Vector Parking Functions with Periodic Boundaries and Rational Parking Functions

Yue Cai[a], Catherine H. Yan[b,*]

[a] *School of Information Technology, Jiangxi University of Finance and Economics, Jiangxi, China, 330032*

[b] *Department of Mathematics, Texas A&M University, College Station, TX 77843, United States*

Abstract

Vector parking functions are sequences of non-negative integers whose order statistics are bounded by a given integer sequence $\mathbf{u} = (u_0, u_1, u_2, \dots)$. Using the theory of fractional power series and an analog of Newton-Puiseux Theorem, we derive the exponential generating function for the number of $\mathbf{u}$-parking functions when $\mathbf{u}$ is periodic. Our method is to convert an Appell relation of Gončarov polynomials to a system of linear equations. Solving the system we obtain an explicit formula of the exponential generating function in terms of Schur functions of certain fractional power series. In particular, we apply our methods to rational parking functions for which the boundary is induced by a linear function with rational slope.

Keywords: vector parking functions, rational parking functions, periodic boundary

1. Introduction

The notion of parking functions was introduced by Konheim and Weiss in the analysis of a well-known computer algorithm: the hashing with linear probing. In [8] Konheim and Weiss gave a picturesque description of parking functions via a parking process on a one-way street. There are several equivalent definitions for parking functions. In this paper we adopt the one in terms of order statistics. For a sequence $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$, let $\mathrm{inc}(\mathbf{a}) = (a_{(0)}, a_{(1)}, \dots, a_{(n-1)})$ be its non-decreasing rearrangement, that is, $a_{(0)} \le a_{(1)} \le \cdots \le a_{(n-1)}$, where $a_{(i)}$ is called the *i-th order statistic* of the sequence $\mathbf{a}$.

Definition 1.1. An integer sequence $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$ is a *parking function* if and only if its order statistics satisfy the inequalities $0 \le a_{(i)} < i + 1$ for all i.

The set of parking functions is an object lying in the center of combinatorics and appearing in many discrete and algebraic structures. In addition to hashing and linear probing, they are also related to enumeration of trees and forests, hyperplane arrangements, noncrossing partitions, monomial ideals, and combinatorial theory of Macdonald polynomials, to list a few. See [22] for a survey on its history and some recent developments. There are various generalizations in literature, for example, vector parking functions that depend on a vector $\mathbf{u}$ [11, 19], G-parking functions related to the critical configurations of the sandpile model on a directed graph G [15], parking functions on trees and directed graphs that generalize the parking process of Konheim and Weiss [2, 7, 12], and parking sequences allowing cars of different sizes [4]. In this paper we are concerned with vector parking functions.

*Corresponding author

Email addresses: caiyue@jxufe.edu.cn (Yue Cai), cyan@math.tamu.edu (Catherine H. Yan)

Definition 1.2. Let $\mathbf{u} = (u_0, u_1, \dots)$ be a sequence of non-decreasing positive integers. A $\mathbf{u}$-*parking function* of length n is a sequence $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{N}^n$ whose order statistics satisfy $0 \le a_{(i)} < u_i$ for $i = 0, 1, \dots, n-1$.

We denote by $\mathcal{PF}_n(\mathbf{u})$ the set of $\mathbf{u}$-parking functions of length n and by $PF_n(\mathbf{u})$ the cardinality of $\mathcal{PF}_n(\mathbf{u})$. When $u_i = i+1$ for all i, we recover Definition 1.1 and the sequences in $\mathcal{PF}_n(\mathbf{u})$ are referred to as *classical parking functions*.

A sequence of polynomials from interpolation theory gives a natural algebraic tool to study $\mathbf{u}$-parking functions. Those polynomials are called *Gončarov polynomials*, which are the basis of solutions to the Gončarov interpolation problem in numerical analysis. The n-th Gončarov polynomial $g_n(x; \mathbf{u}) = g_n(x; u_0, \dots, u_{n-1})$ is a monic polynomial of degree n depending on the parameters $u_0, u_1, \dots, u_{n-1}$. The detailed description of the algebraic and combinatorial theory of Gončarov polynomials can be found in [11]. The main result connecting $\mathbf{u}$-parking functions and Gončarov polynomials is the following equation.

Theorem 1.3 ([11])**.**

$$PF_n(\mathbf{u}) = g_n(0; -u_0, \dots, -u_{n-1}) = (-1)^n g_n(0; u_0, \dots, u_{n-1}). \tag{1.1}$$

In particular, when $\mathbf{u}$ is an arithmetic progression with $u_i = a + bi$ for some positive integers a and b, $PK_n(\mathbf{u}) = a(a+bn)^{n-1}$. For general $\mathbf{u}$, the only formula available for computing $PK_n(\mathbf{u})$ is a determinantal formula, which is not easy to evaluate. The goal of the present paper is to give an explicit formula of the exponential generating function of $PF_n(\mathbf{u})$ when $\mathbf{u}$ is a periodic sequence. Here a sequence $\mathbf{u} = (u_0, u_1, \dots)$ is *periodic with period k and height ℓ* if there are positive integers k and ℓ such that $u_0 \le u_1 \le \cdots \le u_{k-1} \le u_0 + \ell$ and $u_m = q\ell + u_r$ whenever $m = qk + r$ with $0 \le r < k$. Our analysis will start with an identity called *the Appell relation* for Gončarov polynomials.

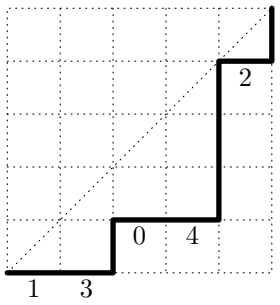
Theorem 1.4 ([11])**.** *Appell relation.*

$$e^{xt} = \sum_{n=0}^{\infty} g_n(x; \mathbf{u}) \, \frac{t^n e^{u_n t}}{n!}. \tag{1.2}$$

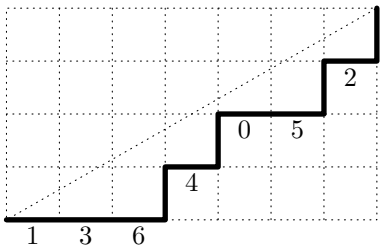
We will combine Eqs.(1.1) and (1.2) to get explicit generating functions of $PF_n(\mathbf{u})$ for periodic $\mathbf{u}$. The idea is to use the theory of fractional power series and an analog of Newton-Puiseux Theorem. This method was used previously in [10] to show the algebraicity of the (ordinary) generating functions of lattice paths with periodic boundaries. In this paper we extend it to vector parking functions. In particular, our result covers rational parking functions, which arose in the study of diagonal harmonic and combinatorial theory of Macdonald polynomials. The notion of rational parking functions come from an encoding of parking functions as labeled Dyck paths. Next we recall the necessary definitions and explain how to fit it into the notion of vector parking functions.

For any sequence $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{N}^n$, $\mathrm{inc}(\mathbf{a})$ can be represented by a lattice path from $(0,0)$ to (n,n) with north step N and east step E. If $\mathrm{inc}(\mathbf{a}) = (a_{(0)}, a_{(1)}, \dots, a_{(n-1)})$, the lattice path has E-steps from $(i, a_{(i)})$ to $(i+1, a_{(i)})$ for each i. The sequence $\mathbf{a}$ is a classical parking function if and only if the corresponding lattice path is a Dyck path, which stays weakly below the diagonal $y = x$. To extend this representation to a bijection to all parking functions, we assign each E-step in the Dyck path a *label*. If $a_{j_1} = a_{j_2} = \cdots = a_{j_k} = i$ for $j_1 < j_2 < \cdots < j_k$ we label the E-steps in the Dyck path at height i as $j_1, j_2, \dots, j_k$ from left to right. Conversely, given a Dyck path whose E-steps are labeled by $\{0, 1, \dots, n-1\}$ and the labels on each consecutive run of E-steps are from small to large, we can recover the parking function by taking $a_j = i$ whenever the E-step with label j is at height i. Figure 1(a) gives an example of a labeled Dyck path and its corresponding parking function.

From this encoding of parking functions Armstrong, Loehr, and Warrington introduced *rational parking*



(a) Labeled Dyck path for the parking function $(1, 0, 4, 0, 1)$



(b) Labeled Dyck path for the $(4, 7)$-parking function $(2, 0, 3, 0, 1, 2, 0)$

Figure 1: Parking function and rational parking function.

functions [1]. [1]

Definition 1.5. Let a and b be coprime positive integers. An (a, b)-*Dyck path* is a lattice path from $(0, 0)$ to (b, a) with steps $\{N, E\}$ that stays weakly below the diagonal $y = ax/b$. An (a, b)-*parking function* of length b is an (a, b)-Dyck path together with a labeling of the E-steps by the set $\{0, 1, \ldots, b-1\}$ such that labels increase in each consecutive run of E-steps.

In [1] Armstrong, Loehr, and Warrington considered the Frobenius characteristic of parking functions under an action of the symmetric group and showed the following theorem using representation theoretical method.

Theorem 1.6. *When* $\gcd(a, b) = 1$*, the number of* (a, b)*-parking functions of length* b *is* a^{b-1}.

Note that when $(a, b) = (n + 1, n)$ this recovers the result on classical parking functions: the number of parking functions of length n is given by the famous Cayley's formula $(n + 1)^{n-1}$.

For an (a, b)-Dyck path labeled as described above, define a sequence $(x_0, x_1, \ldots, x_{b-1})$ by letting $x_j = i$ whenever the E-step with label j is at height i. Figure 1(b) shows a $(4, 7)$-parking function, which corresponds to the sequence $(2, 0, 3, 0, 1, 2, 0)$. It is easy to see that a sequence $(x_0, x_1, \ldots, x_{b-1})$ corresponds to an (a, b)-parking function if and only if the height of the i-th E-step is weakly below ia/b for $0 \le i < b$. Hence we can extend the notion of (a, b)-parking functions to arbitrary lengths.

Definition 1.7. A sequence $(x_0, x_1, \ldots, x_{n-1})$ is an (a, b)-parking function of length n if and only if its order statistics $x_{(0)} \le x_{(1)} \le \cdots \le x_{(n-1)}$ satisfy $0 \le x_{(i)} \le ia/b$ for all $i = 0, 1, \ldots, n-1$. Given an (a, b)-parking function $\mathbf{x}$, denote by $D(\mathbf{x})$ its corresponding labeled Dyck path.

Comparing to Definition 1.2, one sees that (a, b)-parking functions are exactly the vector parking functions associated with the integer sequence $\mathbf{u} = (1 + \lfloor ia/b \rfloor)_{i \ge 0}$. If a is not a multiple of b, this sequence is not an arithmetic progression. Nevertheless, it is periodic with period b and height a, to which our techniques apply.

The paper is organized as follows. We will describe the method and proofs for rational parking functions first, since the results are most clear. In section 2 we prove that the equation $z = t^b e^{-at}$ has b fractional power series solutions. Then we convert Eq.(1.2) associated with the vector $\mathbf{u} = (1 + \lfloor ia/b \rfloor)_{i \ge 0}$ into a system of linear equations, and we solve the system to get an explicit formula for the exponential generating functions of

[1] The definition here is essentially the one given in [1], except that we change the orientation and put the labels on E-steps to make the notation consistent with the rest of the paper.

$PF_n(\mathbf{u})$ in terms of the elementary symmetric functions of the fractional power series solutions. Section 3 gives a combinatorial approach to enumerate the (a,b)-parking functions of length nb. In Section 4 we apply the techniques of fractional power series to vector parking functions with general periodic boundaries and present various applications. In Section 5 we discuss the cases when integers a, b are not relatively prime, and in Section 6 we extend the technique to eventually periodic $\mathbf{u}$, i.e., $\mathbf{u}$ is periodic except for finitely many terms. We end this paper with some open questions in Section 7.

For future use, we briefly recall the definition of Schur functions.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$. Let $\delta = (n-1, n-2, \ldots, 1, 0)$, then we can write $\alpha = \lambda + \delta$ where λ is an integer partition of length no greater than n. Here we use the definition and notation from [17, Section 1.7]. In particular, the length of an integer partition is the number of non-zero terms in this partition. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a set of variables. Define the determinant

$$a_\alpha(\mathbf{x}) = a_{\lambda+\delta}(\mathbf{x}) = \det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^{n} \tag{1.3}$$

When $\lambda = \emptyset$,

$$a_\delta(\mathbf{x}) = \prod_{1\leq i<j\leq n} (x_i - x_j) \tag{1.4}$$

is the Vandermonde determinant.

The *Schur function* is defined as

$$s_\lambda(\mathbf{x}) = a_{\lambda+\delta}(\mathbf{x})/a_\delta(\mathbf{x}). \tag{1.5}$$

In particular, $s_{(1^k)}(\mathbf{x}) = e_k(x)$, the k-th elementary symmetric function of degree k, and $s_{(\ell)}(\mathbf{x}) = h_\ell(\mathbf{x})$, the complete homogeneous symmetric function of degree ℓ. For more details on symmetric functions, we refer the readers to [14, Chapter 1].

2. Rational parking functions: An algebraic approach

In this section, we give an algebraic approach to compute the exponential generating function of $PF_n(\mathbf{u})$, where $u_i = \lfloor ia/b \rfloor + 1$. This is done in three steps: First, we find a system of linear equations of generating functions of $PF_n(\mathbf{u})$ using Appell relation. Next we simplify the equations using a generalization of Newton-Puiseux Theorem that appeared in [10]. Lastly, by Cramer's rule and computing the determinants for certain matrices, we obtain explicit expression of the generating function in terms of symmetric functions.

Let $PF_0(\mathbf{u}) = 1$. Combining Theorems 1.3 and 1.4 and letting $x = 0$, we have

$$1 = \sum_{n=0}^{\infty} PF_n(\mathbf{u})\ \frac{t^n e^{-u_n t}}{n!}. \tag{2.1}$$

Expanding the right-hand side with $u_i = \lfloor ia/b \rfloor + 1$, we get

$$\begin{aligned}
1 &= \sum_{n\geq 0} \sum_{0\leq i\leq b-1} PF_{nb+i}(\mathbf{u}) \cdot \frac{t^{nb+i} \cdot e^{-(na+u_i)\cdot t}}{(nb+i)!} \\
&= \sum_{n\geq 0} \frac{PF_{nb}(\mathbf{u})}{(nb)!} \cdot t^{nb} e^{-(na+u_0)\cdot t} + \sum_{n\geq 0} \frac{PF_{nb+1}(\mathbf{u})}{(nb+1)!} \cdot t^{nb+1} e^{-(na+u_1)\cdot t} + \cdots \\
&\quad + \sum_{n\geq 0} \frac{PF_{nb+b-1}(\mathbf{u})}{(nb+b-1)!} \cdot t^{nb+b-1} e^{-(na+u_{b-1})\cdot t} \\
&= \sum_{n\geq 0} \frac{PF_{nb}(\mathbf{u})}{(nb)!} \cdot z^n \cdot e^{-u_0 t} + \sum_{n\geq 0} \frac{PF_{nb+1}(\mathbf{u})}{(nb+1)!} \cdot z^n \cdot t \cdot e^{-u_1 t} + \cdots
\end{aligned}$$

$$+\sum_{n\geq 0}\frac{PF_{nb+b-1}(\mathbf{u})}{(nb+b-1)!}\cdot z^n\cdot t^{b-1}\cdot e^{-u_{b-1}t}$$

where $z=t^b\cdot e^{-at}$. Let $Q_i(z)=\sum_{n\geq 0}\frac{PF_{bn+i}(\mathbf{u})}{(bn+i)!}\cdot z^n$ be the exponential generating function for $PF_{bn+i}(\mathbf{u})$, then the above equation can be rewritten as

$$\sum_{i=0}^{b-1}Q_i(z)\cdot t^i e^{-u_i t}=1. \tag{2.2}$$

In [10, Lemma 4.1], the authors gave a generalization of the Newton-Puiseux Theorem on certain power series.

Lemma 2.1. *Let $h(t)$ be a power series such that $h(0)=1$. Then the equation*

$$z=t^k h(t)$$

has k fractional power series solutions $\tau_m(z)$, $0\leq m\leq k-1$ such that

$$\tau_0(z)=z^{1/k}+\sum_{i\geq 2}c_i z^{i/k}$$

and

$$\tau_m(z)=\omega^m z^{1/k}+\sum_{i\geq 2}c_i\omega^{mi}z^{i/k}, 1\leq m\leq k-1,$$

where ω is a primitive k-th root of unity.

This lemma guarantees that there exist b distinct solutions $t_0(z),\ldots,t_{b-1}(z)$ to $z=t^b\cdot e^{-at}$. Thus we obtain b linear equations from Eq.(2.2):

$$\begin{pmatrix} e^{-u_0t_0} & t_0e^{-u_1t_0} & \cdots & t_0^{b-1}e^{-u_{b-1}t_0}\\ e^{-u_0t_1} & t_1e^{-u_1t_1} & \cdots & t_1^{b-1}e^{-u_{b-1}t_1}\\ \vdots & \vdots & \ddots & \vdots\\ e^{-u_0t_{b-1}} & t_{b-1}e^{-u_1t_{b-1}} & \cdots & t_{b-1}^{b-1}e^{-u_{b-1}t_{b-1}}\end{pmatrix}\begin{pmatrix}Q_0(z)\\ Q_1(z)\\ \vdots\\ Q_{b-1}(z)\end{pmatrix}=\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}. \tag{2.3}$$

Denote the coefficient matrix by

$$A=\left(t_i^j\cdot e^{-u_jt_i}\right)_{i,j=0}^{b-1}. \tag{2.4}$$

If $\det(A)\neq 0$, then by Cramer's rule, $Q_i(z)=\det A_i/\det A$, where A_i is the matrix obtained by replacing the i-th column of A by a column vector of ones. Note that our index starts from 0.

In order to find the determinants, we first simplify the matrix A by finding $t_i(z)$.

Following the notation in [18, Section 5.3], we denote $R(x)=\sum_{n\geq 1}n^{n-1}\frac{x^n}{n!}$ as the generating function for labeled rooted trees. Then

$$R(x)=xe^{R(x)}. \tag{2.5}$$

Moreover, fixing w as a primitive b-th root of unity, we let

$$\gamma_i(z)=z^{1/ab}\omega^i\exp\left(\frac{1}{a}R\Big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\Big)\right). \tag{2.6}$$

We have the following lemma.

Lemma 2.2. *For $0 \le i \le b-1$, $t_i(z) = \gamma_i(z)^a$ are b solutions to $z = t^b \cdot e^{-at}$.*

Proof. Simplify the expression γ_i^a:

$$\begin{aligned}\gamma_i^a &= z^{1/b}\omega^{ia}\exp\left(R\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big)\right)\\ &= z^{1/b}\omega^{ia}\cdot R\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big)\cdot\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big)^{-1}\\ &= \frac{b}{a}R\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big).\end{aligned}$$

Since $\gcd(a,b)=1$, $\{\omega^{ai} : 0 \le i \le b-1\}$ are b distinct roots for $x^b = 1$. Hence $t_i(z)$ are all distinct.

On the other hand, since $R(x) = xe^{R(x)}$, we can verify

$$\begin{aligned}t_i^b\cdot e^{-at} &= \left(\frac{b}{a}R\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big)\right)^b\cdot\exp\left(-a\cdot\frac{b}{a}R\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big)\right)\\ &= \left(\frac{b}{a}\right)^b\left(R\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big)\cdot\exp\big(-R\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big)\big)\right)^b\\ &= \left(\frac{b}{a}\right)^b\cdot\big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\big)^b\\ &= z.\end{aligned}$$

Hence $t_i(z)$ is a solution to $z = t^b\cdot e^{-at}$ for $i = 0, 1, \ldots, b-1$. □

We remark that Lemma 2.2 agrees with Lemma 2.1. Just notice that t_i can be obtained from t_0 by substituting $\omega^{ai}z^{1/b}$ for $z^{1/b}$, where ω^a is again a primitive b-th root of unity. The format in Lemma 2.2 allows us to simplify the entries in the matrix A.

From $z = t_i^b\cdot e^{-at_i}$ we have $e^{-t_i} = \eta z^{1/a}\cdot\gamma_i^{-b}$ where η is an a-th root of unity. Expanding both sides as fractional power series and comparing the constant coefficient, we have $\eta = 1$. Hence $e^{-t_i} = z^{1/a}\cdot\gamma_i^{-b}$ and consequently

$$A = \big(z^{u_j/a}\cdot\gamma_i^{aj-u_jb}\big)_{i,j=0}^{b-1}. \tag{2.7}$$

Next, we claim that

Lemma 2.3. *Let $\lambda_i = u_ib - ia$, then $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{b-1}) \in \mathfrak{S}_b$. Here $\mathfrak{S}_n$ denotes the set of permutations of $\{1, 2, \ldots, n\}$*

Proof. Assume $\lfloor ia/b\rfloor = j$, then $ia = jb + r$ where $0 \le r < b$. This gives $\lambda_i = (j+1)b - ia = b - r$. So $0 < \lambda_i \le b$ for all $0 \le i \le b$. The values of λ_i are all distinct because they are in different congruence classes modulo b. Hence $\lambda \in \mathfrak{S}_b$. □

Now we are ready to compute $\det A$.

Proposition 2.4. *Let A be the matrix defined in Eq.(2.4), then*

$$\det A = \operatorname{sgn}(\lambda)\cdot z^c\cdot a_\delta(\boldsymbol{\gamma})\cdot\prod_{i=0}^{b-1}\gamma_i^{-b}.$$

Here $c = \sum_{i=0}^{b-1}u_i/a$, $a_\delta(\boldsymbol{\gamma})$ is as defined in Eq.(1.4), and $\operatorname{sgn}(\lambda)$ is the sign of permutation λ. That is, $\operatorname{sgn}(\sigma) = (-1)^m$ where m is the number of transpositions to map σ to $\operatorname{inc}(\sigma)$.

Proof. Using Eq.(2.7) and Lemma 2.3, we have

$$\begin{aligned}\det A &= \det\left(z^{u_j/a}\cdot\gamma_i^{aj-u_jb}\right)_{i,j=0}^{b-1}\\ &= z^c\cdot\det\left(\gamma_i^{-\lambda_j}\right)_{i,j=0}^{b-1}\\ &= \operatorname{sgn}(\lambda)\cdot z^c\cdot\det\left(\gamma_i^{-1-j}\right)_{i,j=0}^{b-1}\\ &= \operatorname{sgn}(\lambda)\cdot z^c\cdot\prod_{i=0}^{b-1}\gamma_i^{-b}\cdot\det\left(\gamma_i^{b-1-j}\right)_{i,j=0}^{b-1}\\ &= \operatorname{sgn}(\lambda)\cdot z^c\cdot a_\delta(\boldsymbol{\gamma})\cdot\prod_{i=0}^{b-1}\gamma_i^{-b}\end{aligned}$$

as desired. □

Using a similar idea, we compute $\det A_i$.

Proposition 2.5. *For $0\le i\le b-1$, let A_i be the matrix obtained by replacing the i-th column of A by a column vector of ones, then*

$$\det A_i = (-1)^{\lambda_i-1}\cdot\operatorname{sgn}(\lambda)\cdot z^{c-u_i/a}\cdot\prod_{j=0}^{b-1}\gamma_j^{-b}\cdot e_{\lambda_i}(\boldsymbol{\gamma})\cdot a_\delta(\boldsymbol{\gamma})$$

where $\boldsymbol{\gamma}=(\gamma_0,\gamma_1,\dots,\gamma_{b-1})$ as defined in Eq.(2.6).

Proof. By definition,

$$\det A_i = \det(a_{jk})_{j,k=0}^{b-1}$$

where $a_{jk}=z^{u_j/a}\cdot\gamma_k^{-\lambda_j}$ when $k\ne i$, and $a_{ji}=1$. Thus

$$\det A_i = (-1)^{\lambda_i-1}\cdot\operatorname{sgn}(\lambda)\cdot z^{c-u_i/a}\cdot\prod_{j=0}^{b-1}\gamma_j^{-b}\cdot\det\left(\gamma_i^{\mu_j}\right)_{i,j=0}^{b-1},$$

where $(\mu_0,\mu_1,\dots,\mu_{b-1})=(b,b-1,\dots,b-\lambda_i+1,b-\lambda_i-1,\dots,0)=\mu+\delta$ and $\mu=(1^{\lambda_i})$, a partition with all nonzero parts equal to one. By Eqs.(1.3) and (1.5) we have

$$\begin{aligned}\det A_i &= (-1)^{\lambda_i-1}\cdot\operatorname{sgn}(\lambda)\cdot z^{c-u_i/a}\cdot\prod_{j=0}^{b-1}\gamma_j^{-b}\cdot s_{(1^{\lambda_i})}(\boldsymbol{\gamma})\cdot a_\delta(\boldsymbol{\gamma})\\ &= (-1)^{\lambda_i-1}\cdot\operatorname{sgn}(\lambda)\cdot z^{c-u_i/a}\cdot\prod_{j=0}^{b-1}\gamma_j^{-b}\cdot e_{\lambda_i}(\boldsymbol{\gamma})\cdot a_\delta(\boldsymbol{\gamma}),\end{aligned}$$

which proves the proposition. □

Combining Propositions 2.4 and 2.5, we find the generating function for (a,b)-parking functions of any length.

Theorem 2.6. *Assume a,b are two coprime positive integers and $\mathbf{u}$ is the sequence given by $u_i=\lfloor ia/b\rfloor+1$. Let $Q_i(z)=\sum_{n\ge0}\dfrac{PF_{bn+i}(\mathbf{u})}{(bn+i)!}\cdot z^n$. Then*

$$Q_i(z) = (-1)^{\lambda_i-1}z^{-u_i/a}e_{\lambda_i}(\boldsymbol{\gamma}), \tag{2.8}$$

where $\lambda_i = u_i b - ia$ *and* $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{b-1})$ *is given by Eq.(2.6). Consequently, the exponential generating function of* $PF_n(\mathbf{u})$ *can be computed by*

$$\sum_{n=0}^{\infty} PF_n(\mathbf{u}) \frac{z^n}{n!} = \sum_{i=0}^{b-1} z^i Q_i(z^b).$$

Note that when $b = 1$, (a, b)-parking functions are $\mathbf{u}$-parking functions where $\mathbf{u}$ is the arithmetic progression with $u_i = 1 + ai$. In this case Theorem 2.6 gives

$$\sum_{n\geq 0} PF_n((1+ai)_{i\geq 0}) \frac{z^n}{n!} = \exp\left(\frac{1}{a} R(az)\right).$$

We can extract the value of $PF_n((1+ai)_{i\geq 0})$ by the Lagrange inversion formula and hence recover the following formula in [11, Corollary 5.5].

Corollary 2.7. *Let* $u_i = 1 + ai$ *for a positive integer* a*. Then* $PF_n(\mathbf{u}) = (1 + an)^{n-1}$.

Proof. Let $g(z) = \frac{1}{a} R(az)$. Then $R(x) = x \exp(R(x))$ implies $g(z) = z \exp(ag(z))$. Applying a generalized version of the Lagrange inversion formula, for example, see [18, Corollary 5.4.3] with $G(z) = e^{az}$ and $H(z) = e^z$, we have

$$[z^n] \exp(g(z)) = [z^n] H(g(z)) = \frac{1}{n} [z^{n-1}] H'(z) G(z)^n = \frac{1}{n} [z^{n-1}] \exp((1 + an)z),$$

which gives $PF_n(\mathbf{u}) = (1 + an)^{n-1}$. □

When $b > 1$, we have the following special cases.

Corollary 2.8. *Assume* $a = \ell \cdot b + r$*, where* $b > 1, \ell \geq 0$ *and* $r \leq b - 1$*, then*

$$Q_0(z) = (-1)^{b-1} z^{-1/a} \prod_{i=0}^{b-1} \gamma_i \tag{2.9}$$

and

$$Q_{b-1}(z) = (-1)^{r-1} z^{-1} e_r(\boldsymbol{\gamma}). \tag{2.10}$$

Proof. Apply $u_0 = 1$ and $u_{b-1} = a$ to Theorem 2.6. Note that $\lambda_0 = b$ and $\lambda_{b-1} = r$. □

By simplifying the coefficient of z in Eq.(2.9), we obtain a new proof of Theorem 1.6.

Corollary 2.9. *Let* a, b *be two positive integers such that* $\gcd(a, b) = 1$.

(i) The number of (a, b)*-parking functions of length* b *is* a^{b-1}.

(ii) The number of (a, b)*-parking functions of length* $2b$ *is* $(2a)^{2b-1} + \frac{1}{2} \cdot \binom{2b}{b} a^{2b-2}$.

Proof. Using Formula (2.6), we have

$$Q_0(z) = \prod_{i=0}^{b-1} \exp\left(\frac{1}{a} R\left(\frac{\omega^{ai} a}{b} z^{1/b}\right)\right) = \exp\left(\frac{1}{a} \sum_{i=0}^{b-1} R\left(\frac{\omega^{ai} a}{b} z^{1/b}\right)\right).$$

Let $\eta = \omega^a$. Note that η is also a b-th primitive root of unity. Thus $\sum_{i=0}^{b-1}(\eta^i)^n = b$ if n is a multiple of b, and the sum is 0 otherwise. Hence

$$\begin{aligned}
\sum_{i=0}^{b-1} R\left(\frac{\omega^{ai}a}{b}z^{1/b}\right) &= \sum_{i=0}^{b-1}\sum_{n\geq 1}\frac{n^{n-1}}{n!}\left(\frac{\omega^{ai}a}{b}z^{1/b}\right)^n \\
&= \sum_{n\geq 1}\frac{n^{n-1}}{n!}\left(\sum_{i=0}^{b-1}\omega^{ain}\right)\left(\frac{a}{b}\right)^n z^{n/b} \\
&= \sum_{k\geq 1}\frac{(kb)^{kb-1}}{(kb)!}\; b\;\left(\frac{a}{b}\right)^{kb} z^k \\
&= \frac{a^b}{b!}z + \frac{a\cdot(2a)^{2b-1}}{(2b)!}z^2 + Kz^3,
\end{aligned}$$

where K is a formal power series of z. Therefore

$$\begin{aligned}
Q_0(z) &= \exp\left(\frac{a^{b-1}}{b!}z + \frac{(2a)^{2b-1}}{(2b)!}z^2 + Kz^3/a\right) \\
&= 1 + \frac{a^{b-1}}{b!}z + \frac{1}{(2b)!}\left((2a)^{2b-1} + \frac{1}{2}\binom{2b}{b}a^{2b-2}\right)z^2 + \text{higher powers of } z,
\end{aligned}$$

which gives the desired results. □

Another special case is when $a = 1$. In this case we can simplify γ_i as

$$\gamma_i(z) = z^{1/b}\omega^i \exp\left(R(\frac{\omega^i}{b}z^{1/b})\right) = b\cdot R(\frac{\omega^i}{b}z^{1/b}),$$

where the last equation follows from Eq.(2.5). Thus Theorem 2.6 has the form

Theorem 2.10. *When $a = 1$, we have*

$$Q_i(z) = (-1)^{b-i-1}z^{-1}e_{b-i}(\boldsymbol{\gamma}).$$

In particular,

$$Q_0(z) = (-1)^{b-1}z^{-1}b^b\prod_{i=0}^{b-1}R(\frac{\omega^i}{b}z^{1/b}) \tag{2.11}$$

and

$$Q_{b-1}(z) = z^{-1}b\sum_{i=0}^{b-1}R(\frac{\omega^i}{b}z^{1/b}). \tag{2.12}$$

Computing the coefficients in Eq.(2.12) we get the following corollary.

Corollary 2.11. *When $a = 1$, that is, when $\mathbf{u} = (1, 1, \ldots, 1, 2, 2, \ldots, 2, 3, \ldots)$, the number of $(1, b)$-parking functions of length $nb + b - 1$ is given by*

$$PF_{nb+b-1}(\mathbf{u}) = (n+1)^{nb+b-2}.$$

Proof. Expanding Eq.(2.12) we obtain

$$\begin{aligned}
Q_{b-1}(z) &= z^{-1}b\cdot\sum_{i=0}^{b-1}\sum_{n\geq 1}\frac{n^{n-1}}{n!}\left(\frac{\omega^i}{b}z^{1/b}\right)^n \\
&= z^{-1}b\cdot\sum_{n\geq 1}\frac{n^{n-1}}{n!b^n}z^{n/b}\sum_{i=0}^{b-1}\omega^{ni}.
\end{aligned}$$

Note that when $b \not| n$, $\sum_{i=0}^{b-1} \omega^{ni} = 0$, thus the only non-zero terms will be when $b|n$, in which case $\omega^{ni} = 1$. So we have

$$\begin{aligned} Q_{b-1}(z) &= z^{-1}b \cdot \sum_{n\geq 1} \frac{(nb)^{nb-1}}{(nb)!b^{nb}} z^n \cdot b \\ &= b \cdot \sum_{n\geq 1} \frac{n^{nb-1}}{(nb)!} z^{n-1} \\ &= b \cdot \sum_{n\geq 0} \frac{(n+1)^{nb+b-1}}{(nb+b)!} z^n \\ &= b \cdot \sum_{n\geq 0} \frac{(n+1)^{nb+b-1}}{b(n+1)(nb+b-1)!} z^n \\ &= \sum_{n\geq 0} \frac{(n+1)^{nb+b-2}}{(nb+b-1)!} z^n. \end{aligned}$$

Comparing the coefficients of z^n, we obtain the desired result. □

Theorem 2.10 and Corollary 2.11 recover the work of Blake and Konheim [3, Section 2], who considered a computer storage problem similar to that of the classical parking functions, except that each address (parking space) is capable of holding k items.

3. Rational parking functions: A combinatorial approach

In this section, we give a combinatorial proof for the cardinality of $PF_{nb}(\mathbf{u})$ for $u_i = \lfloor ia/b \rfloor + 1$. Again assume a, b are positive integers with $\gcd(a, b) = 1$. From Section 1 we see that $PF_{nb}(\mathbf{u})$ counts the number of certain labeled (a, b)-Dyck paths. We adapt the techniques in lattice path counting and consider how the labels would contribute. In particular, we use the well-known cycle lemma; see [9] for a survey and many related references.

For convenience, we will use PF_{nb} to denote the number of (a, b)-parking functions of length nb. Moreover, denote by $[n]$ the set $\{1, 2, \ldots, n\}$ and $[n]_0$ the set $\{0, 1, 2, \ldots, n\}$.

First we present a purely combinatorial proof of Theorem 1.6, which is an extension of Pollak's classical proof for the formula $(n + 1)^{n-1}$ for classical parking functions; see [16]. This proof for rational parking functions, written in terms of cosets in the group $(\mathbb{Z}/a\mathbb{Z})^b$, was presented in [21, Section 3.1.2] and used in [20, Section 2.4]. For completeness, we include the proof here again.

A combinatorial proof of Theorem 1.6. We build a bijection $\phi : \mathcal{PF}_b(\mathbf{u}) \longrightarrow [a-1]_0^{b-1}$ as follows.

For any (a, b)-parking function $\mathbf{c} = (c_1, c_2, \ldots, c_b) \in \mathcal{PF}_b(\mathbf{u})$, let $\phi(\mathbf{c})$ be its *difference sequence* $\mathbf{d} = (d_1, d_2, \ldots, d_{b-1})$, where $d_i = c_{i+1} - c_i \mod a$. Clearly $\phi(\mathbf{c})$ is in the set $[a-1]_0^{b-1}$.

To see ϕ is a bijection, we prove that for any sequence $\mathbf{d} \in [a-1]_0^{b-1}$, there is a unique integer $k \mod a$ such that $(k, k + d_1, k + d_1 + d_2, \ldots, k + d_1 + \cdots + d_{b-1}) \mod a$ is an (a, b)-parking function.

To show the existence of k, we use the following construction. Given a sequence $\mathbf{d} \in [a-1]_0^{b-1}$, consider the sequence $\mathbf{c} = (c_0, c_1, \ldots, c_{b-1})$ where $c_0 = 0$ and $c_i = \sum_{j=0}^{i} d_j \mod a$. Let $c_{(i)}$ be its i-th order statistic. Since $\gcd(a, b) = 1$, the numbers $\{c_{(i)} - ia/b : 0 \leq i \leq b-1\}$ are all distinct. Let i be the index that maximizes $c_{(i)} - ia/b$, and let $\ell = c_{(i)}$. Note that if $\{c_{(i)} : c_{(i)} - ia/b > 0\} = \emptyset$, then $\ell = 0$. We claim that $\mathbf{c} - \ell = (c_0 - \ell, c_1 - \ell, \ldots, c_{b-1} - \ell) \mod a$ is an (a, b)-parking function, that is, $k = -\ell \mod a$.

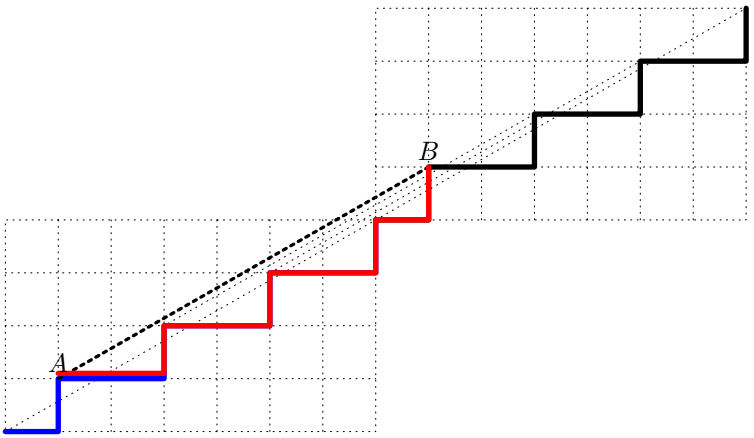


Figure 2: A graphical explanation of the proof.

For $j < i$, we have $c_{(j)} - \ell = c_{(j)} + a - \ell \mod a$ and the order statistic for $\mathbf{c} - \ell$ is then $(0 = c_{(i)} - \ell, c_{(i+1)} - \ell, \ldots, c_{(b-1)} - \ell, c_{(0)} + a - \ell, \ldots, c_{(i-1)} + a - \ell) := (c'_{(0)}, c'_{(1)}, \ldots, c'_{(b-1)})$. In formula,

$$c'_{(j)} = \begin{cases} c_{(j+i)} - \ell, & \text{if } 0 \le j \le b-i-1 \\ c_{j-(b-i)} + a - \ell, & \text{if } b-i \le j \le b-1. \end{cases}$$

Now let's check the condition that $c'_{(j)} \le ja/b$.

(i) The case $0 \le j \le b-i-1$. Since $c_{(i)} - ia/b > c_{(i+j)} - (i+j)a/b$ by our choice of i, we have $c_{(i+j)} - \ell = c_{(i+j)} - c_{(i)} \le ja/b$, that is $c'_{(j)} \le jb/a$.

(ii) The case $b-i \le j \le b-1$. Using $c_{(i)} - ia/b > c_{(i-\beta)} - (i-\beta)a/b$, we have $c'_{(j)} = c_{(i-(b-j))} + a - \ell < c_{(i)} - (b-j)a/b + a - \ell = a - (b-j)a/b = ja/b$.

Thus the sequence $\mathbf{c} - \ell$ is an (a, b)-parking function as desired.

Figure 2 gives a graphical explanation of the above proof: The lattice path from $(0, 0)$ to (b, a) corresponds to the order statistics $\mathrm{inc}(\mathbf{c})$. Given any lattice path from $(0, 0)$ to (b, a), find the peak that is furthest from the diagonal $y = ax/b$. Starting from that vertex and cyclically permute the lattice path, we would obtain an (a, b)-Dyck path that is fully under the diagonal.

Finally we show the uniqueness of the cyclic shift by proof of contradiction. Assume that in addition to the above $\mathbf{c} - \ell$, there is another $p \neq \ell \mod a$ such that $\mathbf{c} - p$ is also an (a, b)-parking functions. Assume that $\mathrm{inc}(\mathbf{c} - \ell) = 0^{m_0} 1^{m_1} \cdots (a-1)^{m_{a-1}}$, i.e., it contains m_i many i for $i = 0, \ldots, a-1$. Then the inequalities $c'_{(j)} \le ja/b$ are equivalent to the system

$$\begin{cases} \sum_{t=0}^{j-1} m_t > jb/a & \text{for } 0 \le j < a-1 \\ \sum_{t=0}^{a-1} m_t = b. \end{cases}$$

Since $\mathbf{c} - p$ is a cyclic shift of $\mathbf{c} - \ell$, the order statistics $\mathrm{inc}(\mathbf{c} - p)$ is of the form $0^{m_i} 1^{m_{i+1}} \cdots (a-1-i)^{m_{a-1}} (a-i)^{m_0} \cdots (a-1)^{m_{i-1}}$ for some $i \neq 0$. Hence we would have $m_i + m_{i+1} + \cdots + m_{a-1} > (a-i)b/a$. Combining with $\sum_{t=0}^{i-1} m_t > ib/a$, we obtain $\sum_{t=0}^{a-1} m_t > b$, a contradiction. □

Next we generalize this argument to get a recurrence for PF_{nb}.

Theorem 3.1. *For any integer $n \geq 1$ and $\gcd(a,b) = 1$ we have*

$$PF_{nb} = (na)^{nb-1} + \sum_{\substack{\ell_1+\cdots+\ell_k=n \\ \ell_i \geq 1, k \geq 2}} (-1)^k \cdot \binom{nb}{\ell_1 b, \ell_2 b, \ldots, \ell_k b} \cdot \frac{1}{k} \cdot PF_{\ell_1 b} \cdots PF_{\ell_k b}. \tag{3.1}$$

Proof. Consider the map $\phi : \mathcal{PF}_{nb}(\mathbf{u}) \longrightarrow [na-1]_0^{nb-1}$ as follows: Given an (a,b)-parking function $\mathbf{c} = (c_0, c_1, \ldots, c_{nb-1}) \in \mathcal{PF}_{nb}(\mathbf{u})$, $\phi(\mathbf{c})$ is the difference sequence $\mathbf{d} = (d_1, d_2, \ldots, d_{nb-1})$ where $d_i = c_i - c_{i-1} \mod na$.

By a similar proof as the previous one, it is straightforward to check that given $\mathbf{d} \in [na-1]_0^{nb-1}$, we can construct a sequence $\mathbf{c} = (c_0, c_1, \ldots, c_{nb-1})$ where $c_0 = 0$ and $c_i = \sum_{j=0}^{i} d_j$. Then there exists an $\ell \geq 0$ such that $\mathbf{c} - \ell$ is an (a,b)-parking function of length $nb-1$. Thus ϕ is surjective.

However, this map is not injective. Without loss of generality, assume $\mathbf{c} \in \mathcal{PF}_{nb}(\mathbf{u})$ whose difference sequence is $\mathbf{d}$. Then using a similar argument, we can verify that the sequence $\mathbf{c} - j \mod na$ is an (a,b)-parking function with the same difference sequence $\mathbf{d}$ if and only if there is an index $0 < j < nb-1$ such that $c_{(j)} = ja/b$. And since $\gcd(a,b) = 1$, this only happens when j is a multiple of b.

We can also give a graphical explanation of this argument. Let $\mathbf{c} \in \mathcal{PF}_{nb}(\mathbf{u})$, and let $D(\mathbf{c})$ be its corresponding labeled Dyck path. By definition, $D(\mathbf{c})$ is weakly below the line $y = ax/b$. If $D(\mathbf{c})$ touches the diagonal at an internal point (kb, ka) for some $0 < k < n$, we can decompose D into two sub-path: D_1 from $(0,0)$ to (kb, ka), and D_2 from (kb, ka) to (nb, na). Then the Dyck path $D' = (D_2, D_1)$ obtained by attaching the initial point of D_1 to the end point of D_2 is also an (a,b)-Dyck path. On the other hand, keeping the labels on D_1 and D_2 unchanged, we can check that $\mathbf{c}'$ obtained from D' has the same difference sequence as that of $\mathbf{c}$.

In other words, given one difference sequence $\mathbf{d} \in [na-1]_0^{nb-1}$, and let $\mathbf{c}$ be a pre-image of $\mathbf{d}$ under the map ϕ. If $D(\mathbf{c})$ touches the diagonal $y = ax/b$ exactly $k-1$ times at some internal points, then $D(\mathbf{c})$ can be decomposed into sub-paths $D_1, D_2, \ldots, D_k$, and any cyclic permutation $D_i, D_{i+1}, \ldots, D_{i-1}$ with labels unchanged will also recover a parking function in $\phi^{-1}(\mathbf{d})$. Thus $|\phi^{-1}(\mathbf{d})| = k$ and we only need one element from each pre-image to compute $|\phi(\mathcal{PF}_{nb})|$.

Let $S \subseteq [n-1]$ and denote by A_S the number of parking functions whose Dyck paths hits $y = ax/b$ only at the internal points $(s_i b, s_i a)$ for all $s_i \in S$. Note that $|S| = k-1$ implies that the corresponding Dyck path can be decomposed into k segments, each is a shorter (a,b)-Dyck path. Denote by B_S the number of parking functions whose corresponding Dyck paths hits $y = ax/b$ at least at the internal points $(s_i b, s_i a)$ for all $s_i \in S$. Then by Principle of Inclusion-Exclusion, we have

$$B_T = \sum_{S \subseteq T} A_S$$

and

$$A_S = \sum_{T \supseteq S} (-1)^{|T|-|S|} B_T.$$

On the other hand, if $T = \{t_1, t_2, \ldots, t_{k-1}\}$, then any Dyck path $D \in B_T$ hits $y = ax/b$ at least at the points $(t_i b, t_i a)$. Since there are nb many horizontal steps in D, if we denote by $\ell_i b$ the number of horizontal steps between the points $(t_{i-1} b, t_{i-1} a)$ and $(t_i b, t_i a)$ where $t_0 = 0$ and $t_k = n$, then $(\ell_1, \ell_2, \ldots, \ell_k)$ is a positive integer composition of n. Since there are $\binom{nb}{\ell_1 b, \ldots, \ell_k b}$ many ways to choose a labeling of D, and there are $PF_{\ell_1 b} \cdots PF_{\ell_k b}$ many parking functions with the given decomposition. We have that

$$B_T = \binom{nb}{\ell_1 b, \ldots, \ell_k b} \cdot PF_{\ell_1 b} \cdots PF_{\ell_k b}.$$

Thus

$$\begin{aligned}(na)^{nb-1} &= \sum_{S\subseteq[n-1]} \frac{A_S}{|S|+1} \\ &= \sum_{S\subseteq[n-1]} \frac{1}{|S|+1} \cdot \sum_{T\supseteq S} (-1)^{|T|-|S|} B_T \\ &= \sum_{T\subseteq[n-1]} B_T \cdot \sum_{S\subseteq T} \frac{(-1)^{|T|-|S|}}{|S|+1}\end{aligned}$$

Using the identity $\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1}{i+1} = \frac{(-1)^k}{k+1}$, we can simplify the above equation as

$$\begin{aligned}(na)^{nb-1} &= \sum_{T\subseteq[n-1]} (-1)^{|T|}\frac{B_T}{|T|+1} \\ &= \sum_{k=1}^{n}(-1)^{k+1}\cdot\sum_{\ell}\frac{1}{k}\binom{nb}{\ell_1 b,\ldots,\ell_k b} PF_{\ell_1 b}\cdots PF_{\ell_k b},\end{aligned}$$

where ℓ ranges over all positive integer compositions of n. This simplifies to Eq.(3.1). □

Note that when $n = 2$ the above formula simplifies to Corollary 2.9(ii).

Define the generating functions $P(x) = \sum_{n\geq 1}\frac{PF_{nb}}{(nb)!}\cdot x^{nb}$ and let $F(x) = \sum_{n\geq 1}\frac{(na)^{nb-1}}{(nb)!}\cdot x^{nb}$. Multiplying $x^{nb}/(nb)!$ and summing over $n \geq 1$ in Eq.(3.1), we get

$$\begin{aligned}P(x) &= F(x) + \sum_{k\geq 2}\frac{(-1)^k}{k}\prod_{i=1}^{k}\sum_{\ell_i\geq 1}\frac{PF_{\ell_i b}}{(\ell_i b)!}x^{\ell_i b} \\ &= F(x) + \sum_{k\geq 2}\frac{(-1)^k}{k}P^k(x) \\ &= F(x) + P(x) - \ln(1+P(x)).\end{aligned}$$

Thus we obtain the following theorem.

Theorem 3.2. *Let $PF_0 = 1$. Then we have $1 + P(x) = \exp(F(x))$, that is,*

$$\sum_{n\geq 0}\frac{PF_{nb}}{(nb)!}\cdot x^{nb} = \exp\left(\sum_{n\geq 1}\frac{(na)^{nb-1}}{(nb)!}\cdot x^{nb}\right). \tag{3.2}$$

We remark that Eq.(3.2) is equivalent to Eq.(2.9) by noticing that $1 + P(x) = Q_0(x^b)$ and

$$F(x) = \sum_{n\geq 1}\frac{(na)^{nb-1}}{(nb)!}\cdot x^{nb} = \frac{b}{a}\sum_{n\geq 1}\frac{(nb)^{nb-1}}{(nb)!}\left(\frac{ax}{b}\right)^{nb} = \frac{b}{a}\sum_{m:b|m}\frac{m^{m-1}}{m!}\left(\frac{ax}{b}\right)^{m} = \frac{1}{a}\sum_{i=0}^{b-1}R(\omega^i\frac{ax}{b}),$$

where ω is a primitive b-th root of unity and $R(x) = \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}$.

Using a similar idea, we can consider the number of (a, b)-parking functions whose corresponding lattice paths do not touch the line $y = ax/b$ except at the endpoints.

Definition 3.3. A *prime* (a,b)*-parking function of length* nb is an integer sequence $(a_0, a_1, \ldots, a_{nb-1})$ such that the ith order statistic satisfies $a_{(0)} = 0$ and $0 \le a_{(i)} < ia/b$ for $i = 1, \ldots, nb-1$.

Theorem 3.4. *Denote by* PPF_{nb} *the number of prime* (a,b)*-parking functions of length* nb*, then*

$$PPF_{nb} = |PF_{nb}| - \sum_{\ell} (-1)^k \binom{nb}{\ell_1 b, \ell_2 b, \ldots, \ell_k b} \cdot PF_{\ell_1 b} \cdot PF_{\ell_2 b} \cdots PF_{\ell_k b}, \tag{3.3}$$

where the sum is over all compositions of n *with at least two parts. As a consequence, let* $Q(x) = \sum_{n \ge 1} \frac{PPF_{nb}}{(nb)!} x^{nb}$ *be its generating function. Then*

$$Q(x) = \frac{P(x)}{1 + P(x)}.$$

Proof. This follows easily by considering the parking functions whose corresponding Dyck paths touches the diagonal $y = a/b \cdot x$ and using the Inclusion-Exclusion Principle. □

The combinatorial method described in this section provides a different approach to study rational parking functions. We expect to extend the lattice path counting techniques to study parking functions whose underlying paths end at a given point, carry various weight, or have a piecewise linear boundary.

4. Vector parking functions with a periodic boundary

Notice that Eq.(2.3) does not depend on explicit $\mathbf{u}$ values, so it holds for any periodic boundary $\mathbf{u}$ with period b and height a. In this section, we show that when $\gcd(a,b) = 1$, we can still compute the determinant in a similar way and solve the system.

Throughout this section, assume $\gcd(a,b) = 1$.

Lemma 4.1. *Let* $1 \le u_0 \le u_1 \le \cdots \le u_{b-1}$*. Then for* $0 \le i \le b-1$*, the numbers* $\lambda_i = u_i b - ia$ *are distinct.*

Proof. It follows from the fact that λ_i's are in different congruence classes modulo b, □

Lemma 4.2. *Denote* $\mathrm{inc}(\lambda) = (\lambda_{(0)}, \ldots, \lambda_{(b-1)})$ *and let* $c = \sum_{i=0}^{b-1} u_i/a$*. Then*

$$\det A = \mathrm{sgn}(\lambda) \cdot z^c \cdot s_\mu(\boldsymbol{\gamma}) \cdot a_\delta(\boldsymbol{\gamma}) \cdot \prod_{i=0}^{b-1} \gamma_i^{-\lambda_{(b-1)}}, \tag{4.1}$$

Here $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_{b-1})$ *is given by Eq.(2.6),* $s_\pi(\mathbf{x})$ *is as defined in Eq.(1.5), and* $\mu = (\mu_0, \ldots, \mu_{b-1})$ *is a partition where* $\mu_i = \lambda_{(b-1)} - \lambda_{(i)} - (b-i-1)$.

Proof. Rewrite $z = t_i^b \cdot e^{-at_i}$ as $e^{-t_i} = z^{1/a}\gamma_i^{-b}$, then

$$\begin{aligned}
\det A &= \det\left(z^{u_j/a}\cdot\gamma_i^{-\lambda_j}\right)_{i,j=0}^{b-1}\\
&= z^c\cdot\det\left(\gamma_i^{-\lambda_j}\right)_{i,j=0}^{b-1}\\
&= \operatorname{sgn}(\lambda)\cdot z^c\cdot\det\left(\gamma_i^{-\lambda_{(j)}}\right)_{i,j=0}^{b-1}\\
&= \operatorname{sgn}(\lambda)\cdot z^c\prod_{i=0}^{b-1}\gamma_i^{-\lambda_{(b-1)}}\cdot\det(\gamma_i^{\lambda_{(b-1)}-\lambda_{(j)}})_{i,j=0}^{b-1}\\
&= \operatorname{sgn}(\lambda)\cdot z^c\cdot s_\mu(\boldsymbol{\gamma})\cdot a_\delta(\boldsymbol{\gamma})\cdot\prod_{i=0}^{b-1}\gamma_i^{-\lambda_{(b-1)}}.
\end{aligned}$$

Note that by Lemma 4.1, $\lambda_{(i+1)} > \lambda_{(i)}$ and $\lambda_{(b-1)} - \lambda_{(i)} \geq b-i-1$. It implies that $\mu_i \geq \mu_{i+1}$ and $\mu_i = \lambda_{(b-1)} - \lambda_{(i)} - (b-i-1) \geq 0$. Thus $s_\mu(\boldsymbol{\gamma})$ is well-defined. □

Next we consider $\det A_i$.

Consider $\lambda^{(i)} = (\lambda_0, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \lambda_{b-1})$, and let $\operatorname{inc}(\lambda^{(i)}) = (\lambda_0^{(i)}, \lambda_1^{(i)}, \ldots, \lambda_{b-1}^{(i)})$. Note that since $\gcd(a,b) = 1$, we have $u_i b - ia \neq 0$. Thus $\lambda_j^{(i)}$ with $0 \leq j \leq b-1$ are all distinct.

Lemma 4.3.

$$\det A_i = \operatorname{sgn}(\lambda^{(i)})\cdot z^{c-u_i/a}\cdot s_{\mu^{(i)}}(\boldsymbol{\gamma})\cdot a_\delta(\boldsymbol{\gamma})\cdot\prod_{j=0}^{b-1}\gamma_j^{-\lambda_{b-1}^{(i)}}, \tag{4.2}$$

here $\mu^{(i)} = (\mu_0^{(i)}, \mu_1^{(i)}, \ldots, \mu_{b-1}^{(i)})$ *is the partition with parts* $\mu_j^{(i)} = \lambda_{b-1}^{(i)} - \lambda_j^{(i)} - (b-j-1)$.

Proof. Replacing the i-th column of A by 1, we get

$$\begin{aligned}
\det A_i &= \det\begin{pmatrix}
z^{u_0/a}\gamma_0^{-\lambda_0} & \cdots & 1 & \cdots & z^{u_{b-1}/a}\gamma_0^{-\lambda_{b-1}}\\
z^{u_0/a}\gamma_1^{-\lambda_0} & \cdots & 1 & \cdots & z^{u_{b-1}/a}\gamma_1^{-\lambda_{b-1}}\\
\vdots & \ddots & \vdots & \ddots & \vdots\\
z^{u_0/a}\gamma_{b-1}^{-\lambda_0} & \cdots & 1 & \cdots & z^{u_{b-1}/a}\gamma_{b-1}^{-\lambda_{b-1}}
\end{pmatrix}\\
&= \operatorname{sgn}(\lambda^{(i)})\cdot z^{c-u_i/a}\cdot\det\left(\gamma_i^{-\lambda_j^{(i)}}\right)_{i,j=0}^{b-1}\\
&= \operatorname{sgn}(\lambda^{(i)})\cdot z^{c-u_i/a}\cdot\prod_{j=1}^{b-1}\gamma_j^{-\lambda_{b-1}^{(i)}}\cdot\det\left(\gamma_k^{\lambda_{b-1}^{(i)}-\lambda_j^{(i)}}\right)_{k,j=0}^{b-1}.
\end{aligned}$$

By a similar argument as in the proof of Lemma 4.1 we obtain the desired result. □

Combining Lemmas 4.2 and 4.3 we have

Theorem 4.4. *For* $0 \leq i \leq b-1$,

$$Q_i(z) = \operatorname{sgn}(\lambda)\cdot\operatorname{sgn}(\lambda^{(i)})\cdot z^{-u_i/a}\cdot\prod_{j=0}^{b-1}\gamma_j^{\lambda_{(b-1)}-\lambda_{b-1}^{(i)}}\cdot\frac{s_{\mu^{(i)}}(\boldsymbol{\gamma})}{s_\mu(\boldsymbol{\gamma})}. \tag{4.3}$$

Next we look at some examples of Theorem 4.4.

Example 1. $u_i = \lfloor ia/b \rfloor + \ell$. Consider the parking functions whose corresponding lattice paths are weakly below the line $y = a/b \cdot x + \ell - 1$, that is, **u**-parking functions with $u_i = \lfloor ia/b \rfloor + \ell$ for some integer $\ell \geq 1$.

The proof of the following lemma is similar to that of Lemma 2.3 and is omitted here. Interested readers can fill the details themselves.

Lemma 4.5. *The entries $\lambda_i = u_i b - ia$ form a permutation of the numbers $(\ell-1)b+1, (\ell-1)b+2, \ldots, \ell b$.*

Using the notation in Theorem 4.4, we have $\lambda_{(i)} = \ell b - b + i + 1$ and $\mu_i = 0$, thus $s_\mu(\boldsymbol{\gamma}) = 1$. On the other hand, $\mathrm{inc}(\lambda^{(i)}) = (0, (\ell-1)b+1, \ldots, \lambda_i - 1, \lambda_i + 1, \ldots, \ell b - 1, \ell b)$.

Theorem 4.6. *Denote by $d_i = \lambda_i - (\ell-1)b - 1$, then*

$$Q_i(z) = (-1)^{d_i} z^{-u_i/a} s_{((\ell-1)b+1, 1^{d_i})}(\boldsymbol{\gamma}) \tag{4.4}$$

Proof. By Lemma 4.2 we have

$$\det A = \mathrm{sgn}(\lambda) \cdot a_\delta(\boldsymbol{\gamma}) \cdot z^c \prod_{i=0}^{b-1} \gamma_i^{-\ell b}.$$

On the other hand, if the i-th column in A is replaced by a vector of ones, we have

$$\begin{aligned}\det A_i &= \mathrm{sgn}(\lambda) \cdot z^{c-u_i/a} \cdot \det\left(\boldsymbol{\gamma}_j^{-\ell b}, \boldsymbol{\gamma}_j^{-\ell b+1}, \ldots, \boldsymbol{\gamma}_j^{-\lambda_i - 1}, \mathbf{1}, \boldsymbol{\gamma}_j^{-\lambda_i+1}, \ldots, \boldsymbol{\gamma}_j^{-(\ell-1)b-1}\right) \\ &= \mathrm{sgn}(\lambda) \cdot z^{c-u_i/a} \cdot \prod_{i=0}^{b-1} \gamma_i^{-\ell b} \cdot \det\left(\mathbf{1}, \boldsymbol{\gamma}_j, \ldots, \boldsymbol{\gamma}_j^{\ell b - \lambda_i - 1}, \boldsymbol{\gamma}_j^{\ell b}, \boldsymbol{\gamma}_j^{\ell b - \lambda_i + 1}, \ldots, \boldsymbol{\gamma}_j^{b-1}\right),\end{aligned}$$

here $\boldsymbol{\gamma}_j^m$ in the determinant means the corresponding column is of the form $(\gamma_0^m, \gamma_1^m, \ldots, \gamma_{b-1}^m)^T$. Since it takes $d_i = \lambda_i - (\ell-1)b - 1$ transpositions to move the column $\boldsymbol{\gamma}_j^{\ell b}$ to the last, thus

$$\det A_i = (-1)^{d_i} \cdot \mathrm{sgn}(\lambda) \cdot z^{c-u_i/a} s_{((\ell-1)b+1, 1^{d_i})}(\boldsymbol{\gamma}) \cdot a_\delta(\boldsymbol{\gamma}) \cdot \prod_{i=0}^{b-1} \gamma_i^{-\ell b},$$

and we prove the theorem. □

Corollary 4.7. *In particular, when $i = 0$, we can simplify $Q_0(z)$ as*

$$\begin{aligned}Q_0(z) &= (-1)^{b-1} z^{-\ell/a} s_{((\ell-1)b+1, 1^{b-1})}(\boldsymbol{\gamma}) \\ &= (-1)^{b-1} z^{-\ell/a} e_b(\boldsymbol{\gamma}) \cdot h_{(\ell-1)b}(\boldsymbol{\gamma}).\end{aligned}$$

Moreover, when $a = 1$,

$$\begin{aligned}Q_{b-1}(z) &= z^{-\ell} s_{((\ell-1)b+1)}(\boldsymbol{\gamma}) \\ &= z^{-\ell} h_{(\ell-1)b+1}(\boldsymbol{\gamma}).\end{aligned}$$

Example 2. $u_i = \lceil ia/b \rceil + 1$. Next we consider the case where $u_i = \lceil ia/b \rceil + 1$. Using a similar argument as in Lemma 2.3, we can show that

Lemma 4.8. *Let $\lambda_i = u_i b - ia$, then $\lambda = (\lambda_0, \ldots, \lambda_{b-1})$ is a permutation of $b, b+1, \ldots, 2b-1$*

The above lemma gives that $\lambda_{(i)} = b + i$, and $\mu_i = \lambda_{(b-1)} - \lambda_{(i)} - (b - i - 1) = 0$, thus $s_\mu(\boldsymbol{\gamma}) = 1$.

On the other hand, $\mathrm{sgn}(\lambda^{(i)}) = (-1)^{\lambda_i + b} \cdot \mathrm{sgn}(\lambda)$, and $\mathrm{inc}(\lambda^{(i)}) = (0, b, b+1, \ldots, \lambda_i - 1, \lambda_i + 1, \ldots, 2b-1)$. Thus $\mu_i = (b, 1^{\lambda_i - b})$, and

Theorem 4.9. *Let $u_i = \lceil ia/b \rceil + 1$, then*

$$Q_i(z) = (-1)^{\lambda_i + b} \cdot z^{-u_i/a} \cdot s_{(b, 1^{\lambda_i - b})}(\boldsymbol{\gamma}). \tag{4.5}$$

In particular,

$$Q_0(z) = z^{-1/a} h_b(\boldsymbol{\gamma}).$$

5. Extension to non-coprime pairs

In Sections 2 and 4, there are three major steps in our algebraic approach. First, we find b distinct solutions to the equation $z = t^b \cdot e^{-at}$, which allow us to convert the Appell relation (c.f. Theorem 1.4) to a system of linear equations whose unknowns are $Q_i(z)$'s for $0 \le i \le b-1$. Second, the system is non-degenerate, hence $Q_i(z)$'s form a unique solution of the system. And last, we compute the determinants and use Cramer's rule to solve for $Q_i(z)$.

Our method can be extended to the case when $\gcd(a,b) \neq 1$ with some restrictions. In this section, we briefly discuss this case and give some examples. Throughout this section, we assume $\gcd(a,b) = k > 1$.

Let $\mathbf{u} = (u_0, u_1, \ldots)$ be a non-decreasing periodic sequence with period b and height a. Note that the Appell relation Eq.(2.3) still holds in this case, but we need to seek for new solutions $t_i(z)$.

Lemma 5.1. *Let $z = t^b e^{-at}$. Then there are b solutions $t_i(z)$ of the form $t_i(z) = \gamma_i^a(z)$, where*

$$\gamma_i(z) = z^{1/ab}\omega^i \exp\left(\frac{1}{a}R\Big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\Big)\right), \tag{5.1}$$

and ω is a primitive ab-th root of unity. Explicitly,

$$t_i(z) = \frac{b}{a}R\Big(\frac{\omega^{ai}\cdot a}{b}z^{1/b}\Big). \tag{5.2}$$

The proof is similar to that of Lemma 2.2 and is omitted here.

From $z = t^b e^{-at}$, we have $e^{-t_i} = \eta z^{1/a}\gamma_i^{-b}$, where η is an a-th root of unity. By comparing the constant term on both sides, we obtain $\eta = \omega^{ib}$ for $i = 0, 1, \ldots, b-1$. Thus we can simplify the matrix A in Eq.(2.4) as

$$A = \left(z^{u_j/a}\cdot\omega^{ibu_j}\gamma_i^{aj-u_jb}\right)_{i,j=0}^{b-1}. \tag{5.3}$$

Theorem 5.2. *Let $Q_i(z) = \displaystyle\sum_{n\ge 0}\frac{PF_{bn+i}(\mathbf{u})}{(bn+i)!}\cdot z^n$ for $0 \le i \le b-1$ be the generating functions for $\mathbf{u}$-parking functions. If $\det A \neq 0$, then*

$$Q_i(z) = \det A_i / \det A,$$

where $A = \left(z^{u_j/a}\cdot\omega^{ibu_j}\gamma_i^{aj-u_jb}\right)_{i,j=0}^{b-1}$, and A_i is the matrix obtained from A by replacing the i-th column by a column vector of ones.

It is not easy to compute the determinants of A or A_i for general $\mathbf{u}$. However, in some special cases we can derive nice formulas. One of such cases is when $u_i = u$ for all $0 \le i \le b-1$. Then we can simplify the determinant as

$$\begin{aligned}
\det A &= \det\left(z^{u/a}\cdot\omega^{ibu}\gamma_i^{aj-ub}\right)_{i,j=0}^{b-1}\\
&= z^{bu/a}\cdot\det\left(\gamma_i^{aj-ub}\right)_{i,j=0}^{b-1}\\
&= z^{bu/a}\cdot\prod_{i=0}^{b-1}\gamma_i^{-ub}\det\left(\gamma_i^{aj}\right)_{i,j=0}^{b-1}\\
&= z^{bu/a}\cdot\prod_{i=0}^{b-1}\gamma_i^{-ub}\cdot\det\left(t_i^{j}\right)_{i,j=0}^{b-1}\\
&= (-1)^{b(b-1)/2}z^{bu/a}\cdot\prod_{i=0}^{b-1}\gamma_i^{-ub}\cdot a_\delta(\mathbf{t}).
\end{aligned}$$

Note that in the last two steps, we use $t_i = \gamma_i^a$. In this case $\det(A) \neq 0$, so Theorem 5.2 applies with

$$\det A_i = \det \left(a_{j\ell}\right)_{j,\ell=0}^{b-1}$$

where $a_{j\ell} = z^{u/a} \cdot \omega^{jbu} \gamma_j^{a\ell - ub}$ when $\ell \neq i$ and $a_{ji} = 1$. If we further assume u is a multiple of k, then

$$\begin{aligned}\det A_i &= z^{(b-1)u/a} \cdot \prod_{i=0}^{b-1} \gamma_i^{-ub} \cdot \det \begin{pmatrix} 1 & \gamma_0^a & \cdots & \gamma_0^{(i-1)a} & \gamma_0^{ub} & \gamma_0^{(i+1)a} & \cdots & \gamma_0^{(b-1)a} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma_{b-1}^a & \cdots & \gamma_{b-1}^{(i-1)a} & \gamma_{b-1}^{ub} & \gamma_{b-1}^{(i+1)a} & \cdots & \gamma_{b-1}^{(b-1)a} \end{pmatrix} \\ &= (-1)^{b(b-1)/2+N(i)} \cdot z^{(b-1)u/a} \cdot \prod_{i=0}^{b-1} \gamma_i^{-ub} \cdot a_\delta(\boldsymbol{\gamma}) \cdot s_\mu(\boldsymbol{\gamma}).\end{aligned}$$

Here $N(i) = |\{j : i < j \leq b-1 \text{ and } ja < ub\}|$ and $\mu^{(i)}$ is the partition such that the decreasing rearrangement of $(a, \ldots, (i-1)a, ub, \ldots, (b-1)a)$ can be written as $\delta + \mu^{(i)}$.

Combining the above equations, we have

Corollary 5.3. *Let $u_i = u$ be a multiple of k for $i = 0, 1, \ldots, b-1$, and let $\mathbf{u}$ be the non-decreasing sequence of period b and height a with initial terms $\underbrace{u, u, \ldots, u}_{b}$. Then*

$$Q_i(z) = (-1)^{N(i)} \cdot z^{-u/a} \cdot \frac{s_{\mu^{(i)}}(\boldsymbol{\gamma}) a_\delta(\boldsymbol{\gamma})}{a_\delta(\mathbf{t})}. \tag{5.4}$$

In particular, if $u_i = a$ for $i = 0, 1, \ldots, b-1$. Then $ia - u_i b = (i-b)a$ for $0 \leq i \leq b-1$. Moreover, $N(i) = b - i - 1$, and

$$\det A_i = (-1)^{b-i-1} z^{b-1} \prod_{i=0}^{b-1} t_i^{-b} \cdot e_{b-i}(\mathbf{t}) a_\delta(\mathbf{t}).$$

Thus we have

Theorem 5.4. *Let $\mathbf{u} = (a, \ldots, a, 2a, \ldots, 2a, 3a, \ldots, 3a, \ldots)$ be a sequence of period b and height a, denote by $Q_i(z)$ the generating functions of $\{PF_{nb+i}(\mathbf{u}) : n \geq 0\}$. Then*

$$Q_i(z) = (-1)^{b-i-1} z^{-1} e_{b-i}(\mathbf{t}).$$

By computing the coefficients of $Q_{b-1}(z)$, we have

Corollary 5.5. *Let $\mathbf{u} = (a, \ldots, a, 2a, \ldots, 2a, 3a, \ldots, 3a, \ldots)$ be a sequence of period b and height a. Then*

$$PF_{mb+b-1}(\mathbf{u}) = a^{(m+1)b-1} \cdot (m+1)^{(m+1)b-2}.$$

The proof is similar to that of Corollary 2.11 and is omitted here.

Remark 5.6. Note that in Corollary 5.5, $\mathbf{u} = a \cdot (1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3, \ldots)$. In [11, Corollary 5.6], Kung and Yan showed that if $\mathbf{u} = k\mathbf{v}$ for an integer k, then $PF_n(\mathbf{u}) = k^n \ PF_n(\mathbf{v})$. Thus Corollary 5.5 can be derived from the homogeneity of $PF_n(\mathbf{u})$ and Corollary 2.11. This formula is also mentioned by Gaydarov and Hopkins in [6, Section 5] without a proof.

6. Eventually periodic boundaries

In this section we will give an example where the vector $\mathbf{u}$ is *eventually periodic*, that is, $\mathbf{u}$ can be written as the concatenation $\mathbf{w}_0, \mathbf{u}'$, where $\mathbf{w}_0 = (w_0, w_0, \ldots, w_{r-1})$ is a finite initial non-decreasing sequence of length $r \geq 1$, $\mathbf{u}' = (u_0, u_1, \ldots)$ is a periodic sequence, and $w_{r-1} \leq u_0$. We will show that in general, the idea of the previous discussion still works. But Eq.(2.2) would become

$$\sum_{i=0}^{b-1} Q_i(z) \cdot t^{i+r} e^{-u_i t} = 1 - \sum_{i=0}^{r-1} a_i \cdot \frac{t^i e^{-w_i t}}{i!} \tag{6.1}$$

for some integers $a_0, \ldots, a_{r-1}$, where again $z = t^b \cdot e^{-at}$ if $\mathbf{u}'$ is of period b and height a. Then again we just need b distinct solutions of $z = t^b \cdot e^{-at}$ to convert Eq.(6.1) to a linear system and solve it by computing certain determinants. The difference here is the right hand side of Eq.(6.1) contains a finite summation. So the solution would be a linear sum of r Schur functions.

We will illustrate the process via the following example, and interested readers can extend it to general cases.

Let ℓ be a positive integer and let the vector $\mathbf{u}$ be $(1, \underbrace{\ell+1, \ldots, \ell+1}_{b}, \underbrace{2\ell+1, \ldots, 2\ell+1}_{b}, 3\ell+1, \ldots)$, that is,

$$u_n = \begin{cases} 1, & \text{if } n = 0, \\ (k+1)\ell + 1, & \text{if } n = 1 + kb + i \text{ for } k \geq 0, 0 \leq i \leq b-1. \end{cases}$$

Although one can view $\mathbf{u}$ as a periodic sequence of length b and height ℓ, with initial terms $(1, \ell+1, ...\ell+1)$, we would single out the very first term and treat it as "eventually periodic" with first period $(\ell+1, ..., \ell+1)$. An advantage of doing so is that we do not need to worry about whether b and ℓ are coprime or not. We also remark that this sequence has appeared in the literature: the boundary $\mathbf{u}$ and its corresponding Dyck paths yield a solution to the generalized tennis ball problem; see [10, Section 5] for details and references. Enumeration of $PF_n = PF_n(u)$ can be considered as a "labeled" version of that problem. Interested readers can also choose $\mathbf{u}$ to be $(1, 1, \underbrace{\ell+1, \ldots, \ell+1}_{b}, \underbrace{2\ell+1, \ldots, 2\ell+1}_{b}, 3\ell+1, \ldots)$, so that the vector is not periodic. The calculations will be similar to the ones below, but the terms may look more complicated.

Note that $PF_0 = 1$. Then Appell's relation implies

$$\sum_{k \geq 0} \sum_{i=0}^{b-1} PF_{1+kb+i} \frac{t^{1+kb+i} e^{-(1+(k+1)\ell)t}}{(1+kb+i)!} = 1 - e^{-t}$$

If we write $z = t^b e^{-t}$, then

$$\begin{aligned} 1 - e^{-t} &= \sum_{i=0}^{b-1} \sum_{k \geq 0} PF_{1+kb+i} \frac{z^k}{(1+kb+i)!} t^{1+i} e^{-(1+\ell)t} \\ &= \sum_{i=0}^{b-1} Q_i(z) t^{1+i} e^{-(1+\ell)t}, \end{aligned}$$

where

$$Q_i(z) = \sum_{k \geq 0} \frac{PF_{1+kb+i}}{(1+kb+i)!} z^k.$$

Let $t_0, \ldots, t_{b-1}$ be the b distinct solutions to $z = t^b e^{-t}$ given by Lemma 2.2, that is,

$$t_i = b \cdot R\Big(\frac{\omega^i}{b} z^{1/b}\Big).$$

Then the generating functions $Q_i(z)$'s satisfy the system of linear equations

$$\left(t_i^{j+1}e^{-(\ell+1)t_i}\right)_{i,j=0}^{b-1}\cdot\left(Q_i(z)\right)_{i=0}^{b-1}=\left(1-e^{-t_i}\right)_{i=0}^{b-1}.$$

Since $e^{-t}=zt^{-b}$, we can rewrite the above equation as

$$\left(z^{\ell+1}\cdot t_i^{j+1-(\ell+1)b}\right)_{i,j=0}^{b-1}\cdot\left(Q_i(z)\right)_{i=0}^{b-1}=\left(1-zt_i^{-b}\right)_{i=0}^{b-1}.$$

Denote this equation as $B\vec{Q}=\vec{D}$, then

$$\begin{aligned}\det B &= z^{(\ell+1)b}\prod_{i=0}^{b-1}t_i^{1-(\ell+1)b}\det\left(t_i^j\right)_{i,j=0}^{b-1}\\ &= (-1)^{b(b-1)/2}\cdot z^{(\ell+1)b}\cdot a_\delta(\mathbf{t})\cdot\prod_{i=0}^{b-1}t_i^{1-(\ell+1)b}.\end{aligned}$$

On the other hand, replacing the i-th column of B by the column vector $\vec{D}$ and denoting the resulting matrix by B_i, we have

$$\begin{aligned}\det B_i =& \det\begin{pmatrix} z^{\ell+1}t_0^{1-(\ell+1)b} & z^{\ell+1}t_0^{2-(\ell+1)b} & \cdots & 1 & \cdots & z^{\ell+1}t_0^{-\ell b}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ z^{\ell+1}t_{b-1}^{1-(\ell+1)b} & z^{\ell+1}t_{b-1}^{2-(\ell+1)b} & \cdots & 1 & \cdots & z^{\ell+1}t_{b-1}^{-\ell b}\end{pmatrix}\\ &-\det\begin{pmatrix} z^{\ell+1}t_0^{1-(\ell+1)b} & z^{\ell+1}t_0^{2-(\ell+1)b} & \cdots & zt_0^{-b} & \cdots & z^{\ell+1}t_0^{-\ell b}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ z^{\ell+1}t_{b-1}^{1-(\ell+1)b} & z^{\ell+1}t_{b-1}^{2-(\ell+1)b} & \cdots & zt_{b-1}^{-b} & \cdots & z^{\ell+1}t_{b-1}^{-\ell b}\end{pmatrix}\end{aligned}$$

Let $B_i^{(0)}$ and $B_i^{(1)}$ be the two matrices in the above formula. We calculate their determinants.

$$\begin{aligned}\det B_i^{(0)} &= z^{(b-1)(\ell+1)}\prod_{j=0}^{b-1}t_j^{1-(\ell+1)b}\det\begin{pmatrix}1 & t_0 & \cdots & t_0^{(\ell+1)b-1} & \cdots & t_0^{b-1}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ 1 & t_{b-1} & \cdots & t_{b-1}^{(\ell+1)b-1} & \cdots & t_{b-1}^{b-1}\end{pmatrix}\\ &= (-1)^{b-i-1}z^{(b-1)(\ell+1)}\prod_{j=0}^{b-1}t_j^{1-(\ell+1)b}\det\begin{pmatrix}1 & t_0 & \cdots & t_0^{i-1} & t_0^{i+1} & \cdots & t_0^{b-1} & t_0^{(\ell+1)b-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & t_{b-1} & \cdots & t_{b-1}^{i-1} & t_{b-1}^{i+1} & \cdots & t_{b-1}^{b-1} & t_{b-1}^{(\ell+1)b-1}\end{pmatrix}\\ &= (-1)^{b-i-1}\cdot(-1)^{b(b-1)/2}\cdot z^{(b-1)(\ell+1)}\prod_{j=0}^{b-1}t_j^{1-(\ell+1)b}s_{(\ell b,1^{b-i-1})}(\mathbf{t})a_\delta(\mathbf{t}).\end{aligned}$$

Similarly,

$$\begin{aligned}\det B_i^{(1)} &= z^{(b-1)(\ell+1)+1}\prod_{j=0}^{b-1}t_j^{1-(\ell+1)b}\det\begin{pmatrix}1 & t_0 & \cdots & t_0^{\ell b-1} & \cdots & t_0^{b-1}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ 1 & t_{b-1} & \cdots & t_{b-1}^{\ell b-1} & \cdots & t_{b-1}^{b-1}\end{pmatrix}\\ &= (-1)^{b-i-1}z^{(b-1)(\ell+1)+1}\prod_{j=0}^{b-1}t_j^{1-(\ell+1)b}\det\begin{pmatrix}1 & t_0 & \cdots & t_0^{i-1} & t_0^{i+1} & \cdots & t_0^{b-1} & t_0^{\ell b-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & t_{b-1} & \cdots & t_{b-1}^{i-1} & t_{b-1}^{i+1} & \cdots & t_{b-1}^{b-1} & t_{b-1}^{\ell b-1}\end{pmatrix}\end{aligned}$$

$$= (-1)^{b-i-1} \cdot (-1)^{b(b-1)/2} \cdot z^{(b-1)(\ell+1)+1} \prod_{j=0}^{b-1} t_j^{1-(\ell+1)b} s_{((\ell-1)b,1^{b-i-1})}(\mathbf{t}) a_\delta(\mathbf{t}),$$

if $\ell > 1$. When $\ell = 1$, $\det B_i = 0$ if $0 \le i < b-1$, and

$$\det B_{b-1}^{(1)} = (-1)^{b-i-1} \cdot (-1)^{b(b-1)/2} \cdot z^{(b-1)(\ell+1)+1} \prod_{j=0}^{b-1} t_j^{1-(\ell+1)b} a_\delta(\mathbf{t}).$$

In conclusion,

(i) When $\ell > 1$,

$$\begin{aligned} Q_i(z) &= \frac{\det B_i}{\det B} \\ &= (-1)^{b-i-1} \big(z^{-(\ell+1)} s_{(\ell b,1^{b-i-1})}(\mathbf{t}) - z^{-\ell} s_{((\ell-1)b,1^{b-i-1})}(\mathbf{t}) \big) \end{aligned} \tag{6.2}$$

In particular, when $i = b-1$

$$Q_{b-1}(z) = z^{-(\ell+1)} h_{\ell b}(\mathbf{t}) - z^{-\ell} h_{(\ell-1)b}(\mathbf{t}).$$

(ii) When $\ell = 1$,

$$Q_i(z) = \begin{cases} (-1)^{b-i-1} z^{-(\ell+1)} s_{(b,1^{b-i-1})}(\mathbf{t}), & \text{if } 0 \le i \le b-2, \\ z^{-(\ell+1)} h_b(\mathbf{t}) - z^{-\ell}, & \text{if } i = b-1 \end{cases} \tag{6.3}$$

We remark that although Eqs.(6.2) and (6.3) contain negative powers of z, they are indeed formal power series of z. As an example, consider $b = 2$ and $\ell = 1$. Then

$$\begin{aligned} Q_0(z) &= -z^{-2} s_{(2,1)}(\mathbf{t}) = -z^{-2} t_0 t_1 (t_0 + t_1) \\ Q_1(z) &= z^{-2} h_2(\mathbf{t}) - z^{-1} = z^{-2}(t_0^2 + t_0 t_1 + t_1^2) - z^{-1}. \end{aligned}$$

where $t_i = 2R((-1)^i z^{1/2}/2) = (-1)^i z^{1/2} e^{t_i/2}$. Let

$$G(z) = \sum_{m \ge 1} \frac{m^{2m-1}}{(2m)!} z^m.$$

Then $t_0 + t_1 = 2G(z)$ and $t_0 t_1 = -z \exp(G(z))$. Hence we can rewrite $Q_0(z)$ and $Q_1(z)$ as

$$\begin{aligned} Q_0(z) = &= z^{-1}(t_0 + t_1) \exp\left(\frac{1}{2}(t_0 + t_1) \right), \\ &= 2z^{-1} G(z) \exp(G(z)), \end{aligned}$$

and

$$\begin{aligned} Q_1(z) &= z^{-2}(t_0 + t_1)^2 - z^{-2} t_0 t_1 - z^{-1} \\ &= 4z^{-2} G(z)^2 + z^{-1} \exp(G(z)) - z^{-1}. \end{aligned}$$

7. Concluding remarks and open questions

In [11], Kung and Yan found the relation between Gončarov polynomials and $\mathbf{u}$-parking functions.

$$PF_n(\mathbf{u}) = g_n(0; -u_0, \ldots, -u_{n-1}) = (-1)^n g_n(0; u_0, \ldots, u_{n-1}). \tag{7.1}$$

The parameters $u_0, \ldots, u_{n-1}$ for Gončarov polynomials can take any real values, for which the most simple and useful case is when $\mathbf{u}$ is given by a linear function, i.e., $u_i = a + ib$ for some real constants a, b. However, Eq.(7.1) only holds when $\mathbf{u}$ is an integer sequence. One natural question is to ask whether we can have an explicit relation between rational parking functions and Gončarov polynomials with non-integral but linear $\mathbf{u}$? In [13], the authors introduced *generalized Gončarov polynomials* by replacing the differentiation operator with a *delta operator*. One direction of solving this problem is to find an appropriate delta operator or a variation of the generalized Gončarov polynomials that work in the rational case. For more details of Gončarov polynomials and delta operators, we refer the readers to [13].

In our work, when the generating function can be expressed in terms of elementary symmetric functions, we can derive nice enumerative formulas on (a, b)-parking functions of certain lengths by computing the coefficients. However, for the more general case of Schur functions we can hardly simplify the coefficients. In particular, when the generating functions are expressed as complete homogeneous symmetric functions, such as in the examples in Section 4, we would like to find some interpretations of the coefficients. For example, let $a = 1$ and $b = 2$ in Theorem 4.9, then $Q_0(z) = z^{-1}(\gamma_0^2 + \gamma_1^2 + \gamma_0\gamma_1)$. In this case, $\gamma_i = 2R\big((-1)^i z^{1/2}/2\big)$. Using the formula

$$R^k(z) = k \cdot \sum_{n \geq k} \frac{n^{n-k} \cdot (n-1)!}{(n-k)!} \cdot \frac{x^n}{n!}$$

we can simplify $z^{-1}(\gamma_0^2 + \gamma_1^2)$ as

$$\begin{aligned}
z^{-1}(\gamma_0^2 + \gamma_1^2) &= 4z^{-1}\Big(R^2\big(\frac{1}{2}z^{1/2}\big) + R^2\big(-\frac{1}{2}z^{1/2}\big)\Big) \\
&= 4z^{-1} \cdot 2 \cdot \sum_{n \geq 2} \frac{n^{n-2} \cdot (n-1)!}{(n-2)!n!}\Big(\big(\frac{1}{2}z^{1/2}\big)^n + \big(-\frac{1}{2}z^{1/2}\big)^n\Big) \\
&= 4z^{-1}\Big(2\sum_{n \geq 1} \frac{(2n)^{2n-2}(2n-1)!}{(2n-2)!}\big(\frac{1}{2}\big)^{2n-1}\frac{z^n}{(2n)!}\Big) \\
&= 4\sum_{n \geq 1} \frac{n^{2n-2}}{2n \cdot (2n-2)!} z^{n-1} \\
&= 2 \cdot \sum_{n \geq 0} \frac{(n+1)^{2n-1}}{(2n)!} \cdot z^n.
\end{aligned}$$

However, the term $z^{-1}\gamma_0\gamma_1$ can not be simplified easily. It is also interesting to find combinatorial meanings of these coefficients, which may help us to find combinatorial proofs of the results appeared in Section 4.

There are many elegant q-analogues for classical parking functions. For example, the following two results are proved by Foata and Riordan [5]. Given a classical parking function $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$, let $\mathrm{tie}(\mathbf{c}) = \{i : 0 \leq i < n-1 \text{ and } c_i = c_{i+1}\}$. Then we have a q-analogue of Caylay's formula:

$$\sum_{\mathbf{c} \in \mathcal{PF}_n} q^{\mathrm{tie}(\mathbf{c})} = (q+n)^{n-1}.$$

Similarly, if $Z(\mathbf{c})$ is the number of zeros in $\mathbf{c}$, then

$$\sum_{\mathbf{c} \in \mathcal{PF}_n} q^{Z(\mathbf{c})} = q(q+n)^{n-1}.$$

Can we extend these results to rational parking functions? In particular, can we have a q-analogue of the generating functions in terms of the symmetric functions?

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References

[1] D. Armstrong, N. A. Loehr and G. S. Warrington, Rational parking functions, *Ann. Comb.* **12** (2016), 21–58.

[2] S. Butler, R. Graham and C. H. Yan, Parking distributions on trees, *European J. Combin.* **65** (2017), 168–185.

[3] I. F. Blake and A. G. Konheim, Big buckets are (are not) better, *J. Assoc. Comput. Math.* **24** (1977), no. 4, 591–606.

[4] R. Ehrenborg and A. Happ, Parking cars of different sizes, *Amer. Math. Monthly* **123** (2016), no. 10, 1045–1048.

[5] D. Foata and J. Riordan, Mappings of acyclic and parking functions, *Aequationes Math.* **10** (1974), 10–22.

[6] P. Gaydarov and S. Hopkins, Parking functions and tree inversions revisited, *Adv. in Appl. Math* **80** (2016), 151–179.

[7] W. King and C. H. Yan, Prime parking functions on rooted trees, *J. Combin. Theory A.* **168** (2019), 1–25.

[8] A. G. Konheim and B. Weiss, An occupancy discipline and applications, *SIAM J. Applied Math.* **14** (1966), 1266–1274.

[9] C. F. Krattehthaler, "Lattice path enumeration," in *Handbook of Enumerative Combinatorics*, 589–678, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015.

[10] J. P. S. Kung, A. de Mier, X. Sun and C. Yan, Lattice and Schröder paths with periodic boundaries, *J. Statist. Plann. Inference* **139** (2009), no. 6, 2014–2027.

[11] J. P. S. Kung and C. Yan, Gancărov polynomials and parking functions, *J. Combin. Theory Ser. A* **102** (2003), 16–37.

[12] M.-L. Lackner and A. Panholzer, Parking functions for mappings, *J. Combin. Theory Ser. A* **142** (2016), 1–28.

[13] R. Lorentz, S. Tringali and C. Yan, "Generalized Gončarov polynomials," in *Connections in Discrete Mathematics*, 56–85, Cambridge Univ. Press, Cambridge, 2018.

[14] I. G. Macdonald, "Symmetric Functions and Hall Polynomials. Second edition," Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[15] A. Postnikov and B. Shapiro, Trees, parking functions, syzygies, and deformations of monomial ideals, *Trans. Amer. Math. Soc.* **356** (2004), no. 8, 3109–3142.

[16] J. Riordan, Ballots and trees, *J. Combinatorial Theory* **6** (1969), 408–411.

[17] R. P. Stanley, "Enumerative Combinatorics, Vol. I, 2nd edition," Cambridge Studies in Advanced Mathematics (No. 49), Cambridge University Press, 2012.

[18] R. P. Stanley, "Enumerative Combinatorics, Vol. II," Cambridge Studies in Advanced Mathematics (No. 62), Cambridge University Press, 1999.

[19] R. P. Stanley and J. Pitman, A polytope related to empirical distributions, plane trees, parking functions, and the associahedron, *Discrete Comput. Geom.* **27** (2002), no. 4, 603–634.

[20] R. Sulzgruber, "Cayley-Catalan Combinatorics of Affine Permutation Groups," Doctoral dissertation, University of Vienna, 2017.

[21] M. Thiel, "Catalan combinatorics of crystallographic root systems," Doctoral dissertation, University of Vienna, 2015.

[22] C. Yan, "Parking functions," in *Handbook of Enumerative Combinatorics*, 835–893, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015.