Multiparking Functions, Graph Searching, and the Tutte Polynomial

Dimitrije Kostić[1*] and Catherine H. Yan[2†]

[1,2] Department of Mathematics
Texas A&M University, College Station, TX 77843
[2] Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P.R. China

[1] dkostic@math.tamu.edu, [2]cyan@math.tamu.edu

Key words and phrases: parking functions, breadth-first search, Tutte polynomial, spanning forest

Mathematics Subject Classification. 05C30, 05C05

Abstract

A parking function of length n is a sequence $(b_1, b_2, \ldots, b_n)$ of nonnegative integers for which there is a permutation $\pi \in S_n$ so that $0 \leq b_{\pi(i)} < i$ for all i. A well-known result about parking functions is that the polynomial $P_n(q)$, which enumerates the complements of parking functions by the sum of their terms, is the generating function for the number of connected graphs by the number of excess edges when evaluated at $1+q$. In this paper we extend this result to arbitrary connected graphs G. In general the polynomial that encodes information about subgraphs of G is the Tutte polynomial $t_G(x, y)$, which is the generating function for two parameters, namely the internal and external activities, associated with the spanning trees of G. We define G-multiparking functions, which generalize the G-parking functions that Postnikov and Shapiro introduced in the study of certain quotients of the polynomial ring. We construct a family of algorithmic bijections between the spanning forests of a graph G and the G-multiparking functions. In particular, the bijection induced by the breadth-first search leads to a new characterization of external activity, and hence a representation of Tutte polynomial by the reversed sum of G-multiparking functions.

*Partially supported by Texas A&M's NSF VIGRE grant.
†Partially support by NSF grant DMS 0245526.

1 Introduction

The (classical) parking functions of length n are sequences $(b_1, b_2, \ldots, b_n)$ of nonnegative integers for which there is a permutation $\pi \in S_n$ so that $0 \le b_{\pi(i)} < i$ for all i. This notion was first introduced by Konheim and Weiss [11] in the study of the linear probes of random hashing function. The name comes from a picturesque description in [11] of the sequence of preferences of n drivers under certain parking rules. Parking functions have many interesting combinatorial properties. The most notable one is that the number of parking functions of length n is $(n+1)^{n-1}$, Cayley's formula for the number of labeled trees on $n+1$ vertices. This relation motivated much work in the early study of parking functions, in particular, combinatorial bijections between the set of parking functions of length n and labeled trees on $n+1$ vertices. See [8] for an extensive list of references.

There are a number of generalizations of parking functions, for example, see [4] for the double parking functions, [20, 22, 23] for k-parking functions, and [17, 14] for parking functions associated with an arbitrary vector. Recently, Postnikov and Shapiro [18] proposed a new generalization, the *G-parking functions*, associated to a general connected digraph D. Let G be a digraph on $n+1$ vertices indexed by integers from 0 to n. A G-parking function is a function f from $[n]$ to $\mathbb{N}$, the set of non-negative integers, satisfying the following condition: for each subset $U \subseteq [n]$ of vertices of G, there exists a vertex $j \in U$ such that the number of edges from j to vertices outside U is greater than $f(j)$. For the complete graph $G = K_{n+1}$, such defined functions are exactly the classical parking functions, where one views K_{n+1} as the digraph with one directed edge (i, j) for each pair $i \neq j$. In [2] Chebikin and Pylyavskyy constructed a family of bijections between the set of G-parking functions and the (oriented) spanning trees of that graph.

Perhaps the most important statistic of the classical parking functions is the (reversed) sum, that is, $\binom{n}{2} - (x_1 + x_2 + \cdots + x_n)$ for a parking function $(x_1, x_2, \ldots, x_n)$ of length n. It corresponds to the number of linear probes in hashing functions [10], the number of inversions in labeled trees on $[n+1]$ [13], and the number of hyperplanes separating a given region from the base region in the extended Shi arrangements [20], to list a few. It is also closely related to the number of connected graphs on $[n+1]$ with a fixed number of edges. In [23] the second author gave a combinatorial explanation, which revealed the underlying correspondence between the classical parking functions and labeled, connected graphs. The main idea is to use breadth-first search to find a labeled tree on any given connected graph, and record such a search by a queue process.

The objective of the present paper is to extend the result of [23] to arbitrary graphs. For a general graph G, a suitable tool to study all subgraphs of G is the Tutte polynomial. This is a generating function with two parameters, the internal and external activities, which are functions on the collection of spanning trees of G. Evaluating the Tutte polynomial at various points provides combinatorial information about the graph, for example, the number of spanning trees, spanning forests, connected subgraphs, acyclic orientations, subdigraphs, etc. These many valuations make the Tutte polynomial one of the most fundamental tools in algebraic graph theory.

An important approach to getting information about the Tutte polynomial is to use partitions. This approach dates from the 1960's, see Crapo [5]. More information on the history of Tutte polynomial can be found in [9]. Also in that paper Gessel and Sagan proposed a number of new notions of external activity, along with a new way to partition the substructures of a given graph. The basic method is to use depth-first search to associate a spanning forest F with each substructure to be counted. This process partitions the simplicial complex of all substructures (ordered by inclusion) into intervals, one for each F. Every interval turns out to be a Boolean algebra consisting of all ways to add external active edges to F. Expressing the Tutte polynomial in terms of sums over such intervals permits one to extract the necessary combinatorial information.

In [9] Gessel and Sagan also mentioned another search, the *neighbors-first search*, and related the external activity determined by the neighbors-first search on a complete graph with $n+1$ vertices to the sum of (classical) parking functions of length n. This connection was further explained in [23]. In the present paper we extend this result to an arbitrary graph G by developing the connection between Tutte polynomial of G and certain restricted functions defined on $V(G)$, the vertex set of G. This is achieved by combining the two approaches mentioned before. First, we use breadth-first search to get a new partition of all spanning subgraphs of G. Each subgraph is associated with a spanning forest of G, which allows us to get a new expression of the Tutte polynomial in terms of breadth-first external activities of its spanning forests. Second, we construct bijections between the set of all spanning forests of G and the set of functions defined on $V(G)$ with certain restrictions. One of such bijection, namely the one induced by breadth-first search with a queue, leads to the characterization of the (breadth-first) external activity of a spanning forest by the corresponding function.

To work with spanning forests, we propose the notion of a G-multiparking function, a natural extension of the notion of a G-parking function. Let G denote a graph with a totally ordered vertex set $V(G)$. Often we will take $V(G) = [n] = \{1, 2, \ldots, n\}$. For simplicity and clarity, we assume that G is a simple graph in most of the paper, except at the end of Section 3 where we explain how our construction could be modified to apply for general directed graphs, with possible loops and multiple edges. This includes undirected graphs as special cases, as an undirected graph can be viewed as a digraph where each edge $\{u, v\}$ is replaced by a pair of arcs (u, v) and (v, u).

For any subset $U \subseteq V(G)$, and vertex $v \in U$, define $outdeg_U(v)$ to be the cardinality of the set $\{\{v, w\} \in E(G) | w \notin U\}$. Here $E(G)$ is the set of edges of G.

Definition 1. *Let G be a simple graph with $V(G) = [n]$. A* G-multiparking function *is a function $f : V(G) = [n] \to \mathbb{N} \cup \{\infty\}$, such that for every $U \subseteq V(G)$ either* **(A)** *i is the vertex of smallest index in U, (written as $i = \min(U)$), and $f(i) = \infty$, or* **(B)** *there exists a vertex $i \in U$ such that $0 \le f(v_i) < outdeg_U(i)$.*

The vertices which satisfy $f(i) = \infty$ in **(A)** will be called *roots of f* and those that satisfy **(B)** (in U) are said to be *well-behaved* in U, and **(A)** and **(B)** will be used to refer, respectively, to these conditions hereafter. Note that vertex 1 is always a root. The G-multiparking functions with only one root (which is necessarily vertex 1) are exactly the G-parking functions, as defined by Postnikov and Shapiro.

Sections 2 and 3 are devoted to the combinatorial properties of G-multiparking functions. In §2 we construct a family of algorithmic bijections between the set $\mathcal{MP}_G$ of G-multiparking functions and the set $\mathcal{F}_G$ of spanning forests of G. Each bijection is a process based on a choice function, (c.f. §2), which determines how the algorithm proceeds. In §3 we give a number of examples to illustrate various forms of the bijection. This includes the cases where there is a special order on $V(G)$, for instance *depth-first search order*, *breadth-first search order*, and a prefixed linear order, and the cases that the process possesses certain data structure, such as queue and stacks. At the end of §3 we explain how the algorithm works for general directed graphs.

Section 4 is on the relation between G-multiparking functions and the Tutte polynomial of G. First, for each forest F we give a characterization of F-redundant edges, which are edges of $G - F$ that are "irrelevant" in determining the corresponding G-multiparking function. Using that we classify the edges of G, and establish an equation between $|E(G)|$ and sum of G-multiparking function, $|E(F)|$, and the F-redundant edges. Then we use breadth-first search to partition all the subgraphs of G into intervals. Each interval consists of all graphs obtained by adding some breadth-first externally active edges to a spanning forest F. The set of breadth-first externally active edges of F are exactly the F-redundant edges of a certain type, which allows us to express the number of breadth-first externally active edges, and hence the Tutte polynomial, by the values of corresponding G-multiparking function. In section 5 we exhibit some enumerative results related to G-multiparking functions and substructures of graphs.

2 Bijections between multiparking functions and spanning forests

In this section, we construct bijections between the set $\mathcal{MP}_G$ of G-multiparking functions and the set $\mathcal{F}_G$ of spanning forests of G. For simplicity, here we assume G is a simple graph with $V(G) = [n]$. A *sub-forest* F of G is a subgraph of G without cycles. A leaf of F is a vertex $v \in V(F)$ with degree 1 in F. Denote the set of leaves of F by $Leaf(F)$. Let $\prod$ be the set of all ordered pairs (F, W) such that F is a sub-forest of G, and $\emptyset \neq W \subseteq Leaf(F)$. A *choice function* γ is a function from $\prod$ to $V(G)$ such that $\gamma(F, W) \in W$. Examples of various choice functions will be given in §3, where we also explain how the bijections work on a general directed graph, in which loops and multiple edges are allowed. As one can see, loopless undirected graphs can be viewed as special case there.

Fix a choice function γ. Given a G-multiparking function $f \in \mathcal{MP}_G$, we define an algorithm to find a spanning forest $F \in \mathcal{F}_G$. Explicitly, we define quadruples (val_i, P_i, Q_i, F_i) recursively for $i = 0, 1, \ldots, n$, where $val_i : V(G) \to \mathbb{Z}$ is the *value function*, P_i is the set of *processed* vertices, Q_i is the set of vertices *to be processed*, and F_i is a subforest of G with $V(F_i) = P_i \cup Q_i$, $Q_i \subseteq Leaf(F_i)$ or Q_i consists of an isolated vertex of F_i.

Algorithm A.

- **Step 1: initial condition.** Let $val_0 = f$, P_0 be empty, and $F_0 = Q_0 = \{1\}$.
- **Step 2: choose a new vertex v.** At time $i \geq 1$, let $v = \gamma(F_{i-1}, Q_{i-1})$, where γ is the choice function.
- **Step 3: process vertex v.** For every vertex w adjacent to v and $w \notin P_{i-1}$, set $val_i(w) = val_{i-1}(w) - 1$. For any other vertex u, set $val_i(u) = val_{i-1}(u)$. Let $N = \{w | val_i(w) = -1, val_{i-1}(w) \neq -1\}$. Update P_i, Q_i and F_i by letting $P_i = P_{i-1} \cup \{v\}$, $Q_i = Q_{i-1} \cup N \setminus \{v\}$ if $Q_{i-1} \cup N \setminus \{v\} \neq \emptyset$, otherwise $Q_i = \{u\}$ where u is the vertex of the lowest-index in $[n] - P_i$. Let F_i be a graph on $P_i \cup Q_i$ whose edges are obtained from those of F_{i-1} by joining edges $\{w, v\}$ for each $w \in N$. We say that the vertex v is processed at time i.

Iterate steps 2-3 until $i = n$. We must have $P_n = [n]$ and $Q_n = \emptyset$. Define $\Phi = \Phi_{\gamma,G} : \mathcal{MP}_G \to \mathcal{F}_G$ by letting $\Phi(f) = F_n$.

If an edge $\{v, w\}$ is added to the forest F_i as described in Step 3, we say that w *is found by* v, and v is the *parent* of w, if $v \in P_i$. (In this paper, the parent of vertex v will be frequently denoted v^p.) By Step 3, a vertex w is in Q_i because either it is found by some v that has been processed, and $\{v, w\}$ is the only edge of F_i that has w as an endpoint, or w is the lowest-index vertex in $[n] - P_i$ and is an isolated vertex of F_i. Also, it is clear that each F_i is a forest, since every edge $\{u, w\}$ in $F_i \setminus F_{i-1}$ has one endpoint in $V(F_i) \setminus V(F_{i-1})$. Hence $\gamma(F_i, Q_i)$ is well-defined and thus we have a well-defined map Φ from $\mathcal{MP}_G$ to $\mathcal{F}_G$. The following proposition describes the role played by the roots of a G-multiparking function f.

Proposition 2.1. *Let f be a G-multiparking function. Each tree component T of $\Phi(f)$ has exactly one vertex v with $f(v) = \infty$. In particular, v is the least vertex of T.*

Proof. In the algorithm A the value for a root of f never changes, as $\infty - 1 = \infty$. Each nonroot vertex w of T is found by some other vertex v, and $\{v, w\}$ is an edge of T. As any tree has one more vertex than its number of edges, it has exactly one vertex without a parent. By the definition of Algorithm A, this must be a root of f.

To show that the root is the least vertex in each component, let $r_1 < r_2 < \cdots < r_k$ be the roots of f and suppose $T_1, T_2, \ldots, T_k$ are the trees of $F = \Phi(f)$, where $r_i \in T_i$. Let T_j be the tree of smallest index j such that there is a $v \in T_j$ with $v < r_j$. Then $j > 1$ since the vertex 1 is always a root. Define $U := V(T_j \cup T_{j+1} \cup \ldots \cup T_k)$. U is thus a proper subset of $V(G) = [n]$. By assumption, the vertex of least index in U is not a root. Therefore, U must contain a well-behaved vertex; that is, a vertex v such that $0 \leq f(v) < outdeg_U(v)$. Note that all the edges counted by $outdeg_U(v)$ lead to vertices in the trees $T_1, T_2, \ldots, T_{j-1}$. By the structure of algorithm A, all the vertices in the first $j - 1$ trees are processed before the parent of v is processed. But this means that by the time A

processes all the vertices in the first $j-1$ trees, $val_i(v) = f(v) - outdeg_U(v) \leq -1$, so v should be adjacent to some vertex in one of the first $j-1$ trees. This is a contradiction.

$\square$

From the above proof we also see that the forest $F = \Phi(f)$ is built tree by tree by the algorithm A. That is, if T_i and T_j are tree components of F with roots r_i, r_j and $r_i < r_j$, then every vertex of tree T_i is processed before any vertex of T_j.

To show that Φ is a bijection, we define a new algorithm to find a G-multiparking function for any given spanning forest, and prove that it gives the inverse map of Φ.

Let G be a graph on $[n]$ with a spanning forest F. Let $T_1, \ldots, T_k$ be the trees of F with respective minimal vertices $r_1 = 1 < r_2 < \cdots < r_k$.

Algorithm B.

- **Step 1. Determine the process order π.** Define a permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n)) = (v_1 v_2 \ldots v_n)$ on the vertices of G as follows. First, $v_1 = 1$. Assuming $v_1, v_2, \ldots, v_i$ are determined,

  - **Case (1)** If there is no edge of F connecting vertices in $V_i = \{v_1, v_2, \ldots, v_i\}$ to vertices outside V_i, let v_{i+1} be the vertex of smallest index not already in V_i;
  - **Case (2)** Otherwise, let $W = \{v \notin V_i : v \text{ is adjacent to some vertices in } V_i\}$, and F' be the forest obtained by restricting F to $V_i \cup W$. Let $v_{i+1} = \gamma(F', W)$.

  (Hereafter, when discussing process orders, we will write v_i as $\pi(i)$.)

- **Step 2. Define a G-multiparking function $f = f_F$.** Set $f(r_1) = f(r_2) = \cdots = f(r_k) = \infty$. For any other vertex v, let r_v be the minimal vertex in the tree containing v, and $v, v^p, u_1, \ldots, u_t, r_v$ be the unique path from v to r_v. Set $f(v)$ to be the cardinality of the set $\{v_j | \{v, v_j\} \in E(G),\ \pi^{-1}(v_j) < \pi^{-1}(v^p)\}$.

To verify that a function $f = f_F$ defined in this way is a G-multiparking function, we need the following lemma.

Lemma 2.2. *Let $f : V(G) \to \mathbb{N} \cup \{\infty\}$ be a function. If $v \in U \subseteq V(G)$ obeys property* **(A)** *or property* **(B)** *and W is a subset of U containing v, then v obeys the same property in W.*

Proof. If $f(v) = \infty$ and v is the smallest vertex in U, then clearly it will still be the smallest vertex in W. If v is well-behaved in U, then $0 \leq f(v) < outdeg_U(v)$ and as $W \subseteq U$, we have $outdeg_U(v) \leq outdeg_W(v)$. Thus v is well-behaved in W.

$\square$

The *burning algorithm* was developed by Dhar [7] to determine if a function on the vertex set of a graph had a property called *recurrence*. An equivalent description for G-parking functions is given in [2]: We mark vertices of G starting with the root 1. At each iteration of the algorithm, we mark all vertices v that have more marked neighbors than the value of the function at v. The function is a G-parking function if and only if all vertices are marked when this process terminates. Here we extend the burning algorithm to G-multiparking functions, and write it in a linear form.

Proposition 2.3. *A vertex function is a G-multiparking function if and only if there exists an ordering $\pi(1), \pi(2), \ldots, \pi(n)$ of the vertices of a graph G such that for every j, $\pi(j)$ satisfies either condition* **(A)** *or condition* **(B)** *in $U_j := \{\pi(j), \ldots, \pi(n)\}$.*

Proof. We say that the vertices can be "thrown out" of a subset $U \subset V(G)$ if v satisfies either condition **(A)** or condition **(B)** in U. By the definition of G-multiparking function, it is clear that for a G-multiparking function, vertices can be thrown out in some order.

Conversely, suppose that for a vertex function $f : V(G) \to \mathbb{N} \cup \{\infty\}$ the vertices of G can be thrown out in a particular order $\pi(1), \pi(2), \ldots, \pi(n)$. For any subset U of $V(G)$, let k be the maximal index such that $U \subseteq U_k = \{\pi(k), \ldots, \pi(n)\}$. This implies $\pi(k) \in U$. But $\pi(k)$ satisfies either condition **(A)** or condition **(B)** in U_k. By Lemma 2.2, $\pi(k)$ satisfies either condition **(A)** or condition **(B)** in U. Since U is arbitrary, f is a G-multiparking function.

□

Proposition 2.4. *The Algorithm B, when applied to a spanning forest of G, yields a G-multiparking function $f = f_F$.*

Proof. Let π be the permutation defined in Step 1 of Algorithm B. We show that the vertices can be thrown out in the order $\pi(1), \pi(2), \ldots, \pi(n)$. As $\pi(1) = 1$, the vertex $\pi(1)$ clearly can be thrown out. Suppose $\pi(1), \ldots, \pi(k-1)$ can be thrown out, and consider $\pi(k)$.

If $f(\pi(k)) = \infty$, by Case (1) of step 1, $\pi(k)$ is the smallest vertex not in $\{\pi(1), \ldots, \pi(k-1)\}$. Thus it can be thrown out.

If $f(\pi(k)) \neq \infty$, there is an edge of the forest F connecting $\pi(k)$ to a vertex w in $\{\pi(1), \ldots, \pi(k-1)\}$. Suppose $w = \pi(t)$ where $t < k$. By definition of f, there are exactly $f(\pi(k))$ edges connecting $\pi(k)$ to the set $\{\pi(1), \ldots, \pi(t-1)\}$. Hence $f(\pi(k)) < outdeg_{\{\pi(k),\ldots,\pi(n)\}}(\pi(k))$. Thus $\pi(k)$ can be thrown out as well.

By induction the vertices of G can be thrown out in the order $\pi(1), \pi(2), \ldots, \pi(n)$.

□

Define $\Psi_{\gamma,G} : \mathcal{F}_G \to \mathcal{MP}_G$ by letting $\Psi_{\gamma,G}(F) = f_F$. Now we show that $\Phi = \Phi_{\gamma,G}$ and $\Psi = \Psi_{\gamma,G}$ are inverses of each other.

Theorem 2.5. $\Psi(\Phi(f)) = f$ *for any* $f \in \mathcal{MP}_G$ *and* $\Phi(\Psi(F)) = F$ *for any* $F \in \mathcal{F}_G$.

Proof. First, if $f \in \mathcal{MP}_G$ and $F = \Phi(f)$, then by Prop. 2.1 the roots of f are exactly the minimal vertices in each tree component of F. Those in turn are roots for $\Psi(F)$. In applying algorithm B to F, we note that the order $\pi = v_1 v_2 \dots v_n$ is exactly the order in which vertices of G will be processed when running algorithm A on f. That is, $P_i = \{v_1, \dots, v_i\}$, and v_{i+1} is not a root of f, then Q_i is the set of vertices which are adjacent (via edges in F) to those in P_i. By the construction of algorithm A, a vertex w is found by v if and only if there are $f(w)$ many edges connecting w to vertices that are processed before v, or equivalently, to vertices u with $\pi^{-1}(u) < \pi^{-1}(v)$. Since in $\Phi(f)$, $v = w^p$, we have $\Psi(\Phi(f)) = f$.

Conversely, we prove that $\Phi(\Psi(F)) = F$ by showing that $\Phi(\Psi(F))$ and F have the same set of edges. First note that the minimal vertices of the tree components of F are exactly the roots of $f = \Psi(F)$, which then are the minimal elements of trees in $\Phi(f)$. Edges of F are of the form $\{v, v^p\}$, where v is not a minimal vertex in its tree component. We now show that when applying algorithm A to $\Psi(F)$, vertex v is found by v^p. Note that $f(v) = |\{v_j | \{v, v_j\} \in E(G), \pi^{-1}(v_j) < \pi^{-1}(v^p)\}|$. In the implementation of algorithm A, the *valuation* on v drops by 1 for each adjacent vertex that is processed before v. When it is v^p's turn to be processed, $val_i(v)$ drops from 0 to -1. Thus v^p finds v, and $\{v, v^p\}$ is an edge of $\Phi(\Psi(F))$.

□

Since the roots of the G-multiparking function correspond exactly to the minimal vertices in the tree components of the corresponding forest, in the following we will refer to those vertices as *roots of the forest*.

3 Examples of the bijections

The bijections $\Phi_{\gamma,G}$ and $\Psi_{\gamma,G}$, as defined above via algorithms A and B, allow a good deal of freedom in implementation. In algorithm A, as long as γ is well-defined at every iteration of Step 2, one can obtain val_{i+1}, P_{i+1}, Q_{i+1} and F_{i+1} and proceed. Recall that γ is a function from $\prod$, the set of ordered pairs (F, W), to $V(G)$ such that $\gamma(F, W) \in W$, where F is any sub-forest of G (not necessarily spanning) and W is a non-empty subset of $Leaf(F)$ or consists of an isolated point of F.

When restricting to G-parking functions, (i.e., G-multiparking functions with only one root), the descriptions of the bijections Φ and Ψ are basically the same as the ones given by Chebikin and Pylyavskyy [2], where the corresponding sub-structures in G are spanning trees. However our family of bijections, each defined on a choice function γ, is more general than the ones in [2], which rely on a *proper set of tree orders*. A proper set of tree orders is a set $\Pi(G) = \{\pi(T) : T \text{ is a subtree of } G\}$ of linear orders on the vertices of T, such that for any $v \in T$, $v <_{\pi(T)} v^p$, and if T' is a subtree of T containing the least vertex, $\pi(T')$ is a suborder of $\pi(T)$. Our algorithms do not require there to be a linear order on the vertices of each subtree. In fact, for a spanning tree T of a connected graph G, the proper tree order $\pi(T)$, if it exists, must be the same as the one defined in Step 1 of algorithm B. But in general, for two spanning trees T and T' with a common subtree t, the restrictions of $\pi(T)$ and $\pi(T')$ to vertices of t may not agree. Hence in general the choice function cannot be described in terms of proper sets of tree orders. In addition, our description of the map Φ, in terms of a dynamic process, provides a much clearer way to understand the bijection, and leads to a natural classification of the edges of G which plays an important role in connection with the Tutte polynomial (c.f.§4).

Different choice functions γ will induce different bijections between $\mathcal{MP}_G$ and $\mathcal{F}_G$. In this section we give several examples of choice functions that have combinatorial significance. In Example 1 we explain how to translate a proper set of tree orders into a choice function. Hence the family of bijections defined in [2] can be viewed as a subfamily of our bijections restricted to G-parking functions. The next three examples have appeared in [2]. We list them here for their combinatorial significance. Example 5 is the combination of breadth-first search with the Q-sets equipped with certain data structures. It is the one used to establish connections with Tutte polynomial in §4. The last example illustrates a case where γ cannot be expressed as a proper set of tree orders. We illustrate the corresponding map $\Phi_{\gamma,G}$ for examples 2–6 on the graph G in Figure 1. A G-multiparking function f is indicated by "$i/f(i)$" on vertices, where i is the vertex label.

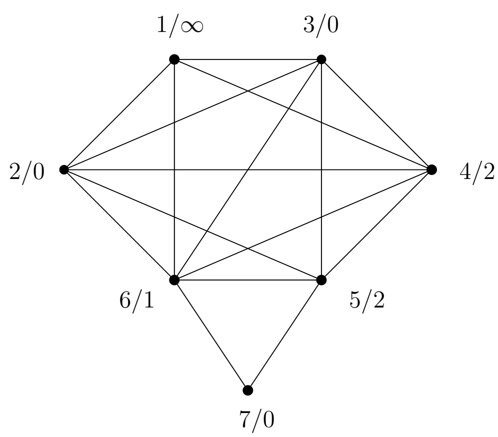


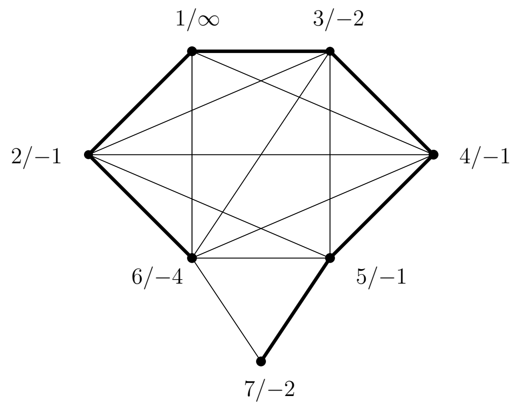
Figure 1: A graph and a multiparking function.

In each example, we will show the resulting spanning forest by darkened edges in G. Again each vertex will be labeled by a pair i/j, where i is the vertex labels, and $j = val_n(i)$, where $n = 7$. Beneath that, a table will record the sets Q_t and P_t for each time t. In each Q_t, the vertex listed first is the next to be processed.

Example 1. γ with a proper set of tree orders.
We define the choice function that corresponds to a proper set of tree orders. Here we should generalize to the proper set of forest orders, i.e., a set of orders $\pi(F)$, defined on the set of vertices for each subforest F of G, such that for any $v \in F$, $v <_{\pi(F)} v^p$, and if F' is a subforest of F with

the same minimal vertex in each tree component, $\pi(F')$ is a suborder of $\pi(F)$. In this case, define $\gamma(F, W) = v$ where v is the minimal element in W under the order $\pi(F)$. Examples 2–4 are special cases of this kind.

Example 2. γ with a given vertex ranking.
Given a vertex ranking $\sigma \in S_n$ define $\gamma_\sigma(F, W) := v$, where v is the vertex in W with minimal ranking. In particular, if σ is the identity permutation, then the vertex processing order is the *vertex-adding order* of [2]. In this case, in Step 2 of algorithm A, we choose v to be the least vertex in Q_{i-1} and process it at time i. The output of algorithm A is



The Q_i and P_i for this instance are as follows.

t	0	1	2	3	4	5	6	7
Q_t	{1}	{2,3}	{3,6}	{4,6}	{5,6}	{6,7}	{7}	$\emptyset$
P_t	$\emptyset$	{1}	{1,2}	{1,2,3}	{1,2,3,4}	{1,2,3,4,5}	{1,2,3,4,5,6}	{1,2,3,4,5,6,7}

Example 3. γ with *depth-first search order.*

The depth-first search order is the order in which vertices of a forest are visited when performing the depth-first search, which is also known as the preorder traversal. Given a forest F with tree components $T_1, T_2, \dots, T_k$, where $1 = r_1 < r_2 < \cdots < r_k$ are the corresponding roots, the order $<_{df}$ is defined as follows. (1) For any $v \in T_i$, $w \in T_j$ and $i < j$, $v <_{df} w$. (2) For any $v \neq r_i$, $v^p <_{df} v$. (3) If $v^p = w^p$ and $v < w$, $v <_{df} w$. (4) For any v, let $F[v]$ be the subtree of F rooted at v. If $v \in F[v']$, $w \in F[w']$ and $v' <_{df} w'$, then $v <_{df} w$. For example, the depth-first search order on the below tree is $1 <_{df} 2 <_{df} 3 <_{df} 6 <_{df} 4 <_{df} 5$.

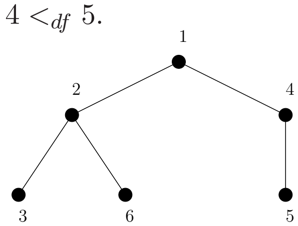
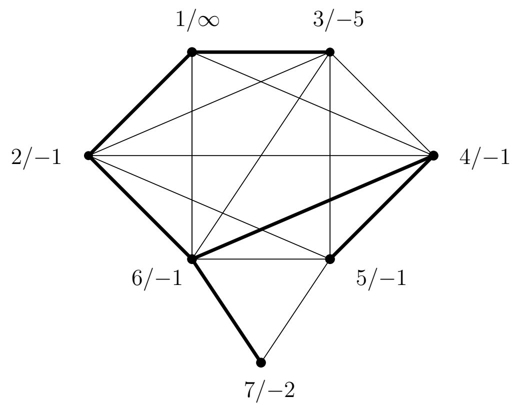


Figure 2: A tree with 6 vertices.

The choice function γ_{df} with depth-first search order is then defined as $\gamma_{df}(F, W) = v$ where v is the minimal element of W under the depth-first search order $<_{df}$ of F. Here is the output of algorithm A with the choice function γ_{df} on the example in Figure 1.

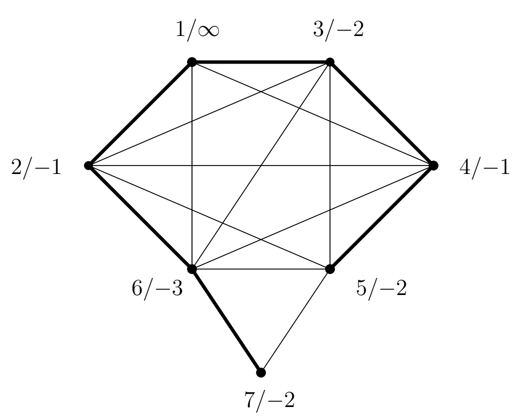


The Q_i and P_i for this instance are as follows.

t	0	1	2	3	4	5	6	7
Q_t	{1}	{2,3}	{6,3}	{4,3,7}	{5,3,7}	{7,3}	{3}	$\emptyset$
P_t	$\emptyset$	{1}	{1,2}	{1,2,6}	{1,2,4,6}	{1,2,4,5,6}	{1,2,4,5,6,7}	{1,2,3,4,5,6,7}

Example 4. γ with *breadth-first search order.*
Breadth-first search is another commonly used tree traversal in computer science. Given a forest F, whose tree components are T_i with roots r_i, $(1 \le i \le k)$, and $1 = r_1 < r_2 < \cdots < r_k$, the order $<_{bf}$ is defined as follows. (1) For any $v \in T_i$, $w \in T_j$ and $i < j$, $v <_{bf} w$. (2) Within tree T_i, for each $v \in T_i$, let *height* $h_{T_i}(v)$ of v be the number of edges in the unique path from v to the root r_i. We set $v <_{bf} w$ if $h_{T_i}(v) < h_{T_i}(w)$, or else if $h_{T_i}(v) = h_{T_i}(w)$ and $v < w$. For example, the the breadth-first search order for the tree in Figure 2 is $1 <_{bf} 2 <_{bf} 4 <_{bf} 3 <_{bf} 5 <_{bf} 6$.

The choice function γ_{bf} with breadth-first search order is defined as $\gamma_{bf}(F, W) = v$ where v is the minimal element of W under the breadth-first search order $<_{bf}$ of F. Here is the output of algorithm A with the choice function γ_{bf} on the example in Figure 1.



The Q_i and P_i for this instance are as follows.

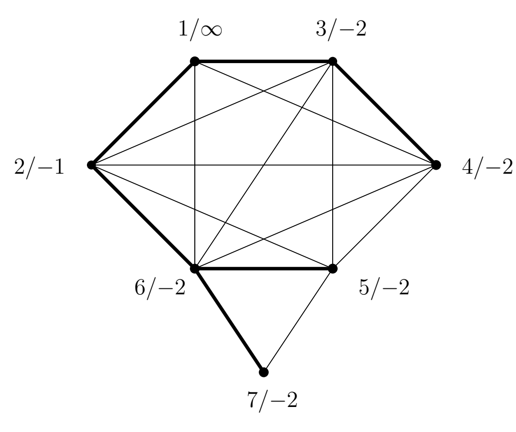
t	0	1	2	3	4	5	6	7
Q_t	{1}	{2,3}	{3,6}	{4,6}	{6,5}	{5,7}	{7}	$\emptyset$
P_t	$\emptyset$	{1}	{1,2}	{1,2,3}	{1,2,3,4}	{1,2,3,4,6}	{1,2,3,4,5,6}	{1,2,3,4,5,6,7}

Example 5. Breadth-first search with a data structure on Q_i.

In this case, new vertices enter the set Q_i in a certain order, and some intrinsic data structure on Q_i decides which vertex of Q_i is to be processed in the next step. A typical example is that of breadth-first search with a queue, in which case each Q_i is an ordered set, (i.e., the stage of a queue at time i). New vertices enter Q_i in numerical order, and γ chooses the vertex that entered the queue earliest.

This example can also be defined by a modified breadth-first search order, which we call *breadth-first order with a queue*, and denote by $<_{bf,q}$. Given a forest F, whose tree components are T_i with root r_i, $(1 \le i \le k)$, and $1 = r_1 < r_2 < \cdots < r_k$, the order $<_{bf,q}$ is defined as follows. (1) For any $v \in T_i$, $w \in T_j$ and $i < j$, $v <_{bf,q} w$. (2) Within tree T_i, the root r_i is minimal under $<_{bf,q}$. (3) $v <_{bf,q} w$ if $v^p <_{bf,q} w^p$. (4) If $v^p = w^p$ and $v < w$, $v <_{bf,q} w$. For example, the breadth-first search order with a queue for the tree in Figure 2 is $1 <_{bf,q} 2 <_{bf,q} 4 <_{bf,q} 3 <_{bf,q} 6 <_{bf,q} 5$.

The choice function γ associated with this order is denoted by $\gamma_{bf,q}$, and is used in §4. The following is the output of algorithm A with $\gamma_{bf,q}$ on the graph in Figure 1.



The Q_i and P_i for this instance are as follows, where each Q_i is an ordered set, and the first element in Q_i is the next one to be processed.

t	0	1	2	3	4	5	6	7
Q_t	(1)	(2,3)	(3,6)	(6,4)	(4,5,7)	(5,7)	(7)	$\emptyset$
P_t	$\emptyset$	{1}	{1,2}	{1,2,3}	{1,2,3,6}	{1,2,3,4,6}	{1,2,3,4,5,6}	{1,2,3,4,5,6,7}

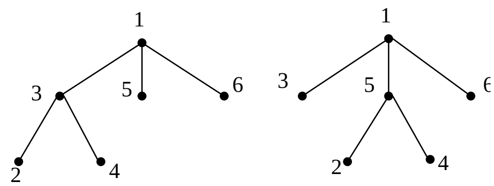
Another typical structure is to let Q_i be the stage of a stack at time i, that is, it pops out the vertex that last entered. We can also combine the other vertex orders with a queue or stack for the Q-sets.

Example 6. A choice function γ that cannot be defined by a proper set of tree orders.

Let

$$\gamma(F, W) = \begin{cases} x & \text{if } W = \{x\}, \\ \text{the second minimal vertex of } W, & \text{if } |W| \geq 2. \end{cases}$$

Then the order on the left tree is 156342, and the one on the right tree is 153462, which do not agree on the subtree consisting of vertices 1356. Hence it can not be defined via a proper set of tree orders.



Remark. Note that the bijection given in [2] applies to a general directed graph. We explain how our algorithms could be slightly modified to apply to that case, too. Let D denote a general directed graph on $[n]$. An oriented spanning forest F of D is a subgraph of G such that (1) the edges, when ignoring the orientation, do not form a cycle, and (2) for each v in a tree component with minimal vertex r, there is a unique directed path from v to r. Again we denote by v^p the vertex lying on the directed path from v to r with $(v, v^p) \in E(D)$. We say that the minimal vertex in each tree component of F is a root of F.

The definition of a D-multiparking function is the same as that of the undirected multiparking function, except that $outdeg_U(i)$ is the number of edges going from i to vertices not in U. Again we say the vertices v with $g(v) = \infty$ are the roots of the D-multiparking function g.

Spanning forests do not contain loops, as loops are a trivial kind of cycle. Hence we can assume D is loopless without loss of generality. We allow D to have multiple edges. But to distinguish between multiple edges of D, we fix a total order on the set of edges going from i to j, for each $i \neq j$.

The maps Φ and Ψ can be modified accordingly to give a bijection between the set of D-multiparking functions to the set of oriented spanning forests of D, which carry the roots of multiparking functions to the roots of spanning forests. The only modifications we need to make are:

For Algorithm A.

In Step 3, lower the value of w by 1 for each directed edge from w to v if w is not a root. Load w to Q_i whenever $val_i(w) < 0$. For each such w, add the $(k+1)$-st edge from w to v if $val_{i-1}(w) = k \geq 0$.

For Algorithm B.

In Step 1. Let W be the set $\{v \notin V_i : \exists w \in V_i \text{ such that } (v, w) \in E(D)\}$.

In Step 2. For any nonroot vertex v lying in the tree with root r_v, and $v \neq r_v$, set $f(v)$ to be $k + |\{v_j | (v, v_j) \in E(G), \pi^{-1}(v_j) < \pi^{-1}(v^p)\}|$ if the edge (v, v^p) in F is the $(k+1)$-st edge in the set of edges from v to v^p in D.

These modified algorithms for directed graphs cover the case for an undirected graph G, provided that one views G as a digraph, where each edge $\{u, v\}$ of G is replaced with two directed edges (u, v) and (v, u).

The notion of G-parking functions, as proposed in [18], is closely related to the critical configurations of the chip-firing games, (also known as sand-pile models). The generalization of chip-firing games with multiple sources is given in [6], where they are called *Dirichlet games*. In [12] the first author shows that modified G-multiparking functions (which in Condition **(A)**, instead of requiring that $i = min(U)$, one requires i to belong to a prefixed subset of vertices), are the corresponding counterpart for critical configurations of the Dirichlet games. In fact, both are in one-to-one correspondence with the set of rooted spanning forests of G, as well as a set of objects called descending multitraversals on G.

4 External activity and the Tutte polynomial

4.1 F-redundant edges

A forest F on $[n]$ may appear as a subgraph of different graphs, and a vertex function f may be a G-multiparking function for different graphs. In this section we characterize the set of graphs which share the same pair (F, f). Again let G be a simple graph on $[n]$, and fix a choice function γ. For a spanning forest F of G, let $f = \Psi_{\gamma,G}(F)$. We say an edge e of $G - F$ is *F-redundant* if $\Psi_{\gamma,G-\{e\}}(F) = f$. Note that we only need to use the value of γ on (F', W) where F' is a sub-forest of F. Hence $\Psi_{\gamma,G-\{e\}}(F)$ is well-defined.

Let π be the order defined in Step 1 of Algorithm B. Note that π only depends on F, not the underlying graph G. Recall that v^p denotes the parent vertex of vertex v in some spanning forest. We have the following proposition.

Proposition 4.1.
An edge $e = \{v, w\}$ of G is F-redundant if and only if e is one of the following types:

1. *Both v and w are roots of F.*
2. *v is a root and w is a non-root of F, and $\pi^{-1}(w) < \pi^{-1}(v)$.*
3. *v and w are non-roots and $\pi^{-1}(v^p) < \pi^{-1}(w) < \pi^{-1}(v)$. In this case v and w must lie in the same tree of F.*

Proof. We first show that each edge of the above three types are F-redundant. Since for any root r of the forest F, $f(r) = \infty$, the edges of the first two types play no role in defining the function f. And clearly those edges are not in F. Hence they are F-redundant.

For edge (v, w) of type 3, clearly it cannot be an edge of F. Since $f(v) = \#\{v_j | \{v, v_j\} \in E(G), \pi^{-1}(v_j) < \pi^{-1}(v^p)\}$, and $\pi^{-1}(w) > \pi^{-1}(v^p)$, removing the edge $\{v, w\}$ would not change the value of $f(v)$. This edges has no contribution in defining $f(u)$ for any other vertex u. Hence it is F-redundant.

For the converse, suppose that $e = \{v, w\}$ is not one of the three type. Assume w is processed before v in π. Then v is not a root, and w appears before v^p. Then removing the edge e will change the value of $f(v)$. Hence it is not F-redundant.

□

Let $R_1(G; F)$, $R_2(G; F)$, and $R_3(G; F)$ denote the sets of F-redundant edges of types 1, 2, and 3, respectively. Among them, $R_3(G; F)$ is the most interesting one, as $R_1(G; F)$ and $R_2(G; F)$ are a consequence of the requirement that $f(r) = \infty$ for any root r. Let $R(G; F)$ be the union of these three sets. Clearly the F-redundant edges are mutually independent, and can be removed one by one without changing the corresponding G-multiparking function. Hence

Theorem 4.2. *Let H be a subgraph of G with $V(H) = V(G)$. Then $\Psi_{\gamma,G}(F) = \Psi_{\gamma,H}(F)$ if and only if $G - R(G; F) \subseteq H \subseteq G$.*

4.2 A classification of the edges of G

The notion of F-redundancy allows us to classify the edges of a graph in terms of the algorithm A. Roughly speaking, the edges of any graph can be thought of as either lowering $val(v)$ for some v to 0, being in the forest, or being F-redundant. Explicitly, we have

Proposition 4.3.

Let f be a G-multiparking function and let $F = \Phi(f)$. Then

$$|E(G)| = \left(\sum_{v: f(v) \neq \infty} f(v) \right) + |E(F)| + |R(G; F)|.$$

Proof. For each non-root vertex v, the number of different values that $val_i(v)$ takes on during the execution of algorithm A is $f(v) + 1 + n_v$, where $n_v = -val_n(v)$. At the beginning, $val_0(v) = f(v)$. The value $val_i(v)$ then is lowered by one whenever there is a vertex w which is adjacent to v and processed before v^p. When v^p is being processed, $val_i(v) = -1$, and the edge $\{v^p, v\}$ contributes to the forest F. Afterward, the value of $val_i(v)$ decreases by 1 for each F-redundant edge $\{u, v\}$ with $\pi^{-1}(v^p) < \pi^{-1}(u) < \pi^{-1}(v)$. See Step 3 of algorithm A for details. Summing over all non-root vertices gives

$$\sum_{v: f(v) \neq \infty} deg_{<_\pi}(v) \quad = \quad \sum_{v: f(v) \neq \infty} f(v) + |E(F)| + \sum_{v: f(v) \neq \infty} n_v,$$

where $deg_{<_\pi}(v) = |\{\{w, v\} \in E(G)|\pi^{-1}(w) < \pi^{-1}(v)\}|$.

The edges that lower $val(v)$ below -1 are exactly the F-redundant edges of type (3) in Prop. 4.1, hence $\sum_{v:f(v)\neq\infty} n_v = |R_3(G; F)|$. On the other hand, $\sum_{v:f(v)\neq\infty} deg_{<_\pi}(v)$ is exactly $|E(G)| - |R_1(G; F)| - |R_2(G; F)|$. The claim follows from the fact that the sets $R_1(G; F), R_2(G; F)$, and $R_3(G; F)$ are mutually exclusive.

□

One notes that for roots of f and $F = \Phi(f)$, $|R_1(G; F)|+|R_2(G; F)|$ is exactly $\sum_{root\ v} deg_{<_\pi}(v)$, where π is the processing order in algorithm A. But it is not necessary to run the full algorithm A to compute $|R_1(G; F)| + |R_2(G; F)|$. Instead, we can apply the burning algorithm in a greedy way to find an ordering $\pi' = v'_1 v'_2 \cdots v'_n$ on $V(G)$: Let $v'_1 = 1$. After determining $v'_1, \ldots, v'_{i-1}$, if $V_i = V(G) - \{v'_1, \ldots, v'_{i-1}\}$ has a well-behaved vertex, let v'_i be one of them; otherwise, let v'_i be the minimal vertex of V_i, (which has to be a root.)

π' may not be the same as π, but they have the following properties:

1. Let $r_1 < r_2 < \cdots < r_k$ be the roots of f. Then $r_1, r_2, \ldots, r_k$ appear in the same positions in both π and π'.

2. The set of vertices lying between r_i and r_{i+1} are the same in π and π'. In fact, they are the vertices of the tree T_i with root r_i in $F = \Phi(f)$.

It follows that for any root vertex v, $deg_{<_\pi}(v) = deg_{<_{\pi'}}(v)$. The value of $deg_{<_\pi}(v)$ (v root) can be characterized by a global description: Let $\mathcal{U}_v$ be the collection of subsets U of $V(G)$ such that $v = \min(U)$, and U does not have a well-behaved vertex. $\mathcal{U}_v$ is nonempty for a root v since $U = \{v\}$ is such a set. Then

$$deg_{<_\pi}(v) = \min_{U\in\mathcal{U}_v} outdeg_U(v).$$

We call $deg_{<_\pi}(v)$ the *record* of the root v, and denote it by $rec(v)$. Then

$$|R_1(G; F)| + |R_2(G; F)| = \sum_{root\ v} deg_{<_{\pi'}}(v) = \sum_{root\ v} rec(v)$$

is the *total root records*. Let $Rec(f) = |R_1(G; F)|+|R_2(G; F)|$. It is the number of F-redundant edges adjacent to a root. By the above greedy burning algorithm, the total root records $Rec(f)$ can be computed in linear time.

4.3 A new expression for Tutte polynomial

In this subsection we relate G-multiparking functions to the Tutte polynomial $t_G(x, y)$ of G. We follow the presentation of [9] for the definition of Tutte polynomial and its basic properties. Although the theory works for general graphs with multiedges, we assume G is a simple connected graph to simplify the discussion. There is no loss of generality by assuming connectedness, since for a disconnected graph, $t_G(x, y)$ is just the product of the Tutte polynomials of the components of G. We restrict ourselves to connected graphs to avoid any possible confusion when we consider their spanning forests. The modification when G has multiple edges is explained at the end of §3.

Suppose we are given G and a total ordering of its edges. Consider a spanning tree T of G. An edge $e \in G - T$ is *externally active* if it is the largest edge in the unique cycle contained in $T \cup e$. We let

$$\mathcal{EA}(T) = \text{set of externally active edges in } T$$

and $ea(T) = |\mathcal{EA}(T)|$. An edge $e \in T$ is *internally active* if it is the largest edge in the unique cocycle contained in $(G - T) \cup e$ (A *cocycle* is a minimal edge cut). We let

$$\mathcal{IA}(T) = \text{set of internally active edges in } T$$

and $ia(T) = |\mathcal{IA}(T)|$. Tutte [21] then defined his polynomial as

$$t_G(x, y) = \sum_{T \subset G} x^{ia(T)} y^{ea(T)}, \tag{1}$$

where the sum is over all spanning trees T of G. Tutte showed that t_G is well-defined, i.e., independent of the total ordering of the edges of G. Henceforth, we will not assume that the edges of G are ordered.

Let H be a (spanning) subgraph of G. Denote by $c(H)$ the number of components of H. Define two invariants associated with H as

$$\sigma(H) = c(H) - 1, \qquad \sigma^*(H) = |E(H)| - |V(G)| + c(H). \tag{2}$$

The following identity is well-known, for example, see [1].

Theorem 4.4.

$$t_G(1 + x, 1 + y) = \sum_{H \subseteq G} x^{\sigma(H)} y^{\sigma^*(H)}, \tag{3}$$

where the sum is over all spanning subgraphs H of G.

Recall that the *breadth-first search* (BFS) is an algorithm that gives a spanning forest in the graph H. Assume $V(G) = [n]$. We will use our favorite description to express the BFS as a queue Q that starts at the least vertex 1. This description was first introduced in [19] to develop an exact formula for the number of labeled connected graphs on $[n]$ with a fixed number of edges, and was used by the second author in [23] to reveal the connection between the classical parking functions (resp. k-parking functions) and the complete graph (resp. multicolored graphs).

Given a subgraph H of G with $V(H) = V(G) = [n]$, we construct a queue Q. At time 0, Q contains only the vertex 1. At each stage we take the vertex x at the head of the queue, remove x from the queue, and add all unvisited neighbors $u_1, \ldots, u_{t_x}$ of x to the queue, in numerical order. We will call this operation "processing x". If the queue becomes empty, add the least unvisited vertex to Q. The output F is the forest whose edge set consists of all edges of the form $\{x, u_i\}$ for $i = 1, \ldots t_x$. We will denote this output as $F = BFS(H)$. Figure 3 shows the spanning forest found by BFS for a graph G.

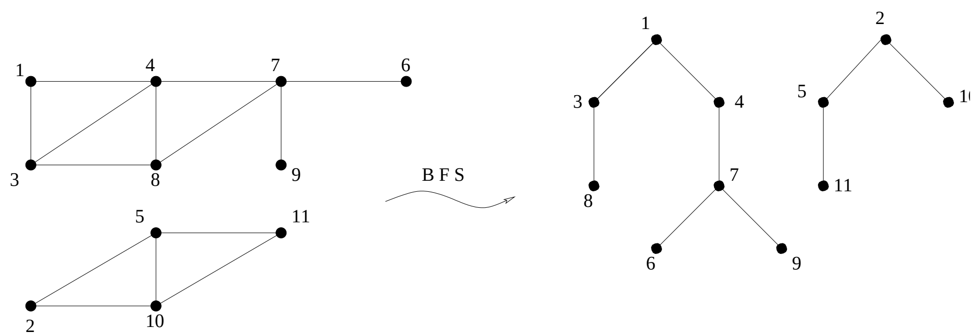


Figure 3: Spanning forest found by BFS.

The queue Q for Figure 3 is

t	0	1	2	3	4	5	6	7	8	9	10	11
Q	(1)	(3,4)	(4,8)	(8,7)	(7)	(6,9)	(9)	(2)	(5,10)	(10,11)	(11)	$\emptyset$

For a spanning forest F of G, let us say that an edge $e \in G - F$ is *BFS-externally active* if $BFS(F \cup e) = F$. A crucial observation is made by Spencer [19]: An edge $\{v, w\}$ can be added to F without changing the spanning forest under the BFS if and only if the two vertices v and w have been present in the queue at the same time. In our example of Figure 3, edges $\{3,4\}, \{4,8\}, \{7,8\}, \{6,9\}, \{5,10\}$ and $\{10,11\}$ could be added back to F. We write $\mathcal{E}(F)$ for the set of BFS-externally active edges.

Proposition 4.5 (Spencer)**.** *If H is any subgraph and F is any spanning forest of G then*

$$BFS(H) = F \textit{ if and only if } F \subseteq H \subseteq F \cup \mathcal{E}(F).$$

Now consider the Tutte polynomial. Note that if $BFS(H) = F$, then $c(H) = c(F)$. So $\sigma(H) = c(F) - 1$ and $\sigma^*(H) = |E(H)| - |E(F)| = |\mathcal{E}(F) \cap H|$. Hence if we fix a forest F and sum over the corresponding interval $[F, F \cup \mathcal{E}(F)]$, we have

$$\sum_{H:BFS(H)=F} x^{\sigma(H)} y^{\sigma^*(H)} = x^{c(F)-1} \sum_{A \subseteq \mathcal{E}(F)} y^{|A|} = x^{c(F)-1}(1+y)^{|\mathcal{E}(F)|}.$$

Summing over all forests F, we get

$$t_G(1+x,1+y) = \sum_{H\subseteq G} x^{\sigma(H)} y^{\sigma^*(H)} = \sum_{F\subseteq G} x^{c(F)-1}(1+y)^{|\mathcal{E}(F)|}.$$

Or, equivalently,

$$t_G(1+x,y) = \sum_{F\subseteq G} x^{c(F)-1} y^{|\mathcal{E}(F)|}. \tag{4}$$

To evaluate $\mathcal{E}(F)$, note that when applying BFS to a graph H, the queue Q only depends on the spanning forest $F = BFS(H)$. Given a forest F, the processing order in Q is a total order $<_Q=<_Q(F)$ on the vertices of F satisfying the following condition: Let $T_1, T_2, \ldots, T_k$ be the tree components of F with minimal elements $r_1 = 1 < r_2 < \cdots < r_k$. Then (1) If v is a vertex in tree T_i, w is a vertex in tree T_j and $i < j$, then $v <_Q w$. (2) Among vertices of each tree T_i, r_i is minimal in the order $<_Q$. (3) For two non-root vertices v, w in the same tree, $v <_Q w$ if $v^p <_Q w^p$. In the case $v^p = w^p$, $v <_Q w$ whenever $v < w$.

Comparing with the examples in §3, we note that $<_Q$ is exactly the order $<_{bf,q}$ described in Example 5 of §3, as *breadth-first order with a queue*. Fix the choice function $\gamma = \gamma_{bf,q}$, the one associated to $<_{bf,q}$ and consider the maps $\Phi_{\gamma,G}$ and $\Psi_{\gamma,G}$. Given F, the condition that two vertices v, w have been present at the queue Q at the same time when applying BFS to F is equivalent to $v^p <_{bf,q} w <_{bf,q} v$ or $w^p <_{bf,q} v <_{bf,q} w$. That is, an edge is BFS-externally active if and only if it is an F-redundant edge of type 3, as defined in §4.1. It follows that $\mathcal{E}(F) = R_3(G;F)$.

Therefore by Prop. 4.3 and the equation at the end of §4.2,

$$|\mathcal{E}(F)| = |R_3(G;F)| = |E(G)| - |E(F)| - \left(\sum_{v:f(v)\neq\infty} f(v)\right) - Rec(f), \tag{5}$$

where $f = \Psi_{\gamma,G}(F)$ is the corresponding G-multiparking function. Note that $|E(F)| = n - c(F)$, and $c(F) = r(f)$, where $r(f)$ is the number of roots of f. Therefore

Theorem 4.6.

$$t_G(1+x,y) = y^{|E(G)|-n} \sum_f x^{r(f)-1} y^{r(f)-Rec(f)-\left(\sum_{v:f(v)\neq\infty} f(v)\right)},$$

where the sum is over all G-multiparking functions.

For a G-multiparking function f, where G is a graph on n vertices, we call the statistics $|E(G)| - n + r(f) - Rec(f) - \sum_{v:f(v)\neq\infty} f(v)$ the *reversed sum* of f, denote by $rsum(f)$. The name comes from the corresponding notation for classical parking functions, see, for example, [15]. Theorem 4.6 expresses Tutte polynomial in terms of generating functions of $r(f)$ and $rsum(f)$. In [9] Gessel and Sagan gave a similar expression, in terms of $\mathcal{E}_{DFS}(F)$, the set of *greatest-neighbor* *externally active* edges of F, which is defined by applying the greatest-neighbor depth-first search on subgraphs of G. Combining the result of [9] (Formula 5), we have

$$xt_G(1+x,y) = \sum_{F\subseteq G} x^{c(F)} y^{|\mathcal{E}_{DFS}(F)|} = \sum_{F\subseteq G} x^{c(F)} y^{|\mathcal{E}(F)|} = \sum_{f\in\mathcal{MP}_G} x^{r(f)} y^{rsum(f)}. \tag{6}$$

That is, the three pairs of statistics, $(c(F), |\mathcal{E}_{DFS}(F)|)$ and $(c(F), |\mathcal{E}(F)|)$ for spanning forests, and $(r(f), rsum(f))$ for G-multiparking functions, are equally distributed.

Remark. Alternatively, one can prove Theorem 4.6 by conducting *neighbors-first search* (NFS), a tree traversal defined in [9, §6], and using $\gamma = \gamma_{df}$, the choice function associated with the depth-first search order, (c.f. Example 3, §3). Here the NFS is another algorithm that builds a spanning forest F given an input graph H. The following description is taken from [9].

NFS1 Let $F = \emptyset$.

NFS2 Let v be the least unmarked vertex in V and mark v.

NFS3 Search v by marking all neighbors of v that have not been marked and adding to F all edges from v to these vertices.

NFS4 Recursively search all the vertices marked in NFS3 in increasing order, stopping when every vertex that has been marked has also been searched.

NFS5 If there are unmarked vertices, then return to NFS2. Otherwise, stop.

The NFS searches vertices of H in a depth-first manner but marks children in a locally breadth-first manner. Figure 4 shows the result of NFS, when applies to the graph on the left of Figure 3.

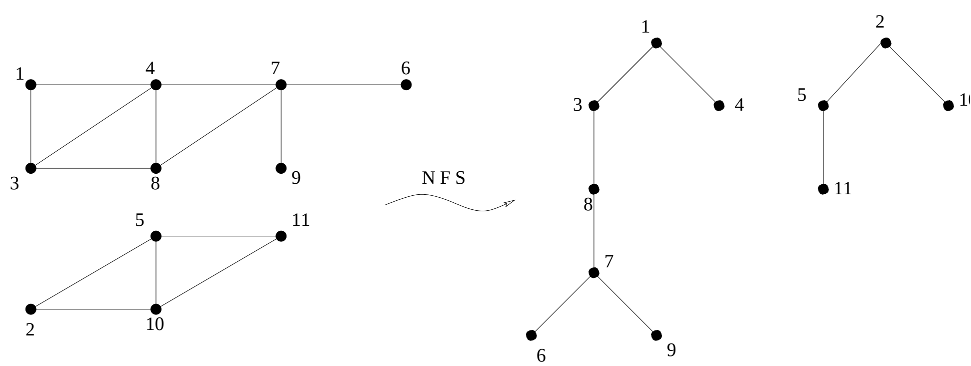


Figure 4: Spanning forest found by NFS.

Similarly, one defines $\mathcal{E}_{NFS}(F)$, the set of edges *externally active with respect to NFS*, to be those edges $e \in G - F$ such that $NFS(F \cup e) = F$. Then Prop. 4.5 and Eq. (4) hold again when we replace BFS with NFS, and $\mathcal{E}(F)$ with $\mathcal{E}_{NFS}(F)$.

Now let $\gamma = \gamma_{df}$ and use the bijections $\phi_{\gamma_{df},G}$ and $\Psi_{\gamma_{df},G}$, one notices again that an edge is externally active with respect to NFS if and only if it is F-redundant of type 3. And hence we get another proof of Theorem 4.6.

An interesting specialization of Theorem 4.6 is to consider $t_G(1, y)$, the restriction to spanning trees of G and G-parking functions. For a G-parking function f, or equivalently a G-multiparking function with exactly one root (which is vertex 1), $r(f) = 1$ and $Rec(f) = 0$. Hence $rsum(f) = |E(G)| - n + 1 - \sum_{v:f(v)\neq\infty} f(v)$. Thus we obtain

$$t_G(1, y) = \sum_{f\,:\,G\text{-parking functions}} y^{rsum(f)}.$$

An equivalent form of this result, in the language of sand-pile models, was first proved by López [16] using a recursive characterization of Tutte polynomial. A bijective proof was given by Cori and Le Borgne in [3] by constructing a one-to-one correspondence between trees with external activity i (in Tutte's sense) to recurrent configurations of level i, which is equivalent to G-parking functions with reversed sum i. Our treatment here provides a new bijective proof.

In [9] it is shown that, restricted to simple graphs, the greatest-neighbor externally active edges of F are in one-to-one correspondence with certain inversions of F. For a simple graph G with a spanning forest F, view each tree T of F as rooted at its smallest vertex. An edge $\{u, v\}$ is greatest-neighbor externally active if and only if v is a descendant of u, and $w > v$ where w is the child of u on the unique $u - v$ path in F, (that is, $u = w^p$). Call such a pair $\{w, v\}$ a G-inversion. And denoted by $Ginv(F)$ the number of G-inversions of the forest F. Then we have the following corollary.

Corollary 4.7. *Let $\mathcal{F}_k(G)$ be the set of spanning forests of G with exactly k tree components. And $\mathcal{MP}_k(G)$ be the set of G-multiparking functions with k roots. Then*

$$\sum_{F\in\mathcal{F}_k(G)} y^{Ginv(F)} = \sum_{f\in\mathcal{MP}_k(G)} y^{rsum(f)}.$$

In particular, when G is the complete graph K_{n+1} and $k = 1$, we have the well-known result on the equal-distribution of inversions over labeled trees, and the reversed sum over all classical parking functions of length n, (for example, see [13, 20])

$$\sum_{T \text{ on } [n+1]} y^{inv(T)} = \sum_{\alpha\in P_n} y^{\binom{n}{2} - \sum_{i=1}^{n} \alpha_i},$$

where P_n is the set of all (classical) parking functions of length n.

5 Enumeration of G-multiparking functions and graphs

In this section we discuss some enumerative results on G-multiparking functions and substructures of graphs.

Theorem 5.1. *The number of G-multiparking functions with k roots equals the number of spanning forests of G with k components. In particular, for connected graph G, the number of G-multiparking functions is $t_G(2,1)$. Among them, those with an odd number of roots is counted by $\frac{1}{2}(t_G(2,1)+t_G(0,1))$, and those with an even number of roots is counted by $\frac{1}{2}(t_G(2,1)-t_G(0,1))$.*

Proof. The first two sentences follow directly from the bijections constructed in §2, and Theorem 4.6. For the third sentence, just note that $t_G(0,1)=\sum_f(-1)^{r(f)-1}$ is the difference between the number of G-multiparking functions with an odd number of roots, and those with an even number of roots. $\square$

Another consequence of Theorem 4.6 and its proof is an expression for the number of spanning subgraphs with a fixed number of components and fixed number of edges, in terms of (BFS)-external activity and G-multiparking functions. It is a generalization of the expectation formula in [19], which is the special case for complete graph K_n.

Theorem 5.2. *Let G be a connected graph. The number $\gamma_{t,k}(G)$ of spanning subgraphs H with t components and $V(G)-1+k$ edges is given by*

$$\gamma_{t,k}(G)=\sum_{F\in\mathcal{F}_t}\binom{\mathcal{E}(F)}{k}=\sum_{f\in\mathcal{MP}_t(G)}\binom{rsum(f)}{k},$$

where the first sum is over all spanning forests with t components, and the second sum is over all G-multiparking functions with t roots.

Proof. For any spanning forest F with k components, the number of spanning subgraphs H with $V(G)-1+k$ edges such that $BFS(H)=F$ is given by $\binom{\mathcal{E}(F)}{k}$. $\square$

Next we give a new expression of the $t_{K_{n+1}}(x,y)$ in terms of classical parking functions. It enumerates the classical parking functions by the number of *critical left-to-right maxima*. Given a classical parking function $\mathbf{b}=(b_1,\ldots,b_n)$, we say that a term $b_i=j$ is *critical* if in $\mathbf{b}$ there are exactly j terms less than j, and exactly $n-1-j$ terms larger than j. For example, in $\mathbf{b}=(3,0,0,2)$, the terms $b_1=3$ and $b_4=2$ are critical. Among them, only $b_1=3$ is also a left-to-right maximum.

Let $\alpha(\mathbf{b})$ be the number of critical left-to-right maxima in a classical parking function $\mathbf{b}$. We have

Theorem 5.3.

$$t_{K_{n+1}}(x,y)=\sum_{\mathbf{b}\in P_n}x^{\alpha(\mathbf{b})}y^{\binom{n}{2}-\sum_i b_i},$$

where P_n is the set of classical parking functions of length n.

Proof. Let F be a spanning forest on $[n+1]$ with tree components $T_1,\ldots,T_k$, where T_i has minimal vertex r_i, and $r_1<r_2<\cdots<r_k$. We define an operation $merge(F)$ which combines the trees

$T_1, \ldots, T_k$ by adding an edge between r_i with w_{i-1} for each $i = 2, ..., k$, where w_{i-1} is the vertex of T_{i-1} that is maximal under the order $<_{bf,q}$. Denote by $T_F = merge(F)$ the resulting tree. We observe that for the forest F and the tree T_F, the queue obtained by applying BFS are exactly the same. This implies that F and T_F have the same set of BFS-externally active edges, that is,

$$\mathcal{E}(F) = \mathcal{E}(T_F). \tag{7}$$

Conversely, given T and an edge $e = \{w, v\} \in T$ where $w <_{bf,q} v$. We say the edge e is *critical* in T if $merge(T \setminus \{e\}) = T$. Assume $T \setminus \{e\} = T_1 \cup T_2$ where $w \in T_1$ and $v \in T_2$. By the definition of the merge operation, e is critical if and only if w is the maximal in T_1 under the order $<_{bf,q}$, and v is vertex of the lowest index in T_2. In terms of the queue obtained by applying BFS to T, it is equivalent to the following two conditions: (1) There is an index i such that $Q_i = \{v\}$, and v does not belong to any other Q_j. (2) v is of minimal index among the set of vertices processed after v.

Consider the maps $\Phi_{\gamma,G}$ and $\Psi_{\gamma,G}$ with $\gamma = \gamma_{bf,q}$ and $G = K_{n+1}$. Let $f = \Psi_{\gamma,G}(T)$, and write f as a sequence $(f(2), f(3), \ldots, f(n+1))$. (There is no need to record $f(1)$, as $f(1) = \infty$ always.) Then an edge $\{w, v\}$ is critical in T if and only if (1') $f(v)$ is critical in the sequence $(f(2), \ldots, f(n+1))$, and (2') $w > v$ for any vertex w with $f(w) > f(v)$. That is, $f(v)$ is a left-to-right maximum in the sequence $(f(2), f(3), \ldots, f(n+1))$. Here the condition (2') follows from the fact that in K_{n+1}, $f(w) > f(v)$ means w enters the queue after v. Hence w is processed after v, and must be larger than v.

Now fix a spanning tree T of K_{n+1} and let $Merge(T)$ be the set of spanning forests F such that $merge(F) = T$. Then an $F \in Merge(T)$ can be obtained from T by removing any subset A of critical edges, in which case $c(F) = c(T) + |A| = 1 + |A|$. This, combined with the equation (7) ($\mathcal{E}(F) = \mathcal{E}(T)$), gives us

$$\sum_{F \in Merge(T)} x^{c(F)-1} y^{|\mathcal{E}(F)|} = y^{|\mathcal{E}(T)|} \sum_{A} x^{|A|}, \tag{8}$$

where A ranges over all subsets of critical edges of T. Under the correspondence $T \to f = \Psi_{\gamma,G}(T)$ and considering f as a sequence $(f(2), \ldots, f(n+1))$, $|\mathcal{E}(T)|$ is just $\binom{n}{2} - \sum_{i=2}^{n+1} f(i)$, (by Equation (5) and $Rec(f) = 0$ for complete graph), and critical edges of T correspond to critical left-to-right maxima of the sequence. Hence the sum in (8) equals

$$y^{|\mathcal{E}(T)|}(1+x)^{\alpha(f_T)} = (1+x)^{\alpha(f_T)} y^{\binom{n}{2} - \sum_{i=2}^{n+1} f(i)}.$$

Theorem 5.3 follows by summing over all trees on $[n+1]$. □

Finally, we use the breadth-first search to re-derive the formula for the number of subdigraphs of G, which was first proved in [9] using DFS, and extend the method to derive a formula for the number of subtraffics of G.

Let G be a graph. A *directed subgraph* or *subdigraph* of G is a digraph D that contains up to one copy of each orientation of every edge of G. Here for an edge $\{u, v\}$ of G we permit both (u, v) and (v, u) to appear in a subdigraph.

For any subdigraph D of G, we apply the BFS to get a spanning forest of D. The only difference from the subgraph case is that when processing a vertex x, we only add those unvisited vertices u such that (x, u) is an edge of D.

If digraph D has BFS forest F, write $\vec{\mathcal{F}}^+(D) = F$. Note that we can view F as an oriented spanning forest, where each edge is pointing away from the root (i.e., the minimal vertex) of the underlying tree component. We say a directed edge $\vec{e} \notin F$ is *directed BFS externally active* with respect to F if $\vec{\mathcal{F}}^+(F \cup \vec{e}) = F$. Denote by $\mathcal{E}^+(F)$ the set of directed BFS-externally active edges. Then we have the following basic proposition, which is the analog in the undirected case.

Proposition 5.4. *If D is any subdigraph and F is any spanning forest of G then*

$$\vec{\mathcal{F}}^+(D) = F \text{ if and only if } F \subseteq D \subseteq F \cup \mathcal{E}^+(F).$$

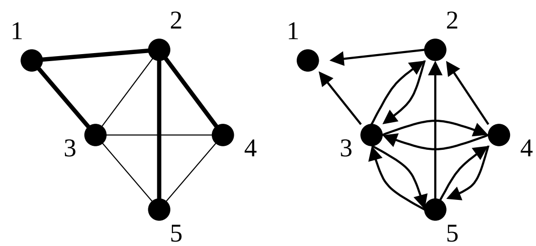
Now we characterize the directed BFS-externally active edges by the set $\mathcal{E}(F)$, the BFS-externally active edges for the undirected graph G. Let $\{u, v\}$ be an edge of G with $u <_{bf,q} v$. If $\{u, v\} \in E(F)$, then the backward edge (v, u) can be added without changing the result of (directed) breadth-first search, that is, $(v, u) \in \mathcal{E}^+(F)$. There are two types of edges in $\mathcal{E}^+(F)$.

1. If $\{u, v\} \in \mathcal{E}(F)$, then both (u, v) and (v, u) are in $\mathcal{E}^+(F)$.
2. If $\{u, v\}$ is not in the forest F or $\mathcal{E}(F)$, then (v, u) is in $\mathcal{E}^+(F)$.

Together we have

$$|E(G)| = |\mathcal{E}^+(F)| - |\mathcal{E}(F)|.$$

Example 7. The illustration below and left is of a graph G and a spanning forest F (whose edges are in bold). On the right is the set $\mathcal{E}^+(F)$, represented as a subdigraph. In the right illustration, there are four pairs of vertices between which there is only one directed edge (these correspond to case 1 above) and four pairs of vertices between which there are two directed edges (these correspond to case 2 above). It is easy to verify that the formula above holds in this case: $|E(G)| = 8 = 12 - 4 = |\mathcal{E}^+(F)| - |\mathcal{E}(F)|$.



From this characterization, we have

Theorem 5.5. *If G has n vertices, then*

$$\sum_D x^{c(D)} y^{|E(D)|} = xy^{n-1}(1+y)^{|E(G)|}\ t_G(1+\frac{x}{y}, 1+y), \tag{9}$$

where the sum is over all subdigraphs of G.

Proof.

$$\begin{aligned}
\sum_D x^{c(D)} y^{|E(D)|} &= \sum_F \sum_{D:\vec{\mathcal{F}}^+(D)=F} x^{c(D)} y^{|E(D)|} \\
&= \sum_F x^{c(F)} y^{|E(F)|}(1+y)^{|\mathcal{E}^+(F)|} \\
&= y^n (1+y)^{|E(G)|} \sum_F \left(\frac{x}{y}\right)^{c(F)} (1+y)^{|\mathcal{E}(F)|} \\
&= xy^{n-1}(1+y)^{|E(G)|}\ t_G(1+\frac{x}{y}, 1+y).
\end{aligned}$$

□

Next we consider a slightly complicated problem. A *subtraffic* K of G, where K is a partially directed graph on $V(G)$, is obtained from G by replacing each edge $\{u,v\}$ of G by (a) $\emptyset$, (b) a directed edge (u,v), (c) a directed edge (v,u), (d) two directed edges (u,v) and (v,u), or (e) an undirected edge $\{u,v\}$. We proceed as we did before. For each subtraffic K, we apply the directed breadth-first search to get a spanning forest F: The queue starts with the minimal vertex 1. At each iteration, we take the vertex x at the head of the queue, remove x from the queue, and add all unvisited vertices u if $(x,u) \in E(K)$ or $\{x,u\} \in E(K)$. Add the directed edge (x,u) to the forest F if $(x,u) \in E(K)$. Otherwise, add the undirected edge $\{x,u\}$ to F. The output is a forest $[n]$ in which each edge is either a directed edge oriented away from the minimal vertex of the underlying tree, or an undirected edge. Let A be the set of directed edges. Denote by (F,A) the output forest and write $BFS(K) = (F,A)$. Note that (F,A) is itself a subtraffic of G.

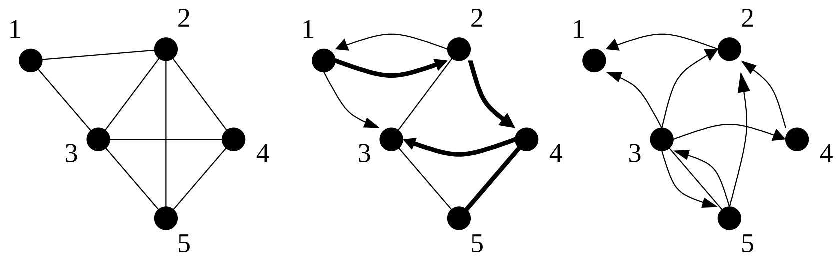
Given a pair (F,A) with directed edges $A \subseteq E(F)$, we have the following characterization of edges that can be added to (F,A), without changing the BFS result, (i.e., $BFS((F,A)\cup e) = BFS(F,A)$.)

1. For each directed edge (u,v) in $A \subseteq E(F)$, we can add back (v,u) without changing the result of the spanning forest.

2. For each BFS-externally active edge $\{u,v\}$ of F, we can add back any one of $(u,v), (v,u)$ and $\{u,v\}$, or both (u,v) and (v,u) at the same time.

3. For each edge not in $F \cup \mathcal{E}(F)$, we can add back the direct edge (u,v) if u is processed after v in the queue.

Example 8. The leftmost illustration below is of a graph G, the middle illustration is of a subtraffic K of G with the darkened edges forming a spanning forest (and the edge $\{2,5\}$ is an $\emptyset$-edge in K), and the rightmost illustration is the graph whose edges set is $\{e \mid BFS((F,A)\cup e) = BFS(F,A)\}$. In this rightmost graph, note that the edges $(2,1)$, $(4,2)$, and $(3,4)$ are of type 1, edges $\{3,5\}$, $(3,5)$, and $(5,3)$ are of type 2, and $(3,1)$, $(3,2)$, and $(5,2)$ are of type 3.



There is no further restriction on how the edges can be added back in addition to the above mentioned cases. Then we have

Theorem 5.6. *Let G be a connected graph. Then*

$$\sum_K x^{c(K)} y^{|E(K)|} = x(y^2+2y)^{n-1}(1+y)^{|E(G)|-n+1}\ t_G(1+\frac{x(1+y)}{y(2+y)}, \frac{1+3y+y^2}{1+y}), \tag{10}$$

where the sum is over all subtraffics of G.

Proof.

$$\sum_K x^{c(K)} y^{|E(K)|} = \sum_F \sum_{A\subseteq E(F)} \sum_{K:BFS(K)=(F,A)} x^{c(K)} y^{|E(K)|},$$

where F is over all spanning forests of G, and A is a subset of the edges of F. A subtraffic K has $BFS(K) = (F,A)$ if and only if it is obtained from F by adjoining some edges as described in the preceding three cases. Considering the contribution of each type, we have

$$\begin{aligned}\sum_{K:BFS(K)=(F,A)} x^{c(K)} y^{|E(K)|} &= x^{c(F)} y^{|E(F)|}(1+y)^{|A|}(1+3y+y^2)^{|\mathcal{E}(F)|}(1+y)^{|E(G)|-|E(F)|-|\mathcal{E}(F)|}\\ &= x^{c(F)}\left(\frac{y}{1+y}\right)^{|E(F)|}(1+y)^{|E(G)|}(1+y)^{|A|}\left(\frac{1+3y+y^2}{1+y}\right)^{|\mathcal{E}(F)|}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_K x^{c(K)} y^{|E(K)|} &= \sum_F x^{c(F)}\left(\frac{y}{1+y}\right)^{|E(F)|}(1+y)^{|E(G)|}\left(\frac{1+3y+y^2}{1+y}\right)^{|\mathcal{E}(F)|}\sum_{A\subseteq E(F)}(1+y)^{|A|}\\ &= \sum_F x^{c(F)}\left(\frac{y}{1+y}\right)^{|E(F)|}(1+y)^{|E(G)|}\left(\frac{1+3y+y^2}{1+y}\right)^{|\mathcal{E}(F)|}(2+y)^{|E(F)|}\\ &= \left(\frac{y(2+y)}{1+y}\right)^n (1+y)^{|E(G)|}\sum_F\left(\frac{x(1+y)}{y(2+y)}\right)^{c(F)}\left(\frac{1+3y+y^2}{1+y}\right)^{|\mathcal{E}(F)|}\\ &= x(y^2+2y)^{n-1}(1+y)^{|E(G)|-n+1}\ t_G\left(1+\frac{x(1+y)}{y(2+y)}, \frac{1+3y+y^2}{1+y}\right).\end{aligned}$$

$\square$

It is easy to see that the number of subtraffics on G is $5^{|E(G)|}$. Using this fact and evaluating equation 10 at $x = y = 1$, we derive the equality $3^{n-1}2^{|E(G)|-n+1}t_G(\frac{5}{3}, \frac{5}{2}) = 5^{|E(G)|}$. Evaluating the equation at $x = 0, y = 1$ proves that the number of connected subtraffics on G is $3^{n-1}2^{|E(G)|-n+1}t_G(1, \frac{5}{2})$.

Acknowledgments

The authors thank Robert Ellis and Jeremy Martin for helpful discussions and comments. We also thank Ira Gessel for useful comments on Tutte polynomials, and for sharing the unpublished portion of a preprint of [9] with us. Finally, we are grateful to William Chen and an anonymous referee for catching a number of mistakes and for thoughtful suggestions on the presentation of the paper.

References

[1] Biggs, N. *Algebraic Graph Theory.* 2^{nd} ed, Cambridge University Press, 1993.

[2] Chebikin, D. and Pylyavskyy, P. *A Family of Bijections Between G-Parking Functions and Spanning Trees.* Journal of Combinatorial Theory A **110** (2005), no. 1, 31-41.

[3] Cori, R. and Le Borgne, Y. *The Sand-Pile Model and Tutte Polynomials.* Advances in Applied Mathematics **30** (2003) 44-52.

[4] Cori, R. and Poulalhon, D. *Enumeration of (p, q)-Parking Functions.* Discrete Math **256** (2002) 609-623.

[5] Crapo, H. H. The Tutte polynomial, *Aequationes Math.* **3**(1969), 211–229.

[6] Ellis, R. *Chip-Firing Games with Dirichlet Eigenvalues and Discrete Green's Functions.* Ph.D. Thesis at University of California, San Diego, 2002. Available at `http://math.iit.edu/~rellis/papers/thesis.pdf`

[7] Dhar, D. *Self-organized critical state of the sandpile automaton models*, Physical Review Letters **64**(1990), no. 14, 1613–1616.

[8] Gilbey, J. D. and Kalikow, L. H. *Parking functions, valet functions and priority queues*, Discrete Mathematics,**197/198** (1999), 351–373.

[9] Gessel, I. and Sagan, B. *The Tutte Polynomial of a Graph, Depth-First Search, and Simplicial Complex Partitions.* Electronic Journal of Combinatorics **3** (no. 2) R9.

[10] Knuth, D. *Linear probing and graphs, average-case analysis for algorithms.* Algorithmica **22**(1998), No.4, 561–568.

[11] Konheim, A. G. and Weiss, B. *An Occupancy Discipline and Applications.* SIAM Journal of Applied Mathematics **14** (1966) 1266–1274.

[12] Kostic, D. *Families of Bijections Between Dirichlet Games and Multiparking Functions*, in preparation.

[13] Kreweras, G. *Une famille de polynômes ayant plusieurs propriétés énumeratives*, Period. Math. Hungar. **11**(1980), 309–320.

[14] Kung, J.P. and Yan, C.H. *Gončarov polynomials and parking functions.* Journal of Combinatorial Theory, Series A, **102**(2003), 16–37.

[15] Kung, J. P. and Yan, C. H. *Exact formula for moments of sums of classical parking functions*, Advances in Applied Mathematics, vol 31(2003), 215–241.

[16] López, C. M. *Chip firing and Tutte polynomials*, Ann. Combin. **3** (1997) 253–259.

[17] Pitman, J. and Stanley, R. P. *A polytope related to empirical distributions, plane trees, parking functions, and the associahedron.* Discrete and Computational Geometry, **27**(2002), no.4, 603–634.

[18] Postnikov, A. and Shapiro, B. *Trees, Parking Functions, Syzygies, and Deformations of Monomial Ideals.* Transactions of the American Mathematical Society **356** (2004).

[19] Spencer, J. *Enumerating graphs and Brownian Motion*, Communications on Pure and Applied Mathematics, Vol. L (1997) 291–294.

[20] Stanley, R. P. *Hyperplane arrangements, parking functions, and tree inversions*, in B. Sagan and R. Stanley, eds., "Mathematical essays in honor of Gian-Carlo Rota," Birkhäuser, Boston and Basel, 1998, pp. 359–375.

[21] Tutte, W. *A Contribution to the Theory of Chromatic Polynomials.* Canadian Journal of Mathematics **6** (1953). 80-91.

[22] Yan, C. H. *Generalized tree inversions and k-parking functions.* Journal of Combinatorial Theory, Series A, **79**(1997), 268–280.

[23] Yan, C. H. *Generalized Parking Functions, Tree Inversions, and Multicolored Graphs.* Advances in Applied Mathematics **27** (2001). 641-670.