between rankings ρ in $\mathcal{R}_i$ and pairs consisting of an i-element subset of $\{1, 2, \ldots, n\}$ (the complement of $\rho^{-1}(i)$) and a ranking ρ' (having order $|\rho| - 1$) on that i-element subset obtained by restricting ρ. Hence,

$$\sum_{\rho \in \mathcal{R}[n]} (-1)^{|\rho|} \prod_{j=1}^{n} a_{\rho(j)} = -\sum_{i=0}^{n-1} a_i^{n-i} \binom{n}{i} \left(\sum_{\rho \in \mathcal{R}[n,i]} (-1)^{|\rho'|} \prod_{j=0}^{i-1} a_{\rho(j)} \right).$$

Since both sides of the equation in Theorem 4.2 satisfy the same recursion and initial condition, they are equal by induction.

By Theorem 4.2 and the shift formula, we obtain the following formula for Gončarov polynomials.

$$\begin{aligned} g_n(x; a_0, a_1, \ldots, a_{n-1}) &= g_n(0; a_0 - x, \ldots, a_{n-1} - x) \\ &= \sum_{\rho} (-1)^{|\rho|} \prod_{i=1}^{n} (a_{\rho(i)} - x). \end{aligned}$$

Abel polynomials are intimately related to the enumeration of trees. In particular, the constant term $(-1)^n g_n(0; 1, 2, \ldots, n)$ is the number of labeled trees on $n+1$ vertices. An interpretation for $(-1)^n g_n(0; u_1, u_2, \ldots, u_n)$ in terms of labeled trees can be obtained by extending the Foata-Riordan bijection [5] between acyclic and ordinary parking functions. There may be other interpretations.

5 A decomposition for sequences of positive integers

In this section, we describe the combinatorial decomposition underlying the theory of parking functions. For us, this decomposition was motivated by the linear recursion for Gončarov polynomials. After discovering this decomposition, we found out from Julian Gilbey that the special case of this decomposition for ordinary parking functions was already used by Konheim and Weiss in the *first* paper [11] on the subject.

(5.1) Theorem. Let $(u_1, u_2, \ldots, u_n)$ be a sequence of non-decreasing positive integers and let x be a positive integer. Then, every sequence $(x_1, x_2, \ldots, x_n)$ of length n with terms x_i integers from the discrete interval $[1, x]$ can be decomposed into a pair of subsequences

$$(x_{i_1}, x_{i_2}, \ldots, x_{i_m}),\ (x_{j_1}, x_{j_2}, \ldots, x_{j_{n-m}})$$

such that the first subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ is a **u**-parking function of length m, and all the terms in the second subsequence, the complementary subsequence of length $n-m$ obtained by removing the terms in the first subsequence from $(x_1, x_2, \ldots, x_n)$ are in the discrete interval $[u_{m+1}+1, x]$. This decomposition provides a bijection between all sequences of length n with terms in $[1, x]$ and all pairs of complementary subsequences of total length n, where the first is a **u**-parking function of length m, and the second is a sequence of length $n-m$ taking values in $[u_{m+1}+1, x]$.

Proof. Consider the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ of order statistics. Let m be the maximum index such that

$$x_{(i)} \leq u_i \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{5.1}$$

Then, the subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ from which the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(m)})$ was obtained by rearrangement is a **u**-parking function of length m. Furthermore, m is the maximum index satisfying condition (5.1) if and only if

$$x_{(n)} \geq x_{(n-1)} \geq \ldots \geq x_{(m+1)} > u_{m+1},$$

or, equivalently, the *complementary* subsequence $(x_{j_1}, x_{j_2}, \ldots, x_{j_{n-m}})$, obtained by deleting the subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ from the original sequence, takes values in the interval $[u_{m+1}+1, x]$. Since the maximum index m and hence, the set $\{i_1, i_2, \ldots, i_m\}$ are uniquely determined by the sequence $(x_1, x_2, \ldots, x_n)$, and any pair of subsequences satisfying the conditions in the theorem can be reassembled into a sequence in $[1, x]^n$, this decomposition yields a bijection.

It will be useful to state the decomposition more explicitly.

(5.2) Corollary. There is a bijection between the set $[1, x]^n$ of all length-n integer sequences with terms in the discrete interval $[1, x]$ and the disjoint union of Cartesian products

$$\bigcup_{\{i_1, i_2, \ldots, i_m\}} \mathrm{Park}(i_1, i_2, \ldots, i_m) \times [u_{m+1} + 1, x]^{n-m},$$

where $\mathrm{Park}(i_1, i_2, \ldots, i_m)$ is the set of length-m $\mathbf{u}$-parking functions indexed by the set $\{i_1, i_2, \ldots, i_m\}$ and $[u_{m+1} + 1, x]^{n-m}$ is the set of length-$(n - m)$ integer sequences with terms in $[u_{m+1} + 1, x]$ indexed by the complement of $\{i_1, i_2, \ldots, i_m\}$.

Let $P_n(\mathbf{u})$ be the number of $\mathbf{u}$-parking functions of length n. Since $P_n(\mathbf{u})$ depends only on the first n terms of $\mathbf{u}$, we will often write $P_n(u_1, u_2, \ldots, u_n)$ instead of $P_n(\mathbf{u})$ to make explicit the parameters on which $P_n(\mathbf{u})$ is dependent. The decomposition in Theorem 5.1 yields the following identity.

(5.3) Corollary. Let x be an integer greater than or equal to u_n. Then

$$x^n = \sum_{m=0}^{n} \binom{n}{m} (x - u_{m+1})^{n-m} P_m(u_1, u_2, \ldots, u_m).$$

Comparing the recursion in Corollary 5.3 with the linear recursion for Gončarov polynomials given in Section 3, we obtain

$$P_n(u_1, u_2, \ldots, u_n) = g_n(x; x - u_1, x - u_2, \ldots, x - u_n).$$

By the shift formula, the Gončarov polynomial equals

$$g_n(0; -u_1, -u_2, \ldots, -u_n).$$

Since the Gončarov polynomial $g_n(x; a_0, a_1, \ldots, a_{n-1})$ is a homogeneous polynomial of total degree n in $x, a_0, a_1, \ldots, a_{n-1}$, we have

$$g_n(0; -u_1, -u_2, \ldots, -u_n) = (-1)^n g_n(0; u_1, u_2, \ldots, u_n).$$

All three forms of the formula for $P_n(\mathbf{u})$ are useful.

(5.4) Theorem.

$$\begin{aligned} P_n(u_1, u_2, \ldots, u_n) &= g_n(x; x - u_1, x - u_2, \ldots, x - u_n) \\ &= g_n(0; -u_1, -u_2, \ldots, -u_n) \\ &= (-1)^n g_n(0; u_1, u_2, \ldots, u_n). \end{aligned}$$

When $u_i = a + (i - 1)b$, we obtain the following special case.

(5.5) Corollary.

$$P_n(a, a+b, a+2b, \ldots, a+(n-1)b) = a(a+nb)^{n-1}.$$

In particular, we have re-derived the classic formula for ordinary parking functions:

$$P_n(1, 2, 3, \ldots, n) = (n+1)^{n-1}.$$

From the fact that Gončarov polynomials are homogeneous, we obtain another consequence of Theorem 5.4.

(5.6) Corollary.

$$P_n(bu_1, bu_2, \ldots, bu_n) = b^n P(u_1, u_2, \ldots, u_n).$$

Any reasonable formula for Gončarov polynomials yields a reasonable formula for parking functions. We give an example which is motivated by results in [18] and [29]. Consider the sequence $a_0, a_1, \ldots, a_{n-m}, c+(n-m+1)d, c+(n-m+2)d, \ldots, c+(n-1)d$ of n nodes. This sequence can be obtained by perturbing the arithmetic progression $c, c+d, \ldots, c+(n-1)d$ by $b_i = a_i - (c+id)$ for $i = 0, 1, \ldots, n-m$. Using the perturbation formula, we have

$$\begin{aligned} & g_n(x; a_0, a_1, \ldots, a_{n-m}, c+(n-m+1)d, c+(n-m+2)d, \ldots, c+(n-1)d) \\ = \; & (x-c)(x-c-nd)^{n-1} \\ & - \sum_{i=0}^{n-m} \binom{n}{i} (a_i - c - id)(a_i - c - nd)^{n-i-1} g_i(x; a_0, a_1, \ldots, a_{i-1}). \end{aligned}$$

Using this and Theorem 5.4, we obtain the following result.

(5.7) Corollary. If $c+(n-m+1)d \geq a_{n-m}$, then

$$\begin{aligned} & P_n(u_1, u_2, \ldots, u_{n-m+1}, c+(n-m+1)d, c+(n-m+2)d, \ldots, c+(n-1)d) \\ = \; & c(c+nd)^{n-1} - \sum_{i=0}^{n-m} \binom{n}{i} (c+id-u_{i+1})(c+id-u_{i+1})^{n-i-1} P_i(u_1, u_2, \ldots, u_i). \end{aligned}$$

Note that c need not be positive and some of the terms in the sum may be negative in Corollary 5.7.

By the determinantal formula for Gončarov polynomials in Section 3, we have the discrete analog of a result for real-valued parking functions usually attributed to Steck [27].

(5.8) Corollary. The number $P_n(u_1, u_2, \ldots, u_n)$ of $\mathbf{u}$-parking functions of length n equals $n! \det \mathcal{D}$, where $\mathcal{D}$ is the matrix with ijth entry equal to

$$\frac{u_i^{j-i+1}}{(j-i+1)!}$$

if $j-i+1 \geq 0$ and 0 otherwise.

Note that Lemma 4.1 and Jacobi's formula for the inverse of a matrix yields another determinantal formula for $P_n(\mathbf{u})$. However, this formula can easily be derived from the formula in Corollary 5.8 by row and column operations.

6 Sum enumerators of parking functions

For ordinary parking functions, two interesting and closely related statistics are the sum S_n of all its terms and the reversed sum or total displacement D_n, defined by $D_n = \binom{n+1}{2} - S_n$. See, for example, [3, 4, 10]). One way to study sums and reversed sums is through their enumerators. The *sum enumerator* $S_n(q; \mathbf{u})$ for the set of $\mathbf{u}$-parking functions is the polynomial in q defined by

$$S_n(q; \mathbf{u}) = \sum_{(a_1, a_2, \ldots, a_n)} q^{a_1 + a_2 + \ldots + a_n - n}$$

where the sum ranges over all $\mathbf{u}$-parking functions $(a_1, a_2, \ldots, a_n)$. The sum enumerator may be regarded as a "q-analogue" of $P_n(\mathbf{u})$. The sum enumerator for a subset $\mathcal{S}$ of $[1, x]^n$ is defined analogously by summing over all sequences in $\mathcal{S}$. Sum enumerators are *multiplicative* in the following sense. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are two sets of subsequences on disjoint index sets. Then the sum enumerator of the Cartesian product $\mathcal{S}_1 \times \mathcal{S}_2$ consisting of all sequences formed by combining a subsequence from $\mathcal{S}_1$ and a subsequence from $\mathcal{S}_1$ is the product of the sum enumerators of $\mathcal{S}_1$ and $\mathcal{S}_2$.

For a $\mathbf{u}$-parking function, the maximum value of the ith order statistic $x_{(i)}$ is at most u_i and hence, $u_i - x_{(i)} \geq 0$. The *reversed sum enumerator* $R_n(q; \mathbf{u})$ is defined by

$$R_n(q; \mathbf{u}) = \sum_{(a_1, a_2, \ldots, a_n)} q^{u_1 + u_2 + \ldots + u_n - (a_1 + a_2 + \ldots + a_n)},$$

where the sum ranges over all $\mathbf{u}$-parking functions $(a_1, a_2, \ldots, a_n)$. Equivalently,

$$R_n(q; \mathbf{u}) = q^{u_1 + u_2 + \ldots + u_n - n} S_n(1/q; \mathbf{u}). \tag{6.1}$$

The reversed sum enumerator is a polynomial in the variable q of degree $u_1 + u_2 + \ldots + u_n - n$.

(6.1) Lemma.

$$(1 + q + q^2 + \ldots + q^{x-1})^n = \sum_{m=0}^{n} \binom{n}{m} (q^{u_{m+1}} + q^{u_{m+1}+1} + \ldots + q^{x-1})^{n-m} S_m(q; \mathbf{u}).$$

Proof. Since sum enumerators are multiplicative, the sum enumerator of $[1, x]^n$ is

$$(1 + q + q^2 + \ldots + q^{x-1})^n.$$

For the same reason, the sum enumerator of functions which are decomposed into a $\mathbf{u}$-parking function of length m and a sequence in $[u_{m+1} + 1, x]^{n-m}$ is

$$(q^{u_{m+1}} + q^{u_{m+1}+1} + \ldots + q^{x-1})^{n-m} S_m(q, \mathbf{u}).$$

The recursion now follows.

Comparing this recursion with the linear recursion in Corollary 5.3, we obtain the following theorem.

(6.2) Theorem.

$$S_n(q;\mathbf{u}) = P_n(1+q+\ldots+q^{u_1-1}, 1+q+\ldots+q^{u_2-1}, \ldots, 1+q+\ldots+q^{u_n-1}).$$

Theorem 6.2 can also be obtain directly using a decomposition for the set of **u**-parking functions due to Pitman and Stanley [19]. Given a **u**-parking function $(\beta_1, \beta_2, \ldots, \beta_n)$, we can associate an ordinary parking function $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ by setting $\alpha_i = r$ if β_i is in the discrete interval $[u_{r-1}+1, u_r]$. Conversely, given an ordinary parking function $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, there are

$$(u_{\alpha_1} - u_{\alpha_1 - 1})(u_{\alpha_2} - u_{\alpha_2 - 1}) \cdots (u_{\alpha_n} - u_{\alpha_n - 1})$$

u-parking functions associated with it. These are obtained by choosing a number from each discrete interval $[u_{\alpha_j - 1}+1, u_{\alpha_j}]$. Here, we use the convention that $u_0 = 0$. Hence,

$$P_n(u_1, u_2, \ldots, u_n) = \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} (u_{\alpha_1} - u_{\alpha_1 - 1})(u_{\alpha_2} - u_{\alpha_2 - 1}) \cdots (u_{\alpha_n} - u_{\alpha_n - 1}),$$

where the sum ranges over all ordinary parking functions of length n. Replacing the number of elements $u_{\alpha_j} - u_{\alpha_j - 1}$ in the discrete interval $[u_{\alpha_{j-1}}+1, u_{\alpha_j}]$ by its sum enumerator and using the fact that sum enumerators are multiplicative, we obtain Theorem 6.2.

Using Theorem 5.3, Theorem 6.1, and the shift formula, we can express sum enumerators in terms of Gončarov polynomials:

$$S_n(q;\mathbf{u}) = g_n\left(\frac{1}{1-q}; \frac{q^{u_1}}{1-q}, \frac{q^{u_2}}{1-q}, \ldots, \frac{q^{u_n}}{1-q}\right).$$

By homogeneity of Gončarov polynomials,

$$(1-q)^n S_n(q;\mathbf{u}) = g_n(1; q^{u_1}, q^{u_2}, \ldots, q^{u_n}).$$

Hence, sum enumerators satisfy the simpler linear recursion

$$1 = \sum_{m=0}^{n} \binom{n}{m} q^{u_{m+1}(n-m)} (1-q)^m S_m(q;\mathbf{u}). \tag{6.2}$$

They also satisfy the following Appell relation

$$\exp(t) = \sum_{n=0}^{\infty} (1-q)^n S_n(q;\mathbf{u}) \exp(q^{u_{n+1}} t) \frac{t^n}{n!}.$$

In the case of ordinary parking functions, $u_i = i$ and we have

$$(1-q)^n S_n(q; 1, 2, \ldots, n) = g_n(1; q, q^2, \ldots, q^n).$$

For example,

$$\begin{aligned}(1-q)^2 S_2(q;1,2) &= 1 - 3q^2 + 2q^3 \\ (1-q)^3 S_3(q;1,2,3) &= 1 - 4q^3 - 3q^4 + 12q^5 - 6q^6.\end{aligned}$$

One does not expect simple generating functions for sum enumerators in general. However, when u_i is an arithmetic progression, we can group terms together to obtain a recursion which yields a simple exponential generating function. We shall show how this can be done for reversed sum enumerators.

Substituting $1/q$ for q in equation (6.2) and using equation (6.1), we obtain

$$q^{u_1+u_2+\ldots+u_n} = \sum_{m=0}^{n} \binom{n}{m} (q-1)^m R_m(q;\mathbf{u}) q^{-(n-m)u_{m+1}+u_{m+1}+u_{m+2}+\ldots+u_n}.$$

If the exponent

$$-(n-m)u_{m+1} + u_{m+1} + u_{m+2} + \ldots + u_n$$

is a function $\tau(n-m)$ depending only on $n-m$, then we have

$$q^{u_1+u_2+\ldots+u_n} = \sum_{m=0}^{n} \binom{n}{m} (q-1)^m R_m(q;\mathbf{u}) q^{\tau(n-m)}.$$

Multiplying this by $t^n/n!$, summing over all non-negative integers n, and manipulating the resulting formal power series, we obtain

$$\sum_{n=0}^{\infty} (q-1)^n R_n(q;\mathbf{u}) \frac{t^n}{n!} = \frac{\displaystyle\sum_{n=0}^{\infty} q^{u_1+u_2+\ldots+u_n} \frac{t^n}{n!}}{\displaystyle\sum_{n=0}^{\infty} q^{\tau(1)+\tau(2)+\ldots+\tau(n)} \frac{t^n}{n!}}.$$

The condition that the exponent is a function $\tau(n-m)$ of $n-m$ is in fact very strong. Consider the case $n-m=2$. Then the condition implies that for all m, $-2u_{m+1}+u_{m+1}+u_{m+2}$ equals a number $\tau(2)$ independently of m, that is, $u_{m+2}-u_{m+1}$ is a constant b for all m. This in turn implies that $\mathbf{u}$ is an arithmetic progression with common difference b. Conversely, if $u_i = a+(i-1)b$, then

$$\sum_{j=1}^{n} u_j = an + b\binom{n}{2}$$

and

$$\sum_{j=1}^{n} \tau(j) = b\binom{n}{2}.$$

We have thus proved the following theorem, which is best possible.

(6.3) Theorem. Let $\mathbf{u}$ be the arithmetic progression $(a, a+b, a+2b, \ldots)$. Then

$$\sum_{n=0}^{\infty} (q-1)^n R_n(q;\mathbf{u}) \frac{t^n}{n!} = \frac{\displaystyle\sum_{n=0}^{\infty} q^{an+b\binom{n}{2}} \frac{t^n}{n!}}{\displaystyle\sum_{n=0}^{\infty} q^{b\binom{n}{2}} \frac{t^n}{n!}}.$$

The reversed sum enumerator $R_n(q;\mathbf{u})$ also enumerates the number of inversions for certain sequences of rooted b-forests, see [30]. As a special case of Theorem 6.3, for $a=b=1$ we obtain the following result of Stanley ([25, 26]):

$$\sum_{n=0}^{\infty}(q-1)^n I_n(q)\frac{t^n}{n!} = \frac{\displaystyle\sum_{n=0}^{\infty} q^{\binom{n+1}{2}}\frac{t^n}{n!}}{\displaystyle\sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{t^n}{n!}},$$

where $I_n(q)$ is the inversion enumerator for labeled trees [16, 12, 25, 26].

7 Factorial moments of sums of parking functions

In this section, we shall use some elementary probability theory. A subset $\mathcal{S}$ of the set $[1,x]^n$ of length-n sequences with terms in the discrete interval $[1,x]$ can be made into a discrete probability space by assigning a probability of $1/|\mathcal{S}|$ to each sequence in $\mathcal{S}$. Given a subset $\mathcal{S}$ of length-n sequences, we define the random variable S_n on $\mathcal{S}$ to be the sum $x_1 + x_2 + \ldots + x_n$ of a random sequence in $\mathcal{S}$. The *expected sum* of a random sequence from $\mathcal{S}$ is the expectation $E[S_n]$. Let $(x)_k$ be the *k-falling factorial* $x(x-1)\cdots(x-k+1)$. The *kth (falling) factorial moment of the sum of a random sequence* in $\mathcal{S}$ is the expectation $E[(S_n)_k]$. Explicitly, $E[(S_n)_k]$ equals

$$\frac{1}{|\mathcal{S}|}\sum_{(x_1,x_2,\ldots,x_n)\in\mathcal{S}}(x_1+x_2+\ldots+x_n)_k.$$

In particular, let $E_k(n;\mathbf{u})$ be the *kth falling factorial moment of the sum of a random $\mathbf{u}$-parking function*, that is,

$$E_k(n;\mathbf{u}) = \frac{1}{P_n(\mathbf{u})}\sum_{(x_1,x_2,\ldots,x_n)}(x_1+x_2+\ldots+x_n)_k,$$

where the sum ranges over all $\mathbf{u}$-parking functions of length n.

Let α and β be integers with $\alpha \le \beta$ and let $U_i(\alpha,\beta)$ be the sum of a random sequence chosen with uniform distribution from the space $[\alpha,\beta]^i$ of all length-i sequences with terms in the discrete interval $[\alpha,\beta]$. A random sequence in $[\alpha,\beta]^i$ can also be thought of as a length-i random sequence obtained by choosing each term independently with uniform distribution from $[\alpha,\beta]$. The factorial moments of $U_i(\alpha,\beta)$ are known and they can be expressed in a compact form by exponential generating functions (see, for example, [8]). Indeed, if

$$\mathcal{U}_i(t;\alpha,\beta) = \sum_{k=0}^{\infty} E[(U_i(\alpha,\beta))_k]\frac{t^k}{k!},$$

then

$$\mathcal{U}_i(t;\alpha,\beta) = \left(\frac{(1+t)^{\beta+1}-(1+t)^{\alpha}}{(\beta-\alpha+1)t}\right)^i.$$

The combinatorial decomposition in Section 5 can also be used to obtain linear recursions for higher factorial moments of sums of random parking functions. Let $\mathbf{a}$ be the sequence defined by $a_j = x - u_{j+1}$ and let

$$e_i^{(k)}(x;a_0,\ldots,a_{n-1}) = E[(S_i)_k],$$

the k-factorial moment of the sum of a random $\mathbf{u}$-parking function as a function of $x, a_0, a_1, \ldots, a_{i-1}$.

(7.1) Theorem. Let k be a positive integer. Then the factorial moments of the sum of a random $\mathbf{u}$-parking function of length n satisfies the following linear recursion:

$$E[(U_n(1,x))_k] = \sum_{m=0}^{n} \binom{n}{m} \frac{a_m^{n-m} g_m(x;\mathbf{a})}{x^n} \left(\sum_{j=0}^{k} \binom{k}{j} e_m^{(j)}(x;\mathbf{a}) E[(U_{n-m}(u_{m+1}+1,x))_{k-j}] \right).$$

Proof. Consider the event that the maximum subsequence forming a $\mathbf{u}$-parking function is indexed by $\{i_1, i_2, \ldots, i_m\}$. Because the length-$m$ $\mathbf{u}$-parking function and the length-$(n-m)$ sequence from $[u_{m+1}+1, x]^{n-m}$ are chosen independently and an analogue of the binomial theorem holds for falling factorials, the expected value of $(U_n(1,x))_k$ conditioned on this event is

$$\sum_{j=0}^{k} \binom{k}{j} e_m^{(j)}(x;\mathbf{a}) E[(U_{n-m}(u_{m+1}+1,x))_{k-j}].$$

Summing over the conditional expectations, we obtain the linear recursion.

The *factorial moment generating function* $\mathcal{S}_i(t;\mathbf{a})$ for $\mathbf{u}$-parking functions of length i is defined by the following formula:

$$\mathcal{S}_i(t;\mathbf{a}) = \sum_{k=0}^{\infty} e_i^{(k)}(x;\mathbf{a}) \frac{t^k}{k!}.$$

Restating Theorem 7.1 in terms of exponential generating functions, we obtain the following linear recursion for the moment generating functions $\mathcal{S}_i(t;\mathbf{a})$:

$$x^n \mathcal{U}_n(t;1,x) = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x;\mathbf{a}) \mathcal{S}_i(t;\mathbf{a}) \mathcal{U}_{n-i}(t;u_{i+1}+1,x), \tag{7.1}$$

From this recursion, we can obtain a simple Appell relation for $\mathcal{S}_i(t;\mathbf{a})$. First observe that

$$\begin{aligned} \sum_{n=0}^{\infty} x^n \mathcal{U}_n(t;1,x) \frac{q^n}{n!} &= \sum_{n=0}^{\infty} \left[\frac{(1+t)^{x+1} - (1+t)}{t} \right]^n \frac{q^n}{n!} \\ &= \exp(\frac{q}{t}((1+t)^{x+1} - (1+t))) \end{aligned}$$

and

$$\sum_{n=i}^{\infty} a_i^{n-i} \mathcal{U}_{n-i}(t;u_{i+1}+1,x) \frac{q^{n-i}}{(n-i)!} = \exp(\frac{q}{t}((1+t)^{x+1} - (1+t)^{1+x-a_i})).$$

Hence, multiplying equation (7.1) by $q^n/n!$, summing over all non-negative integers n, and dividing both sides by $\exp(q(i+t)^{x+1}/t)$, we obtain

$$\exp(-\frac{q}{t}(1+t)) = \sum_{i=0}^{\infty} g_i \mathcal{S}_i(t) \exp(-\frac{q}{t}(1+t)^{1+x-a_i}) \frac{q^i}{i!}.$$

Changing variables from q to qt, we obtain the following Appell relation.

(7.2) Theorem.

$$\exp(-q(1+t)) = \sum_{i=0}^{\infty} g_i(x;\mathbf{a})\mathcal{S}_i(t;\mathbf{a})\exp(-q(1+t)^{1+x-a_i})\frac{t^i q^i}{i!}.$$

The left hand side of the Appell relation does not depend on x (which is not surprising, as the linear recursion from which it is derived holds for all sufficiently large integer x). Hence, simpler Appell relations can be obtained by setting x to be 0 or any convenient constant or variable.

It requires much more work to "solve" the linear recursion and obtain explicit formulas for the moments. This will be done in [14, 15].

Acknowledgment. The authors thank the referee for many valuable comments. In particular, the interpretation of **u**-parking functions in Section 1 was suggested by the referee.

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