Supplement: On the equation in Corollary 4.7

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In Corollary 4.7 of [3] we obtained the following interesting equation of the Narayana numbers
\[ N(n,k) = \frac{1}{k} \binom{n}{k} \binom{n}{k-1}. \]

\[ \sum_{p \geq 0} N(1+p+q,1+p)z^p = \frac{N_q(z)}{(1-z)^{2q+1}}, \tag{1} \]

where \( N_q(z) = \sum_{k=1}^{n} N(n,k)z^{k-1} \).

In a talk presented at the International Conference on Enumerative Combinatorics and Applications Virtual (ICECA 2023, Sep 4–6, 2023), I asked whether anyone has seen this equation or something similar before. Thanks for replies from Mireille Bousquet-Melou, Richard Stanley, and Sergi Elizalde, who provided valuable information on this equation.

1 Parallelogram polyomino

![Figure 1: A parallelogram polyomino having a 6 × 5 bounding box.](image)

Equation (1) has been proved by Mireille Bousquet-Melou [1, 2] as the generating function of the number of parallelogram polyominoes having a fixed width.

A parallelogram polyomino having an \( m \times n \) bounding box is a polyomino in a rectangle consisting of \( m \times n \) cells that is formed by cutting out two (possibly empty) non-touching Young diagrams which have corners at \((0, n)\) and \((m, 0)\). See Figure 1 for a parallelogram polyomino having a 6 × 5 bounding box.
In [2, Eq.(19)] Bousquet-Melou proved that $P_{n,m}$, the number of parallelogram polyominoes within a $p \times q$ bounding box, is the Narayana number $N(p+q-1, p)$. In an accompanying paper [1] she derived explicit expressions for the generating function of parallelogram polyominoes, according to their height, width and area. A specialization of the generating function of parallelogram polyominoes with width $n$, given in Equations (23)–(25) in [1], can be stated as

$$P(y) := \sum_m y^m P_{n,m} = y \frac{N_{n-1}(y)}{(1-y)^{2n-1}}.$$ 

Given a parallelogram polyomino with an $m \times n$ bounding box, removing the two cells at the lower-left corner and the upper-right corner, we obtain two lattice paths in the boundary of the polyomino, namely, $L_1$ from $(1,0)$ to $(m,n-1)$ and $L_2$ from $(0,1)$ to $(m-1,n)$. Moving $L_1$ one unit to the left and $L_2$ one unit down, we obtain a pair of non-crossing lattice paths from $(0,0)$ to $(m-1,n-1)$. Figure 2 shows the non-crossing lattice paths corresponding to the parallelogram polyomino in Figure 1. This above process is invertable and gives a bijection between parallelogram polyominoes with an $m \times n$ bounding box and pairs of non-crossing lattice paths with the box $(m-1) \times (n-1)$. The latter correspond to the set of $(m-1, n-1)$-parking functions. Changing variables, we see that Corollary 4.7 in our paper matches exactly the results of Bousquet-Melou.

Figure 2: A pair of non-crossing lattice paths in a 5 × 4 box.

2 Order polynomial and P-Eulerian polynomial

This interpretation of [1] is communicated by Richard Stanley.

In [3, Prop. 4.3, Cor. 4.4] we obtained that $N(p+q, 1+p)$ is number of multichains of length $p+1$ in the poset $2 \times q$, which also counts the number of order preserving maps from $2 \times q$ to $p+1$. Thus the left side of Equation (1) is the generating function for the order polynomial of the poset $2 \times q$ (up to a factor $z$).

From [5, Theorem 4.5.14], the generating function of the order polynomial of a poset $P$ can be expressed as $P(X)/(1-z)^{n+1}$, where $n = |P|$ and $P(x)$ is the $P$-Eulerian polynomial.

Some background on $P$-Eulerian polynomial: Let $(P, \preceq)$ be a poset on the vertex set $[n]$. The Jordan-Hölder set of $P$ is the set of linear extensions of $P$:

$$\mathcal{L}(P) := \{ \pi \in \Theta_n : \text{if } \pi_i \preceq \pi_j, \text{ then } i \leq j \text{ for all } i, j \in [n] \}.$$
where each permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ in the symmetric group $S_n$ is written in one-line notation. The $P$-Eulerian polynomial is defined by

$$A_P(x) := \sum_{\pi \in \mathcal{L}(P)} x^{\text{des}(\pi) + 1}. \quad (2)$$

When $P = 2 \times q$, we can assume that the vertex set is $[2q]$ and the partial order $\preceq$ is given by $1 \preceq 2 \preceq \cdots \preceq q, q + 1 \preceq q + 2 \preceq \cdots \preceq 2q$ and $i \preceq q + i$ for $i \in [q]$. Hence each linear extension of $P$ corresponds to a Dyck path of semilength $q$, and a descent corresponds to a valley of the Dyck path. Since the number of valleys is one less than the number of peaks, we have that the $P$-Eulerian polynomial is the generating function of Dyck paths by the number of peaks, which is $z N_q(z)$.

The above argument gives the equation

$$\sum_{p \geq 0} N(1 + p + q, 1 + p) z^{p+1} = \frac{z N_q(z)}{(1 - z)^{2q+1}},$$

which is equivalent to Equation (1).

**Remark.** For any finite poset $P$, the order polytope $O(P)$, and a related chain polytope $C(P)$, have the property that their Ehrhart polynomial is the order polynomial of $P$ (shifted by 1). This implies that the $h^*$-vector of $O(P)$ is the $P$-Eulerian polynomial. See [4, Section 4].

Equivalently, $N_q(z)$ is the $h^*$-vector of the order polytope of $2 \times q$. By the hook-content formula, the coefficients of the generating function are

$$(p + 1)(p + 2)^2(p + 3)^2 \cdots (p + q)^2(p + q + 1)/(12^2 \cdots q^2(q + 1)) = N(1 + p + q, 1 + p),$$

agreeing with our result.

## 3 A bijective proof

The following bijective proof of (1) is constructed by Sergi Elizalde.

The coefficient of $z^p$ in the left-hand side of Equation (1) counts Dyck paths of semilength $p + q + 1$ with $p + 1$ peaks.

The coefficient of $z^p$ in the right-hand side counts Dyck paths of semilength $q$ (given by the numerator), together with a sequence of nonnegative integers $(a_0, a_1, \ldots, a_{2q})$ (given by the denominator). If the path has $k$ peaks, then we must have $a_0 + a_1 + \cdots + a_{2q} = p + 1 - k$, to get a $z^p$. We can think of $(a_0, \ldots, a_{2q})$ as labels on the $2q + 1$ vertices of the path.

Now here is a bijection between the two sides. Take a Dyck path of semilength $p + q + 1$ with $p + 1$ peaks. Remove the $p + 1$ peaks $UD$ of this path, and put a label on each vertex indicating how many peaks were removed from that location.

This gives a bijection to Dyck paths of semilength $q$ together with a sequence of nonnegative integers $(b_0, b_1, \ldots, b_{2q})$ indicating how many peaks were removed at each vertex, with the condition that $b_0 + \cdots + b_{2q} = p + 1$, and that each vertex on top of a peak must have at a label $b_i > 0$ (otherwise it would have been a peak of the original path and it would have been removed).
Now, subtract one from each label at a peak.
This gives a bijection to Dyck paths of semilength $q$ together with a sequence of nonnegative integers $(a_0, a_1, \ldots, a_{2q})$ such that $a_0 + \cdots + a_{2q} = p + 1 - k$, where $k$ is the number of peaks of the path.

References


