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Derangements on a Ferrers board

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We study the derangement number on a Ferrers board $B = (n \times n) - \lambda$ with respect to an initial permutation M, that is, the number of permutations on B that share no common points with M. We prove that the derangement number is independent of M if and only if λ is of rectangular shape. We characterize the initial permutations that give the minimal and maximal derangement numbers for a general Ferrers board, and present enumerative results when λ is a rectangle.

Keywords: Derangement; Ferrers board.

Mathematics Subject Classification 2010: 05A05, 05A18

1. Introduction

The classic derangement question can be formulated on an $n \times n$ board, where n is a positive integer. A *permutation* on such an $n \times n$ board is a placement of points on the board such that no two points are in the same row or column, and every row and column contains one point. This is equivalent to placing n nonattacking rooks onto the board, as a rook can only move along rows and columns, so if one has two rooks in the same row or column, they must necessarily be attacking one another. Given an initial permutation M, a *derangement* is a permutation on the $n \times n$ board which shares no common squares with M. It is well-known that the derangement number is independent of the initial permutation M and only dependent on n. This is reflected in the enumeration of the derangements

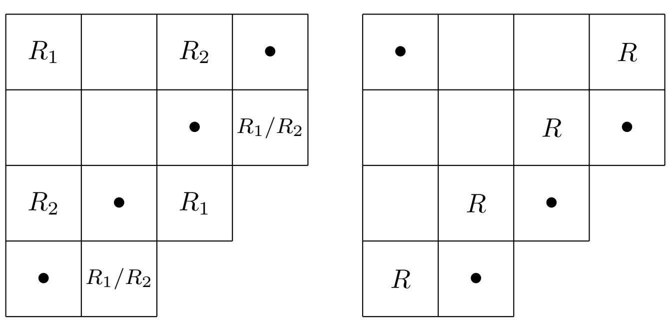
$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}, \tag{1.1}$$

a classical result that can be dated back to de Montmort [4] and Bernoulli in the early 18th century.

This paper investigates the derangement question on a more general class of boards known as *Ferrers Boards*. Given an $n \times n$ board as above, let $r_1, r_2, \dots, r_n$ denote the rows of the board, where r_1 is the bottom row. Further, let $|r_i|$ denote the number of squares in row r_i. Then a Ferrers board B has the property that $1 \le |r_1| \le |r_2| \le \cdots \le |r_n|$. In this paper, we always assume $|r_n| = n$. Thus, a Ferrers board is an $n \times n$ board with a section λ missing in the lower right corner, and so we may write $B = (n \times n) - \lambda$, where λ is an integer partition. Permutations and derangements on a Ferrers board B are defined as in the classic case: A permutation is a placement of n nonattacking rooks on B. Given such a permutation M, a *derangement from M on B* is a permutation on the board that shares no common squares with M.

Denote by $d_B(M)$ the number of derangements from M on a board B. Unlike the classical case, the derangement number $d_B(M)$ for a general Ferrers board B depends not only on the shape of B, (that is, on n and λ), but also on the initial permutation M, as shown in Fig. 1, where the initial permutations M and M' are represented by block dots. In the left figure, there are two derangements, represented by R_1 and R_2 respectively. In the right figure, there is only one derangement, represented by R.

In this paper, we will characterize the extremal permutations for which the derangement number reaches a maximum and minimum. We prove that the two extreme cases happen when the permutations correspond to the noncrossing and nonnesting matchings with given left and right endpoints. This is the content of Sec. 2. As a corollary, the derangement number is independent of the initial permutation M when the missing section λ is of a rectangular shape. The next two sections are devoted to the enumeration of the derangement numbers for the rectangular λ. In Sec. 3, we give an explicit formula and express it in terms of an analog of the classical derangement numbers. In Sec. 4, we establish a recurrence relation, and derive a differential equation satisfied by the generating function of the derangement numbers for the rectangular λ.



(a) $d_B(M) = 2$ (b) $d_B(M') = 1$

Fig. 1. An example showing that $d_B(M)$ is not constant for all M.

2. Bounds for the Derangement Number on a Ferrers Board

Let $\mathcal{S}(B)$ be the set of all permutations on a Ferrers board B (with a given λ). We label the rows of B ascendingly as $r_1, r_2, \ldots, r_n$ and the columns from left to right as $c_1, \ldots, c_n$. Denote the square in the ith row and the jth column as (r_i, c_j). Note that $\mathcal{S}(B) = \emptyset$ unless the board B contains $\{(r_d, c_d) \mid 1 \leq d \leq n\}$, the diagonal from the lower-left corner to the upper-right corner. We subsequently assume that B does contain that diagonal.

To compare the derangement numbers associated with different initial permutations, we introduce two operations on the permutations on B.

2.1. *Left–right and right–left movements*

Definition 1. Let $(r_m, c_i), (r_k, c_j) \in B$ where r_m is above r_k (*i.e.* $m > k$) and c_i is to the left of c_j (*i.e.* $i < j$). Then the *left-right movement* is a transformation LR that replaces the squares (r_m, c_i), (r_k, c_j) with $(r_m, c_j), (r_k, c_i)$. That is,

$$\mathrm{LR}(\{(r_m, c_i), (r_k, c_j)\}) = \{(r_m, c_j), (r_k, c_i)\}. \tag{2.1}$$

See Fig. 2 for an illustration. Note that the rows and columns are not necessarily adjacent.

We can then define the transformation RL, the *right–left movement*, as the inverse of LR, with the additional caveat that if the square (r_k, c_j) is not in B, then RL is undefined for the pair $\{(r_m, c_j), (r_k, c_i)\}$.

We will also make the simple observation that given two squares in any permutation M, one may be able to perform either a left–right or a right–left movement on the squares, but never both simultaneously.

2.2. *Noncrossing and nonnesting permutations*

Given a Ferrers board B, we define two particular permutations using the notation of the previous section: the noncrossing permutation, denoted NC_B, and the nonnesting permutation, denoted NN_B.

Definition 2. The noncrossing permutation will be defined algorithmically:

(1) Let (r_1, c_i) be the rightmost square of row r_1. Then $(r_1, c_i) \in NC_B$.

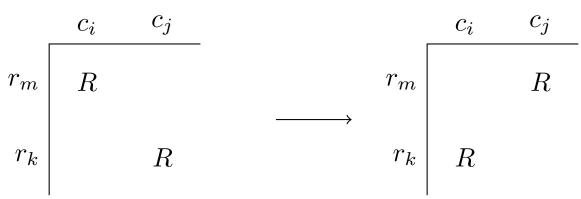


Fig. 2. An LR movement.

(2) For all subsequent rows r_m, let c_p be the rightmost column with $(r_m, c_p) \in B$ that does not contain a square in NC_B yet. Then let $(r_m, c_p) \in NC_B$.

Definition 3. The nonnesting permutation is defined as $NN_B = \{(r_i, c_i) \mid 1 \leq i \leq n\}$. This corresponds to the diagonal from the lower left to the upper right on B.

Referring to the example in Fig. 1, the permutation M in the left figure is NN_B, and the one in the right figure is NC_B.

These definitions come from the well-known noncrossing and nonnesting matchings. Permutations on the set of Ferrers boards of the shapes $\{(n \times n) - \lambda : \lambda \text{ an integer partition}\}$ are in one-to-one correspondence with the set of complete matchings of $\{1, 2, \ldots, 2n\}$, while permutations on a fixed board B correspond to matchings with given sets of minimal elements and maximal elements, as described in [3; 10, Sec. 4.2].

To see the connection, given the board B, starting from the lower-left corner, travel along its southeast boundary until the upper-right corner is reached, and label the steps from 1 to $2n$. Let S be the set of labels on the horizontal steps, and T be the set of labels on the vertical steps. Then a permutation on B can be represented as a matching between S and T, where (r_i, c_j) is in the permutation if and only if the label of row r_i is matched with the label of column c_j. Then NC_B and NN_B correspond exactly to the unique noncrossing and nonnesting matchings from S to T. See Figs. 3 and 4 for illustrations.

Noncrossing and nonnesting matchings are special cases of noncrossing/nonnesting set partitions, which are among the most fundamental structures in combinatorics, and play important roles in other disciplines, for example, in free probability theory [7], representation of Coxeter groups [1], and RNA molecule structures [8], to list a few.

2.3. *Connecting* LR, RL, NC_B *and* NN_B

We establish two lemmas that connect the previous notions together.

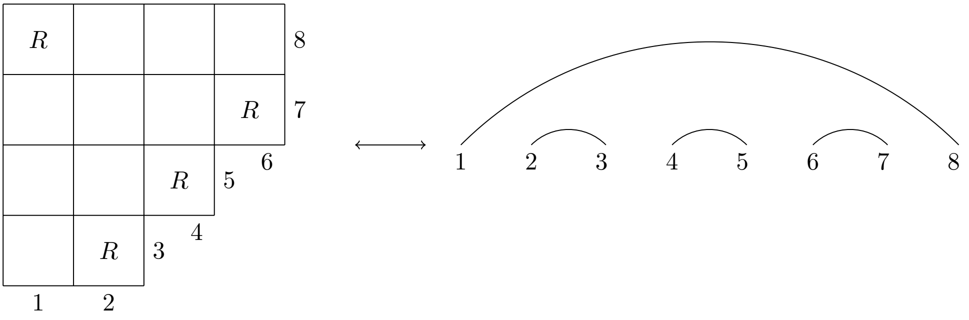


Fig. 3. Equivalence between noncrossing permutation and noncrossing matching.

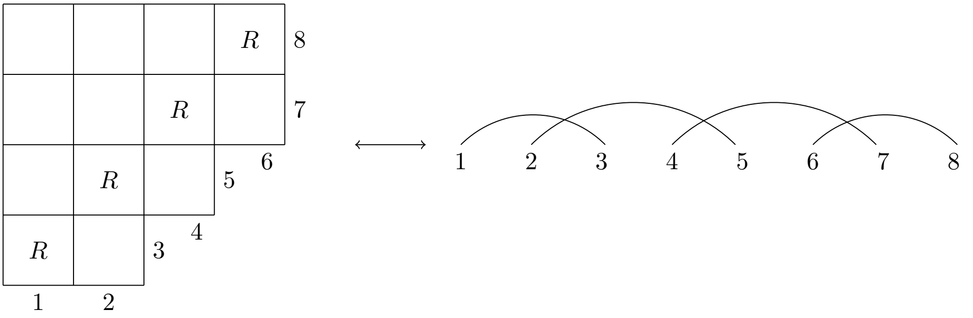


Fig. 4. Equivalence between nonnesting permutation and nonnesting matching.

Lemma 2.1. *The noncrossing permutation is a permutation such that initially only left–right movements can be performed on its elements. Similarly, the nonnesting permutation is a permutation such that initially only right–left movements can be performed on its elements.*

Proof. We begin with NC_B. Assume $(r_m, c_i), (r_k, c_j) \in NC_B$ with $m > k$. If $i < j$, then clearly an LR movement can be performed. On the other hand, if $i > j$, we show that the square $(r_k, c_i) \notin B$, hence it is not possible to perform a RL movement. Prove by contradiction. Assume $(r_k, c_i) \in B$. Then, since $(r_k, c_j) \in NC_B$, we have that (r_k, c_t) cannot be the rightmost square in row r_k not already in NC_B for $j < t \leq i$. But we have that every such $(r_k, c_t) \in B$, hence for every column c_t, there must be a row r_h (with $h < m$) such that $(r_h, c_t) \in NC_B$. In particular, there is a row r_l, such that $(r_l, c_i) \in NC_B$, and $l < k < m$, contradiction to the assumption that $(r_m, c_i) \in NC_B$.

For the NN_B permutation, let $(r_m, c_m), (r_n, c_n) \in NN_B$, with $m < n$. Then, by comparison to Definition 1, the conditions for a LR movement are nowhere satisfied, but the conditions for a RL movement are satisfied if $(r_m, c_n) \in B$. Thus, only RL movements can be initially performed on the NN_B permutation. $\square$

Lemma 2.2. *Let M be any permutation on B. Then, M can be attained by a sequence of LR movements starting from NC_B. Similarly, M can also be attained by a sequence of RL movements starting from NN_B.*

Proof. We present an algorithm for NC_B to attain any other permutation $M = \{(r_m, c_{i_m}) \,|\, 1 \leq m \leq n\}$ by a sequence of LR movements. This algorithm will operate from $m = 1$ to $m = n$. For r_1:

(1) Let $(r_1, c_{j_1}) \in NC_B$. By definition, it is the rightmost cell of r_1. If $c_{i_1} = c_{j_1}$, then move onto the row r_2 as described below. Otherwise, go to Step 2.

(2) Let C_{1j} be the set of columns between c_{i_1} and c_{j_1}. Take the maximal element less than j_1, say k, of C_{1j}. Assume that $(r_{i_k}, c_k) \in NC_B$ for some row r_{i_k}.

Then, do:

$$\mathrm{LR}(\{(r_1, c_{i_1}), (r_{i_k}, c_k)\}) = \{(r_1, c_k), (r_{i_k}, c_{i_1})\}.$$

Update $C_{1,j}$ by deleting k from it.

(3) With the new element (r_1, c_k), continue to find a maximal element t of C_{1j} and a square $(r_{i_t}, c_t) \in NC_B$. As in step 2, perform the LR movement on the two squares $(r_1, c_k), (r_{i_t}, c_t)$. Continue to do this until the square (r_1, c_{i_1}) is reached. Then proceed to r_2 as described below.

For the rows r_m, $m \geq 2$:

(1) In general, apply the algorithm as above. However, when forming the set C_{mj} of columns between c_{j_m} and c_{i_m}, where (r_m, c_{j_m}) is a square attained at some time during the performance of the algorithm on the mth row, ignore columns in the set $\{c_{i_a} : a < m, (r_a, c_{i_a}) \in M\}$.
(2) Apply LR movement to (r_m, c_{j_m}) and (r_{i_t}, c_t) where c_t is the rightmost column in C_{mj}.
(3) Continue this procedure until a permutation with the squares $(r_1, c_{i_1}), \ldots, (r_m, c_{i_m})$ is attained. Then do the same procedure for the row r_{m+1}.

After a square (r_m, c_{i_m}) is attained in the above algorithm, it is subsequently fixed during the succeeding operations. A crucial observation is that after the procedure on row r_1, the squares of $NC_B - \{(r_1, c_{i_1})\}$ are moved to NC_{B_1}, where B_1 is the board obtained from B by removing the first row r_1 and the column c_{i_1}. Hence by induction any permutation M can be generated from a sequence of LR movements starting from NC_B.

The algorithm for NN_B is similar. The algorithm computes a minimal leftmost column greater than the current column with a row greater than the current row at the given stage of the process. Then, an RL movement is performed on the two generated points, and the same process is applied onto the newly generated permutation until the desired (r_m, c_{i_m}) is achieved for a given row r_m. As in the algorithm for NC_B, those points remain fixed. Hence, any permutation M can be achieved by a sequence of RL movements from NN_B. □

We easily attain the following corollary:

Corollary 2.3. *The nonnesting permutation can be achieved following a sequence of left–right movements starting from the noncrossing permutation, and the noncrossing permutation can be achieved following a sequence of right–left movements from the nonnesting permutation.*

2.4. *Producing a bound for the derangement number*

Let $M \in \mathcal{S}(B)$. We now present and prove a general bound for $d_B(M)$.

Theorem 2.4. *If $M \in \mathcal{S}(B)$, and NC_B and NN_B are the noncrossing and nonnesting permutations on B, then the following inequality holds:*

$$d_B(NC_B) \leq d_B(M) \leq d_B(NN_B).$$

First, we will establish a lemma.

Lemma 2.5. *Let $(r_m, c_i), (r_k, c_j) \in B$ with $m > k$ and $i < j$. Let $M, M' \in \mathcal{S}(B)$ so that M contains (r_m, c_i) and (r_k, c_j), and M' be obtained from M by $\mathrm{LR}(\{(r_m, c_i), (r_k, c_j)\})$. Let $D_B(M)$ denote the set of all derangements on B from the permutation M, and let $D_B(M')$ denote the set of all derangements on B from the permutation M'. Then there is an injection from $D_B(M)$ to $D_B(M')$. It follows that $d_B(M) \leq d_B(M')$.*

Proof. We will define an injective map $\phi : D_B(M) \to D_B(M')$, and demonstrate that in certain cases the map fails to be surjective.

We partition the derangements of $D_B(M)$ into three disjoint sets: let $S_i(M)$ be the set of all derangements from M on B that contains exactly i squares from $\{(r_k, c_i), (r_m, c_j)\}$, for $i = 0, 1, 2$. Further divide $S_1(M)$ into two subsets: S_{1a}, the subset of all derangements which contain (r_m, c_j), but not (r_k, c_i); and S_{1b}, the subset of all derangements which contain (r_k, c_i), but not (r_m, c_j).

We define a map $\phi : D_B(M) \to D_B(M')$ as follows:

For $\rho \in S_0(M)$, set $\phi(\rho) = \rho$.
For $\rho_2 \in S_2(M)$, set

$$\phi(\rho_2) = \rho_2 \cup \{(r_m, c_i), (r_k, c_j)\} - \{(r_m, c_j), (r_k, c_i)\}.$$

For $\rho_1 \in S_{1a}$, let (r_k, c_e) be the square of ρ_1 in the row r_k. Set

$$\phi(\rho_1) = \rho_1 \cup \{(r_m, c_e), (r_k, c_j)\} - \{(r_m, c_i), (r_k, c_e)\}.$$

The square $(r_m, c_e) \in B$ because B is a Ferrers board and $m > k$.

For $\rho_1' \in S_{1b}$, we have that $(r_k, c_i) \in \rho_1'$ but $(r_m, c_j) \notin \rho_1'$. Let (r_m, c_e) and (r_d, c_j) be the squares of ρ_1' that lie in row r_m and column c_j, respectively. Then set

$$\phi(\rho_1') = \begin{cases} \rho_1' \cup \{(r_k, c_j), (r_d, c_i)\} - \{(r_k, c_i), (r_d, c_j)\} & \text{if } (r_k, c_e) \notin B, \\ \rho_1' \cup \{(r_m, c_i), (r_k, c_e), (r_d, c_j)\} & \\ \qquad - \{(r_m, c_e), (r_k, c_i), (r_k, c_j)\} & \text{if } (r_k, c_e) \in B. \end{cases}$$

See Fig. 5 for a possible arrangement of r_m, r_k, r_d and c_i, c_j, c_e.

We show that ϕ defined in this way is injective. Note that ϕ maps $S_i(M)$ to $S_i(M')$, and is invertible when restricted to $S_0(M)$ and $S_2(M)$. Hence we only need to check ϕ on $S_1(M)$.

By definition, ϕ maps $S_{1a}(M)$ to derangements α of M' such that α contains (r_k, c_j) but not (r_m, c_i), and the square (r_k, c_e) is in B, if (r_m, c_e) is the square of α

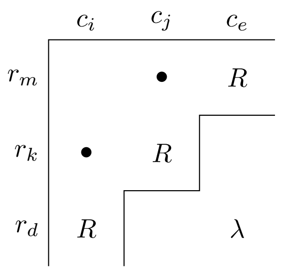


Fig. 5. The map ϕ may not be surjective.

in row r_m. On the other hand, ϕ maps $S_{1b}(M)$ to derangements β of M' such that

— either β contains (r_m, c_i) but not (r_k, c_j),
— or β contains (r_k, c_j), but not (r_m, c_i), and the square (r_k, c_e) is not in B, where (r_m, c_e) is the square of β in row r_m.

Hence no two derangements of $D_B(M)$ have the same image and ϕ is injective.

To see that the map ϕ is not necessarily surjective, consider a board with rows r_d, r_k, r_m $(d < k < m)$ and columns c_i, c_j, c_e $(i < j < e)$, where $(r_m, c_e), (r_k, c_j), (r_d, c_i) \in B$ but $(r_d, c_j), (r_d, c_e), (r_k, c_e) \notin B$. (See Fig. 5, where block dots are squares of M' and R's belong to a derangement of M'.) Then any derangement containing $(r_m, c_e), (r_k, c_j), (r_d, c_i)$ is not an image of ϕ. In this case $d_B(M) < d_B(M')$. □

Proof of Theorem 2.4. By Lemma 2.2, any permutation M can be generated from a sequence of left–right moves beginning with NC_B, hence that $d_B(NC_B) \leq d_B(M)$. Again from Lemma 2.2 any permutation M can be achieved from a sequence of right–left moves starting from NN_B, hence that $d_B(M) \leq d_B(NN_B)$. □

There is now a corollary which will aid the enumeration of the derangement numbers.

Corollary 2.6. *If λ is rectangular, then $d_B(M)$ is independent of the initial choice of M (corresponding to a property in the classic derangement case). The following converse also holds: if n is sufficiently large, and $d_B(M)$ is independent of M, then λ must be rectangular.*

Proof. The proof of Lemma 2.5 demonstrates that the function ϕ is bijective unless each of $(r_d, c_j), (r_d, c_e), (r_k, c_e) \in \lambda$, which is impossible if λ is rectangular. Thus if λ is rectangular, ϕ is bijective and so $d_B(M)$ is independent of the choice of M.

To prove the converse, suppose that λ is nonrectangular. From the proof of Lemma 2.5 and Fig. 5, it is sufficient to show that when n is sufficiently large, there exist two permutations M' and ρ on B, and rows r_d, r_k, r_m, columns c_i, c_j, c_e

such that

(1) $d < k < m$ and $i < j < e$,
(2) $(r_m, c_e), (r_k, c_j), (r_d, c_i) \in B$ but $(r_d, c_j), (r_d, c_e), (r_k, c_e) \notin B$.
(3) $(r_m, c_j), (r_k, c_i) \in M'$, and $(r_m, c_e), (r_k, c_j), (r_d, c_i) \in \rho$.

Assume the minimal rectangle containing λ is of size $a \times b$. Since λ is nonrectangular, we have $a \geq 2$ and $b \geq 2$. Let $n \geq a + b$. We will construct the desired permutations M' and ρ in $B = (n \times n) - \lambda$ by the following steps.

Step 1. Take a derangement ρ_1 in the first a rows and a columns containing the square $R_1 = (r_1, c_a)$. Take a derangement ρ_2 in the last $(n - a)$ rows and $(n - a)$ columns containing the square $R_2 = (r_{a+1}, c_n)$.

Step 2. Assume $R_3 = (r_f, c_t)$ is a corner of the board B such that $1 < f \leq a$ and $|r_{f-1}| < |r_f| < |r_{a+1}| = n$. Let (r_f, c_{f_1}) be the square of ρ_1 in row r_f and (r_{t_1}, c_t) be the square of ρ_2 in column c_t. Set

$$\rho = \rho_1 \cup \rho_2 \cup \{(r_f, c_t), (r_{t_1}, c_{f_1})\} - \{(r_f, c_{f_1}), (r_{t_1}, c_t)\}.$$

Then ρ is a permutation on the board B containing squares R_1, R_2, R_3.

Step 3. In the lower-left $a \times a$ sub-board of B there are only $a - 1$ squares belonging to ρ. Take a permutation M_1 of size a that contains (r_f, c_a) and avoids squares of ρ. Similarly, in the upper-right $(n - a) \times (n - a)$ sub-board of B there are $n - a - 1$ square belonging to ρ. Take a permutation M_2 of size $n - a$ that contains (r_{a+1}, c_t) and avoids squares of ρ. Finally set $M' = M_1 \cup M_2$.

See Example 1 for an illustration with $a = b = 3$ and $n = 6$. The permutations M' and ρ satisfy the desired conditions with the involved rows r_1, r_f, r_{a+1} and columns c_a, c_t, c_n. The existence of permutations ρ_1, ρ_2, M_1, M_2 in Steps 1 and 3 follows from the following property of classical derangements: Given a permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ of length $n \geq 2$, then there are $D_n/(n - 1) = D_{n-1} + D_{n-2}$ derangements σ from π with $\sigma_1 = k$, for all $k \neq \pi_1$. See, for example, [2, Chap. 6.3]. $\square$

Example 1. Figure 6 shows the construction of M' and ρ for a board, where $a = b = 3$ and $n = 6$. In the first figure, ρ_1 and ρ_2 are represented by R where $R_1 = (1, 3)$ and $R_2 = (4, 6)$. The second figure shows the permutation ρ, where the two new squares contain boldface Rs. In the third figure the black dots represent M'. The involved rows and columns are r_1, r_2, r_4 and c_3, c_4, c_6.

3. Enumeration of Derangements in Ferrers Board with a Missing Rectangular Section

The exact enumeration of the derangement numbers may be described as a rook placement on a board with given forbidden positions. For a Ferrers board as we have defined it, the forbidden positions are the squares in $F = \lambda \cup M$, where λ is the

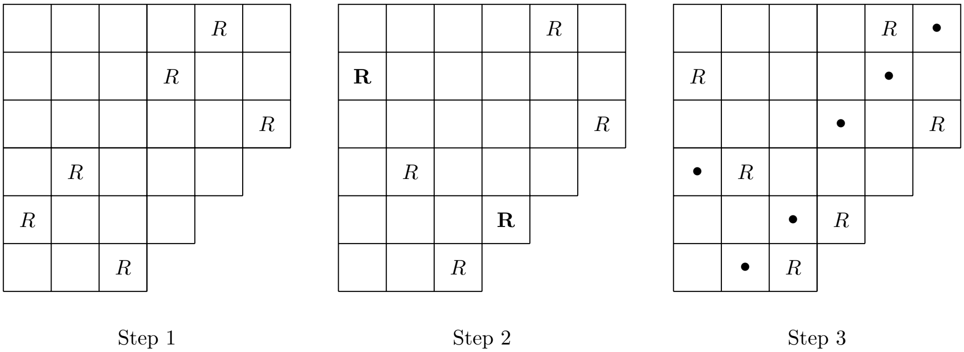


Fig. 6. The construction of M' and ρ.

missing section and M is the initial permutation. There is a well-known theorem connecting the rook coefficients r_k and the number of permutations avoiding given forbidden positions. The theorem can be found in [9]; the result is due to Kaplansky and Riordan in [5].

Theorem 3.1. *Let N_0 be the number of ways to place n nonattacking rooks on an $n \times n$ board avoiding the forbidden positions F. Then, the following equality holds*:

$$N_0 = \sum_{k=0}^{n} r_k (-1)^k (n-k)!$$

where the rook coefficient r_k is the number of ways to place k nonattacking rooks onto the forbidden positions F.

This theorem follows from the Principle of Inclusion–Exclusion. We will investigate specifically the cases where λ is rectangular. In the rest of the paper, we will assume that $\lambda = r \times s$ for some fixed $r, s \geq 0$.

3.1. *Rook coefficients for a rectangular $\boldsymbol{\lambda}$*

Let $F = \lambda \cup M$ be the set of forbidden positions. We will first compute the rook coefficients for F.

Proposition 3.2. *Let r_k be the kth rook coefficient for the forbidden area F. Then*,

$$r_k = \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n-2i}{k-i}.$$

Proof. The two sections of the forbidden positions, λ and M, are disjoint, and so we may place points on λ first, and then points on M. Suppose that we place i points on λ and $k-i$ points on M. The number of ways to place i points on λ is denoted r_λ^i, analogously to the other rook coefficients. Then, note that each of the i points on λ "attacks" two points of M (they attack a particular row and a particular column). Since no two of the i points on λ can be in the same row or column, there are then $2i$ points on M that are attacked. The number of ways to choose $k-i$ points on M from $n-2i$ possible points is given by the binomial coefficient $\binom{n-2i}{k-i}$. Hence, summing over all possible i, we have that

$$r_k = \sum_{i=0}^{k} r_\lambda^i \binom{n-2i}{k-i}.$$

We now need to merely compute r_λ^i. But this is easily done. Since $\lambda = r \times s$ is rectangular, choose i of the r rows and i of the s columns on which to place points. If the rows are fixed, then there are $i!$ possible permutation of the columns with those rows. Hence, $r_\lambda^i = \binom{r}{i}\binom{s}{i} i!$, and so

$$r_k = \sum_{i=0}^{k} \binom{r}{i}\binom{s}{i} i! \binom{n-2i}{k-i}$$

as desired. □

Using an Abel-type inverse relation, we derive from Proposition 3.2 a recurrence relation for r_k.

Proposition 3.3. *The rook coefficients r_k for the forbidden area F satisfy the recurrence*

$$\binom{r}{k}\binom{s}{k} k! = \sum_{i=0}^{n} \frac{n-2i}{n-2k} \binom{-n+2k}{k-i} r_i.$$

Proof. Let $t_i = \binom{r}{i}\binom{s}{i} i!$. From Proposition 3.2, we have $r_k = \sum_{i=0}^{k} A_{k,i} t_i$ where $A_{k,i} = \binom{n-2i}{k-i}$. Hence $t_k = \sum_{i=0}^{k} B_{k,i} r_i$ where $(B_{k,i})$ is the inverse matrix of $(A_{k,i})$.

The formula of $B_{k,i}$ can be found in [6, Ex. 3.1(28)], which states that if $A_{k,i} = \binom{c+kp+iq}{k-i}$, then the entries of the inverse matrix are given by

$$B_{k,i} = \frac{[c+k(p+q)][c+i(p+q)] + (k-i)pq + (c+kp+iq)}{(c+ip+kq)(c+ip+kq+1)} \binom{-c-ip-kq}{k-i}.$$

Substituting $c = n, p = 0$ and $q = -2$, we obtain

$$B_{k,i} = \frac{n-2i}{n-2k} \binom{-n+2k}{k-i},$$

which proves the recurrence. □

We will now compute the generating function for the rook coefficients $\sum_{k=0}^{n} r_k x^k$.

Theorem 3.4. *Let $R(x) = \sum_{k=0}^{n} r_k x^k$ be the ordinary generating function for the rook coefficients as above. Then,*

$$R(x) = \sum_{k=0}^{n} r_k x^k = (1+x)^n \sum_{i=0}^{\min(r,s)} \binom{r}{i}\binom{s}{i} i! \left(\frac{x}{(1+x)^2}\right)^i$$

for sufficiently large n.

Proof. Using the formula for the rook coefficients and some well-known generating functions, we obtain

$$\begin{aligned} R(x) = \sum_{k=0}^{n} r_k x^k &= \sum_{k=0}^{n}\sum_{i=0}^{k} \binom{r}{i}\binom{s}{i} i! \binom{n-2i}{k-i} x^k \\ &= \sum_{i=0}^{n} \binom{r}{i}\binom{s}{i} i! x^i \sum_{k=i}^{n} \binom{n-2i}{k-i} x^{k-i} \\ &= \sum_{i=0}^{n} \binom{r}{i}\binom{s}{i} i! x^i (1+x)^{n-2i} \\ &= (1+x)^n \sum_{i=0}^{n} \binom{r}{i}\binom{s}{i} i! \left(\frac{x}{(1+x)^2}\right)^i. \end{aligned}$$

For $i > \min(r,s)$, note that $\binom{r}{i}\binom{s}{i} = 0$, so the sum is 0 for such i. Hence, the desired equality is attained. □

3.2. *An explicit formula for the derangement numbers for rectangular $\boldsymbol{\lambda}$*

Let $d_{n,r,s} = d_B(M)$ denote the number of derangements on the Ferrers board $B = (n \times n) - (r \times s)$, which is independent of M. Applying Theorem 3.1, we get

$$d_{n,r,s} = \sum_{k=0}^{n}\sum_{i=0}^{k} \binom{r}{i}\binom{s}{i} i! \binom{n-2i}{k-i} (-1)^k (n-k)!$$

We will present an equivalent form of this sum in which $d_{n,r,s}$ is expressed as a finite sum of $\min(r,s)+1$ terms. To do so, we need to introduce a new kind of derangement number, which is a straightforward generalization of the classical derangement number D_n.

For $q \le p$, let $D_{p,q}$ denote the number of permutations of length p such that the first q numbers satisfy the derangement property: i.e. 1 is not in the first position, 2 is not in the second position, …, q is not in the qth position. Clearly when $p = q$ we get the classical derangement numbers. In general, we have the following formula.

Proposition 3.5.

$$D_{p,q} = \sum_{i=0}^{q} (-1)^i \binom{q}{i} (p-i)!$$

Proof. This is a direct computation from the principle of inclusion–exclusion, and closely resembles the computation for the classic derangement numbers. Let P_i be the property that $a_i = i$ (where each permutation is written as $a_1 a_2 \ldots a_p$). Suppose that j of the P_i are satisfied. Then, there are $\binom{q}{j}$ ways to select j properties, and $(p-j)!$ ways to permute the other elements. Hence, by PIE, summing over all possible j, we obtain

$$D_{p,q} = \sum_{j=0}^{q} (-1)^j \binom{q}{j} (p-j)!$$

as desired. $\square$

We will now establish an equivalent form of $d_{n,r,s}$ in terms of $D_{p,q}$.

Proposition 3.6.

$$d_{n,r,s} = \sum_{i=0}^{\min(r,s)} (-1)^i i! \binom{r}{i} \binom{s}{i} D_{n-i,n-2i}$$

for sufficiently large n.

Proof. We compare to the form given above and show the two are equivalent. To wit:

$$\begin{aligned} d_{n,r,s} &= \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n-2i}{k-i} (-1)^k (n-k)! \\ &= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! \sum_{k=i}^{n} \binom{n-2i}{k-i} (-1)^k (n-k)! \\ &= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! \sum_{k=0}^{n-i} \binom{n-2i}{k} (-1)^{k+i} (n-(k+i))! \\ &= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! (-1)^i D_{n-i,n-2i}. \end{aligned}$$

As before, if $i > \min(r,s)$, then $\binom{r}{i}\binom{s}{i} = 0$, so the sum can be displayed in the desired form if $n \geq \min(r,s)$. $\square$

This form is interesting because the numbers $D_{p,q}$ have an easily computable exponential generating function, similar to the classical derangement numbers.

3.3. *A recurrence for the derangement numbers*

In this subsection, we present a linear recurrence and partial differential equation satisfied by the derangements numbers $d_{n,r,s}$ associated with the Ferrers board $B = (n \times n) - (r \times s)$.

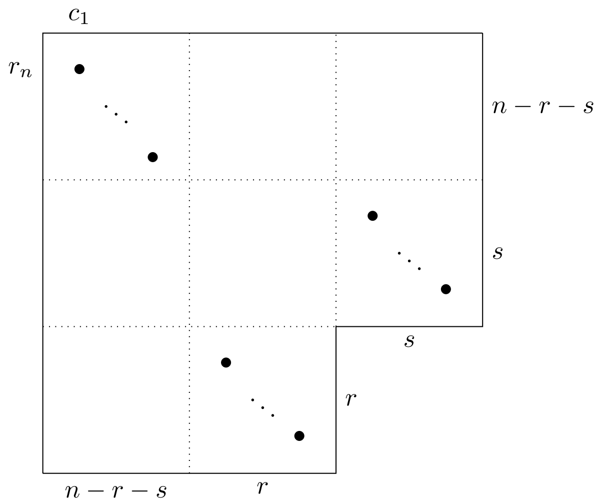


Fig. 7. NC_B on a board with forbidden area $r \times s$.

Theorem 3.7. *The derangement numbers $d_{n,r,s}$ satisfy the following linear recurrence*:

$$\begin{aligned} d_{n,r,s} &= (n-r-s-1)(d_{n-1,r,s} + d_{n-2,r,s}) \\ &\quad + s(d_{n-1,r,s} + d_{n-2,r,s-1}) \\ &\quad + r(d_{n-1,r-1,s} + d_{n-2,r-1,s}) \end{aligned} \tag{3.1}$$

for $n > r+s$, and $d_{r+s,r,s} = D_r D_s$, where D_n is the classical derangement number given in (1.1). *By convention, we have $d_{n,r,s} = 0$ if $n \le 1$ or $r, s < 0$ or $n < r+s$.*

Proof. As λ is rectangular, by Corollary 2.6 the number of derangements from any initial permutation on B is constant. Therefore, we are free to choose any permutation M on B from which to compute $d_B(M)$, so we choose $M = NC_B$. For $n \ge r+s$, in reference to Fig. 7, we see that the noncrossing permutations NC_B has the following pattern: the first r squares proceed diagonally to the northwest from (r_1, c_{n-s}); directly above the rth row are s squares proceeding diagonally to the northwest from (r_{r+1}, c_n), and the remaining squares are again diagonal and to the left of the first r squares of NC_B.

When $n = r+s$, a derangement in $D_B(NC_B)$ is a union of a derangement in the lower-left $r \times r$ sub-board avoiding the southeast–northwest diagonal, together with a derangement in the upper-right $s \times s$ sub-board avoiding the southeast–northwest diagonal. Hence $d_{r+s,r,s} = D_r D_s$.

Now assume $n > r+s$. Let A be the square of a derangement ρ in $D_B(NC_b)$ in the first column. To establish the recurrence, we discuss by cases based on the location of A. Assume A is in row r_a. These are the three cases.

The first case is that A is in one of the top $n-r-s$ rows, i.e. $a > r+s$. We consider the square of ρ in the last row r_n. There are two possibilities: $(r_n, c_a) \notin \rho$ or $(r_n, c_a) \in \rho$. If $(r_n, c_a) \notin \rho$, removing column c_1 and row r_a, we have a board $B_1 = (n-1)\times(n-1)-r\times s$. The derangement ρ, when restricted to B_1, is a derangement avoiding $(NC_B \cap B_1) \cup \{(r_n, c_a)\}$. There are $d_{n-1,r,s}$ such derangements, On the other hand, if $(r_n, c_a) \in \rho$, we can remove rows r_a, r_n and columns c_1, c_a to obtain a board B_2 of size $(n-2)\times(n-2)-r\times s$, with a permutation $NC_B \cap B_2$ forbidden. By hypothesis there are $d_{n-2,r,s}$ derangements on B_2. Combining the two cases and noting that there are $n-r-s-1$ choices for r_a as $a \neq n$, we have that there are $(n-r-s-1)(d_{n-1,r,s}+d_{n-2,r,s})$ derangements when $a > r+s$.

The next two cases are that A is in one of the middle s rows, or A is in one of the bottom r rows. That is, $r < a \leq r+s$ or $a \leq r$. A similar discussion on whether ρ contains the square (r_n, c_a) leads to the terms $s(d_{n-2,r,s-1}+d_{n-1,r,s})$ for the second case, and the terms $r(d_{n-2,r-1,s}+d_{n-1,r-1,s})$ for the third case.

Combining these three cases gives the desired recurrence. □

Remark. It is clear that $d_{n,r,s} = D_n$ if r or s is zero. In that case the recurrence (3.1) reduces to the well-known equation

$$D_n = (n-1)(D_{n-1}+D_{n-2}).$$

Although a linear recurrence has been established, there does not seem to be a simple formula for a multivariate generating function of $d_{n,r,s}$. Nevertheless, let

$$D(x,y,z) = \sum_{r,s\geq 0}\sum_{n\geq r+s} d_{n,r,s}x^n \frac{y^r}{r!}\frac{z^s}{s!}. \tag{3.2}$$

There is a partial differential equation which such a generating function would have to satisfy.

Proposition 3.8. *Let $D(x,y,z)$ be defined as above. Then, $D(x,y,z)$ satisfies the following partial differential equation*:

$$\begin{gathered}(1-xy)T(x,y,z) + (x^2y + x^2z + xy + x^2 - 1)D(x,y,z) \\ + (x^3+x^2)\frac{\partial D}{\partial x} - (xy + x^2y)\frac{\partial D}{\partial y} - x^2z\frac{\partial D}{\partial z} = 0,\end{gathered}$$

where

$$T(x,y,z) = \frac{e^{-xy-xz}}{(1-xy)(1-xz)}.$$

Proof. (Sketch of Proof). First note that when $n = r+s$,

$$\begin{aligned}\sum_{r,s\geq 0} d_{r+s,r,s}x^{r+s}\frac{y^r}{r!}\frac{z^s}{s!} &= \sum_{r,s\geq 0} D_rD_s\frac{(xy)^r}{r!}\frac{(xz)^s}{s!} \\ &= \frac{e^{-xy-xz}}{(1-xy)(1-xz)} = T(x,y,z),\end{aligned}$$

where we use the classical result

$$\sum_{n\geq 0} \frac{D(n)x^n}{n!} = \frac{e^{-x}}{1-x}.$$

Hence

$$D(x,y,z) = T(x,y,z) + \sum_{r,s\geq 0} \sum_{n>r+s} d_{n,r,s} x^n \frac{y^r}{r!} \frac{z^s}{s!}.$$

Applying the recurrence (3.1) to $d_{n,r,s}$ in the second summation, and using that

$$\begin{aligned}
\frac{\partial D}{\partial x} &= \sum_{r,s\geq 0} \sum_{n\geq r+s} n d_{n,r,s} x^{n-1} \frac{y^r}{r!} \frac{z^s}{s!} \\
&= \sum_{r,s\geq 0} \sum_{n>r+s} (n-1) d_{n-1,r,s} x^{n-2} \frac{y^r}{r!} \frac{z^s}{s!}, \\
\frac{\partial D}{\partial y} &= \sum_{r\geq 1, s\geq 0} \sum_{n\geq r+s} d_{n,r,s} x^n \frac{y^{r-1}}{(r-1)!} \frac{z^s}{s!}, \\
\frac{\partial D}{\partial z} &= \sum_{r\geq 0, s\geq 1} \sum_{n\geq r+s} d_{n,r,s} x^n \frac{y^r}{r!} \frac{z^{s-1}}{(s-1)!},
\end{aligned}$$

together with

$$\begin{aligned}
&\sum_{r,s\geq 0} \sum_{n>r+s} r d_{n-1,r-1,s} x^n \frac{y^r}{r!} \frac{z^s}{s!} \\
&\quad = \sum_{r,s\geq 0} \sum_{n>r+s} d_{n,r,s} x^{n+1} \frac{y^{r+1}}{r!} \frac{z^s}{s!} \\
&\quad = \sum_{r,s\geq 0} \sum_{n\geq r+s} d_{n,r,s} x^{n+1} \frac{y^{r+1}}{r!} \frac{z^s}{s!} - \sum_{r,s\geq 0} D_r D_s x^{r+s+1} \frac{y^{r+1}}{r!} \frac{z^s}{s!} \\
&\quad = xyD(x,y,z) - xyT(x,y,z),
\end{aligned}$$

we obtain the desired equation. □

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