# $\boldsymbol{U}$-Parking Functions and $(p, q)$-Parking Functions

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**Abstract**

The notions of $\boldsymbol{U}$-parking functions and $(p, q)$-parking functions are two high-dimensional generalizations of the classical parking functions. $\boldsymbol{U}$-parking functions are defined via a special family of interpolation polynomials called Gončarov polynomials, while $(p, q)$-parking functions can be interpreted as recurrent configurations in the sandpile model for a complete bipartite graph with an additional root, as introduced by Cori and Poulalhon. In this paper we show that $(p, q)$-parking functions can be obtained as a specialization of $\boldsymbol{U}$-parking functions and characterized by a pair of weakly disjoint lattice paths in the grid $p \times q$. Then we present various enumerative results for increasing $(p, q)$-parking functions.

## 1 Introduction

Classical parking functions were initially defined by Konheim and Weiss as a tool in their study of random hashing functions [12], though in recent decades parking functions have been well-studied and recognized for their numerous connections to various combinatorial objects, from labeled trees and noncrossing partitions, to hyperplane arrangements and the sandpile model [23]. To give a simple yet formal definition, a *parking function of length* $n$ is a sequence $\boldsymbol{a} = a_1 a_2 \dots a_n$ of $n$ non-negative integers whose non-decreasing rearrangement $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ satisfies $a_{(i)} < i$ for each $1 \leq i \leq n$. Here, we call the non-decreasing sequence $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ the *order statistics of* $\boldsymbol{a}$, with $a_{(i)}$ the *i-th order statistic of* $\boldsymbol{a}$.

Among the various generalizations of parking functions which have appeared since the classical definition, two higher dimensional generalizations exist in the literature: the $\boldsymbol{U}$-parking function and the $(p, q)$-parking function. The more recent of the two, the $\boldsymbol{U}$-parking function, was described by Khare, Lorentz, and Yan [11] in their study of bivariate Gončarov polynomials, a class of polynomials that form the basis for the solutions of the two-dimensional Gončarov Interpolation Problem. Combinatorially, bivariate Gončarov polynomials enumerate the number of pairs of integer sequences whose order statistics satisfy certain constraints. These pairs of integer sequences arising naturally from this combinatorial interpretation are what Khare et al. referred to as *2-dimensional* $\boldsymbol{U}$*-parking functions*, with multidimensional $\boldsymbol{U}$-parking functions analogously derived from multivariate Gončarov polynomials. For a precise definition and further discussion on 2-dimensional $\boldsymbol{U}$-parking functions, see Subsection 2.1.

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Motivated by the connection between classical parking functions and the Abelian sandpile model on complete graphs, Cori and Poulalhon [7] introduced $(p, q)$-parking functions, which can be realized as recurrent states of the sandpile model on complete bipartite graphs having an additional distinguished root vertex. These 2-dimensional parking functions admit a definition in terms of a *certificate* permutation $\sigma \in \mathfrak{S}_{p+q}$, as well as in terms of parking cars of two distinct colors. It should be noted that classical parking functions might also be defined in terms of a certificate: the integer sequence $\boldsymbol{a} = a_1a_2 \dots a_n$ is a parking function if and only if there exists a permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n$ such that $0 \leq a_i < \sigma_i$ for all $i$. A straightforward extension of these definitions of $(p, q)$-parking functions to $k$ dimensions for an arbitrary integer $k > 2$ yields what are more generally referred to as $(p_1, p_2, \dots, p_k)$-parking functions. More on $(p, q)$-parking functions, including formal definitions, is provided in Subsection 2.2.

Beyond the preliminary definitions of Section 2, the remainder of this paper seeks to bring together the two notions of higher dimensional parking functions, as well as share several results concerning increasing $(p, q)$-parking functions. The first of these objectives is addressed in Section 3, in which $(p, q)$-parking functions are shown to be $\boldsymbol{U}$-parking functions for a specific node-set $\boldsymbol{U}$. Section 4 introduces the sum-enumerator of increasing $(p, q)$-parking functions, with Subsections 4.1-4.2 presenting a recurrence relation and then utilizing a plane partition representation to derive further enumerative results. A $q$-analog of the Narayana numbers is defined in Subsection 4.3. Finally, Section 5 combines several results on noncrossing partitions and parallelogram polyominoes to relate the usual area-statistic on Catalan paths to two area-statistics on a pair of weakly disjoint lattice paths, from which we are able to associate the sum-enumerator of Section 4 to certain well-known Catalan $q$-analogs in the literature.

# 2 Two-dimensional parking functions

## 2.1 Multidimensional $\boldsymbol{U}$-parking functions

We begin with a brief overview of the multidimensional $\boldsymbol{U}$-parking function, which originated in the study of bivariate Gončarov polynomials [11] and their relationship to pairs of integer sequences whose order statistics satisfy certain constraints. For general $p, q \in \mathbb{N}$, let $\boldsymbol{U} \subset \mathbb{N}^2$ be a set of nodes $\boldsymbol{U} = \{(u_{i,j}, v_{i,j}) : 0 \leq i \leq p, 0 \leq j \leq q\}$. Define $G_{p,q}(\boldsymbol{U})$ to be the directed graph having as vertices the lattice points $\{(i, j) : 0 \leq i \leq p, 0 \leq j \leq q\}$ and having as edges all north steps $N = (0, 1)$ and east steps $E = (1, 0)$ connecting its vertices. Every edge $e$ of $G_{p,q}(\boldsymbol{U})$ is assigned a weight $wt(e)$ given by

$$wt(e) = \begin{cases} u_{i,j} & \text{if } e \text{ is an east step from } (i,j) \text{ to } (i+1,j), \\ v_{i,j} & \text{if } e \text{ is a north step from } (i,j) \text{ to } (i,j+1). \end{cases}$$

For a lattice path $P$ from the origin $(0, 0)$ to the point $(p, q)$, we write $P = e_1e_2 \dots e_{p+q}$, where $e_i \in \{E, N\}$, to record the sequence of steps of $P$. Note that $P$ must have exactly $p$ east steps and $q$ north steps. Let $(\boldsymbol{a}, \boldsymbol{b})$ be a pair of non-negative integer sequences with $\boldsymbol{a} = a_1a_2 \dots a_p$ and $\boldsymbol{b} = b_1b_2 \dots b_q$. Then we say that the order statistics of $(\boldsymbol{a}, \boldsymbol{b})$ are bounded by $P$ with respect to the set $\boldsymbol{U}$ if and only if, for $r = 1, 2, \ldots, p+q$,

$$\begin{cases} a_{(i)} < wt(e_r) & \text{if } e_r \text{ is the } i\text{-th east step of } P, \\ b_{(j)} < wt(e_r) & \text{if } e_r \text{ is the } j\text{-th north step of } P, \end{cases}$$

where $a_{(1)} \leq \cdots \leq a_{(p)}$ and $b_{(1)} \leq \cdots \leq b_{(q)}$ are the order statistics of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

**Definition 2.1.** [11] Suppose $\boldsymbol{U} = \{(u_{i,j}, v_{i,j}) : 0 \leq i \leq p, 0 \leq j \leq q\} \subset \mathbb{N}^2$ is a set of nodes satisfying $u_{i,j} \leq u_{i',j'}$ and $v_{i,j} \leq v_{i',j'}$ when $i \leq i'$ and $j \leq j'$. A pair of sequences of non-negative integers $(\boldsymbol{a}, \boldsymbol{b}) = (a_1 a_2 \ldots a_p, b_1 b_2 \ldots b_q)$ is a *2-dimensional $\boldsymbol{U}$-parking function* if and only if the order statistics of $(\boldsymbol{a}, \boldsymbol{b})$ are bounded by some lattice path from $(0,0)$ to $(p,q)$ with respect to $\boldsymbol{U}$.

**Remark.** The lattice path, if it exists, may not be unique.

The set of 2-dimensional $\boldsymbol{U}$-parking functions with the above parameters will be denoted $\mathcal{PF}^{(2)}_{p,q}(\boldsymbol{U})$. We may analogously define a $k$-dimensional $\boldsymbol{U}$-parking function for $k > 2$: for $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, given a set of nodes $\boldsymbol{U} = \{z_{\boldsymbol{i}} = (z_{i_1}, \ldots, z_{i_k}) \in \mathbb{N}^k : \ 0 \leq i_j \leq n_j \text{ for all } j\}$, we define a weighted directed graph $G_{\boldsymbol{n}}(\boldsymbol{U})$ whose vertices are the integer points $\{\boldsymbol{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k : i_j \leq n_j \text{ for all } j\}$ and whose edges consist of all unit steps connecting those integer points, with the $k$ directed edges starting from point $\boldsymbol{i}$ having weights $z_{i_1}, \ldots, z_{i_k}$. A $k$-tuple $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ of non-negative integer sequences of respective lengths $n_1, \ldots, n_k$ is a $k$-dimensional $\boldsymbol{U}$-parking function if its order statistics are bounded by some lattice path from the origin to the point $\boldsymbol{n} = (n_1, \ldots, n_k)$ with respect to $\boldsymbol{U}$. As with the 2-dimensional case, the set of all $k$-dimensional $\boldsymbol{U}$-parking functions will be denoted by $\mathcal{PF}^{(k)}_{\boldsymbol{n}}(\boldsymbol{U})$. Multidimensional $\boldsymbol{U}$-parking functions can be enumerated by specializations of multivariate Gončarov polynomials, as described in [11].

Since the definition of a 2-dimensional $\boldsymbol{U}$-parking function $(\boldsymbol{a}, \boldsymbol{b})$ is dependent on the order statistics of both sequences $\boldsymbol{a}$ and $\boldsymbol{b}$, the set $\mathcal{PF}^{(2)}_{p,q}(\boldsymbol{U})$ is invariant under the action of the product $\mathfrak{S}_p \times \mathfrak{S}_q$ of symmetric groups. That is, for any $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$, $(\sigma(\boldsymbol{a}), \tau(\boldsymbol{b}))$ is also a 2-dimensional $\boldsymbol{U}$-parking function. In particular, we will say a 2-dimensional $\boldsymbol{U}$-parking function $(\boldsymbol{a}, \boldsymbol{b})$ is *increasing* if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-decreasing sequences. Let $\mathcal{IPF}^{(2)}_{p,q}(\boldsymbol{U})$ denote the subset of 2-dimensional $\boldsymbol{U}$-parking functions which are increasing. Thus, we might consider the set $\mathcal{IPF}^{(2)}_{p,q}(\boldsymbol{U})$ as representatives of the orbits of the action of $\mathfrak{S}_p \times \mathfrak{S}_q$ on $\mathcal{PF}^{(2)}_{p,q}(\boldsymbol{U})$. Of course, this definition of increasing $\boldsymbol{U}$-parking functions is readily extended to the $k$-dimensional case.

## 2.2 $(p,q)$-parking functions

An alternative 2-dimensional generalization of parking functions, which predates the aforementioned $\boldsymbol{U}$-parking functions, was introduced by Cori and Poulalhon in [7]. For positive integers $p$ and $q$, a $(p,q)$-*sequence* is a pair $(\boldsymbol{a}, \boldsymbol{b})$ of sequences of non-negative integers such that $\boldsymbol{a} = a_1 a_2 \ldots a_p$ is of length $p$ with $0 \leq a_i \leq q$ for all $i \in [p]$, and $\boldsymbol{b} = b_1 b_2 \ldots b_q$ is of length $q$ with $0 \leq b_j \leq p$ for all $j \in [q]$. Let $\mathcal{S}_{p,q}$ denote the set of all $(p,q)$-sequences. We will first provide a definition of $(p,q)$-parking functions in terms of permutations in $\mathfrak{S}_n$, where $n = p+q$, followed by an interpretation of this definition in terms of parking cars. First, define a map $\varphi : \mathfrak{S}_n \to \mathcal{S}_{p,q}$ as follows: for

$\sigma = \sigma_1\sigma_2\dots\sigma_n \in \mathfrak{S}_n$, set $\varphi(\sigma) = (u_1u_2\dots u_p, v_1v_2\dots v_q)$, where $u_i = \#\{r : p+1 \le r \le n, \sigma_r < \sigma_i\}$ for $i \in [p]$ and $v_j = \#\{r : 1 \le r \le p, \sigma_r < \sigma_{p+j}\}$ for $j \in [q]$.

**Example 2.2.** Let $p = 4$, $q = 6$, and $\sigma = 1\ 3\ 6\ 8\ 2\ 4\ 5\ 7\ 9\ 10$. Then $\varphi(\sigma) = (0134, 122344)$.

**Definition 2.3.** [7] The $(p, q)$-sequence $(\boldsymbol{a}, \boldsymbol{b})$ is a *(p,q)-parking function* if there exists a permutation $\sigma \in \mathfrak{S}_n$ with $\varphi(\sigma) = (u_1u_2\dots u_p, v_1v_2\dots v_q)$ satisfying $a_i \le u_i$ for all $i \in [p]$ and $b_j \le v_j$ for all $j \in [q]$. In this case, the permutation $\sigma$ is called a *certificate* for $(\boldsymbol{a}, \boldsymbol{b})$.

We may interpret this definition in terms of a set of parking conditions on $n$ cars. Consider the following scenario: $p$ red cars and $q$ blue cars wish to park along a one-way street with $n$ parking spots. Each driver $i$ of a red car prefers to have at least $a_i$ blue cars parked before him, and each driver $j$ of a blue car prefers to have at least $b_j$ red cars parked before him. Then the $(p, q)$-sequence $(\boldsymbol{a}, \boldsymbol{b})$ is a *(p,q)-parking function* if there exists a parking arrangement which accommodates the preferences of all $n$ drivers.

**Example 2.4.** The $(4, 6)$-sequence $(0124, 011344)$ is a $(4, 6)$-parking function since the permutation $\sigma = 1\ 3\ 6\ 8\ 2\ 4\ 5\ 7\ 9\ 10$ is a certificate. Alternatively, we see that the parking arrangement depicted below, where **R** represents a parked red car and **B** a parked blue car, accommodates the preferences of all drivers.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **R** | **B** | **R** | **B** | **B** | **R** | **B** | **R** | **B** | **B** |

Note that the set of all $(p, q)$-parking functions is invariant under the action of $\mathfrak{S}_p \times \mathfrak{S}_q$. We say a $(p, q)$-sequence $(\boldsymbol{a}, \boldsymbol{b})$ is an *increasing* $(p, q)$*-parking function* if it is a $(p, q)$-parking function with both $\boldsymbol{a}$ and $\boldsymbol{b}$ non-decreasing sequences. Then as with 2-dimensional $\boldsymbol{U}$-parking functions, the set of increasing $(p, q)$-parking functions may be considered as representatives of the orbits of the action of $\mathfrak{S}_p \times \mathfrak{S}_q$ on the set of all $(p, q)$-parking functions.

The following results are obtained in [7].

**Theorem 2.5.** The number of $(p, q)$-parking functions is $(p + q + 1)(p + 1)^{q-1}(q + 1)^{p-1}$, and the number of increasing $(p, q)$-parking functions is the Narayana number $N(p + q + 1, p + 1) = \frac{1}{p+q+1}\binom{p+q+1}{p+1}\binom{p+q+1}{p}$.

# 3 Relation between the two higher dimensional parking functions

In this section we show that $(p, q)$-parking functions are in fact 2-dimensional $\boldsymbol{U}$-parking functions for a particular set $\boldsymbol{U}$, and then use the latter to characterize $(p, q)$-parking functions by pairs of weakly disjoint lattice paths. This specialization holds analogously in higher dimensions. In addition, for any dimension $k$, the ratio of increasing $\boldsymbol{U}$-parking functions to all $\boldsymbol{U}$-parking functions is the same as the ratio of $k$-tuples of increasing sequences to $k$-tuples of integer sequences in a proper range.

**Theorem 3.1.** Set $\boldsymbol{U} = \{(u_{i,j}, v_{i,j}) : 0 \le i \le p, 0 \le j \le q\}$ with $u_{i,j} = j+1$ and $v_{i,j} = i+1$. A $(p,q)$-sequence $(\boldsymbol{a}, \boldsymbol{b})$ is a $(p,q)$-parking function if and only if $(\boldsymbol{a}, \boldsymbol{b})$ is a 2-dimensional $\boldsymbol{U}$-parking function.

*Proof.* Suppose $(\boldsymbol{a}, \boldsymbol{b}) = (a_1a_2\dots a_p, b_1b_2\dots b_q)$ is a $(p,q)$-parking function. Without loss of generality, we may assume $(\boldsymbol{a}, \boldsymbol{b})$ is increasing. Thus, $(\boldsymbol{a}, \boldsymbol{b})$ corresponds to a successful parking arrangement of $p$ red cars and $q$ blue cars in which the driver of the $i$-th red car ($i \in [p]$) has at least $a_i = a_{(i)}$ blue cars parked before him and the driver of the $j$-th blue car ($j \in [q]$) has at least $b_j = b_{(j)}$ red cars parked before him. From this parking arrangement, construct a lattice path $P = e_1e_2\dots e_n$ from (0,0) to $(p,q)$ as follows: beginning with the car parked in the first spot and continuing down the line, if the $i$-th parked car is red, record step $e_i$ of $P$ as $E$; and if the $i$-th parked car is blue, record step $e_i$ of $P$ as $N$. This means for $r = 1, 2, \dots, n$, if $e_r$ is the $i$-th east step from $(i-1, j)$ to $(i,j)$, then the $i$-th red car is parked in the $r$-th spot and exactly $j$ blue cars are parked before it. Hence, we must have $a_i \le j < j+1 = wt(e_r)$. Similarly, if $e_r$ is the $j$-th north step from $(i, j-1)$ to $(i,j)$, then the $j$-th blue car is parked in the $r$-th spot and exactly $i$ red cars are parked before it. Thus, $b_j \le i < i+1 = wt(e_r)$. Therefore, $(\boldsymbol{a}, \boldsymbol{b})$ is a 2-dimensional $\boldsymbol{U}$-parking function.

On the other hand, if $(\boldsymbol{a}, \boldsymbol{b})$ is a 2-dimensional $\boldsymbol{U}$-parking function with $u_{i,j} = j+1$ and $v_{i,j} = i+1$, then by definition the order statistics of $(\boldsymbol{a}, \boldsymbol{b})$ are bounded by a lattice path $P$ from $(0,0)$ to $(p,q)$ with respect to $\boldsymbol{U}$. Write $P = e_1e_2\dots e_{p+q}$ with $e_i \in \{E, N\}$. Assume the indices for the east steps are $i_1 < i_2 < \cdots < i_p$ and the indices for the north steps are $j_1 < \cdots < j_q$. Rearrange the terms $i_1, \dots, i_p$ as $i'_1 \cdots i'_p$ so that it is order isomorphic to $\boldsymbol{a}$, and similarly define $j'_1 \dots j'_q$. Let $\sigma = i'_1 \dots i'_p j'_1 \dots j'_q$. Then $(\boldsymbol{a}, \boldsymbol{b})$ is a $(p,q)$-parking function with $\sigma$ being a certificate. $\square$

**Example 3.2.** To demonstrate the second part of the above proof, consider the 2-dimensional $\boldsymbol{U}$-parking function $(\boldsymbol{a}, \boldsymbol{b}) = (1021, 224313)$, whose order statistics are bounded by the lattice path $P = ENENNENENN$ with respect to $\boldsymbol{U}$ as described in Proposition 3.1. The indices for the east steps in $P$ are $1, 3, 6, 8$; rearranged according to the pattern of $\boldsymbol{a}$, we have $i'_1i'_2i'_3i'_4 = 3\ 1\ 8\ 6$. Similarly, $j'_1 \dots j'_6 = 4\ 5\ 10\ 7\ 2\ 9$. Hence, $\sigma = 3\ 1\ 8\ 6\ 4\ 5\ 10\ 7\ 2\ 9$ is a certificate for the $(4,6)$-parking function $(\boldsymbol{a}, \boldsymbol{b})$.

Instead of using one lattice path to bound the order statistics of the sequence $(\boldsymbol{a}, \boldsymbol{b})$, we can characterize $(p,q)$-parking functions via a pair of lattice paths. This suggests a direct method for checking whether a given $(p,q)$-sequence is in fact a $(p,q)$-parking function. Without loss of generality, we can just work with the order statistics of given sequences. Let $(\boldsymbol{a}, \boldsymbol{b})$ be an increasing $(p,q)$-sequence. Construct a lattice path $P_{\boldsymbol{a}}$ from (0,0) to $(p,q)$ which has the $i$-th east step ($i \in [p]$) from $(i-1, a_i)$ to $(i, a_i)$ and then north steps that connect the east steps. Similarly, construct a lattice path $P_{\boldsymbol{b}}$ from (0,0) to $(p,q)$ which has the $j$-th north step ($j \in [q]$) from $(b_j, j-1)$ to $(b_j, j)$ and then east steps that connect the north steps. See the left figure in Figure 1.

**Definition 3.3.** For two lattice paths $P_1$ and $P_2$ from $(0,0)$ to $(p,q)$, we say that $P_1$ lies *weakly below* $P_2$ (equivalently, $P_2$ lies *weakly above* $P_1$) if the cells enclosed by $P_1$, $y = 0$, and $x = p$ are disjoint from the cells enclosed by $P_2$, $x = 0$, and $y = q$. In this case, we also say that the pair $(P_1, P_2)$ is a pair of *weakly disjoint* lattice paths.

**Theorem 3.4.** A $(p, q)$-sequence $(\boldsymbol{a}', \boldsymbol{b}')$ with corresponding increasing $(p, q)$-sequence $(\boldsymbol{a}, \boldsymbol{b})$ is a $(p, q)$-parking function if and only if $P_{\boldsymbol{a}}$ lies weakly below $P_{\boldsymbol{b}}$.

*Proof.* Let $(\mathbf{a}, \mathbf{b})$ be an increasing $(p, q)$-parking function. Then there is a parking arrangement $R \in \{\boldsymbol{R}, \boldsymbol{B}\}^{p+q}$ of $p$ red cars and $q$ blue cars that accommodates the preferences of all drivers. Let $P$ be the lattice path obtained by replacing $\mathbf{R}$ with $E$ and $\mathbf{B}$ with $N$. If $a_i = j$, then there are at least $j$ blue cars parked before the $i$-th red car, and consequently the $i$-th east step of $P$ has height at least $j$. This is true for all $i = 1, \ldots, p$. Hence $P$ lies weakly above $P_{\boldsymbol{a}}$. Similarly, $P$ lies weakly below $P_{\boldsymbol{b}}$. In summary, if $(\boldsymbol{a}, \boldsymbol{b})$ is an increasing $(p, q)$-parking function, we must have that $P_{\boldsymbol{a}}$ lies weakly below $P_{\boldsymbol{b}}$. Moreover, this result holds for all $(p, q)$-parking functions due to the invariance under the action of $\mathfrak{S}_p \times \mathfrak{S}_q$.

Now if $(\boldsymbol{a}, \boldsymbol{b})$ is an increasing $(p, q)$-sequence such that $P_{\boldsymbol{a}}$ lies weakly below $P_{\boldsymbol{b}}$, then the lattice path $P_{\boldsymbol{b}}$ translates to a parking arrangement of red and blue cars: let an $E$-step indicate a parked red car and an $N$-step indicate a parked blue car. Then by construction, this parking arrangement must accommodate the preferences of all drivers of blue cars. But also, since $P_{\boldsymbol{a}}$ lies weakly below $P_{\boldsymbol{b}}$, it must accommodate the preferences of all drivers of red cars. Hence, $(\boldsymbol{a}, \boldsymbol{b})$ is a $(p, q)$-parking function. $\square$

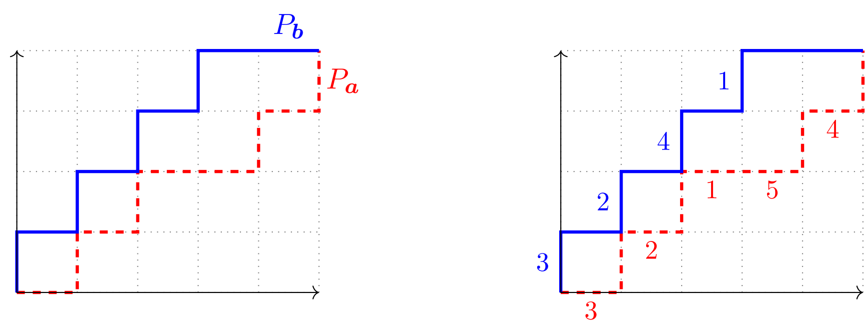


Figure 1: $P_{\boldsymbol{a}}$ and $P_{\boldsymbol{b}}$ for the $(5, 4)$-parking function $(\boldsymbol{a}, \boldsymbol{b}) = (01223, 0123)$, and the labeled paths for $(\boldsymbol{a}', \boldsymbol{b}') = (21032, 3102)$.

The left figure in Figure 1 shows the pair of lattice paths corresponding to an increasing $(p, q)$-parking function. General $(p, q)$-parking functions can be depicted by labeling the lattice paths $P_{\boldsymbol{a}}$ and $P_{\boldsymbol{b}}$, analogous to the manner in which classical parking functions are depicted by labeled Dyck paths. Explicitly, the east steps of $P_{\boldsymbol{a}}$ at $y = k$ are labeled by the indices $\{r : a_r = k\}$, from left to right. Similarly, the north steps of $P_{\boldsymbol{b}}$ at $x = k$ are labeled by the indices $\{r : b_r = k\}$, from bottom to top. For an example, see the right figure in Figure 1.

The following is an immediate corollary of Theorem 2.5.

**Corollary 3.5.** The number of pairs $(P_1, P_2)$ of lattice paths from $(0, 0)$ to $(p, q)$ such that $P_1$ lies weakly below $P_2$ is given by the Narayana number $N(p + q + 1, p + 1)$. The number of such lattice paths with labels as described in the proceeding paragraph is given by $(p+q+1)(p+1)^{q-1}(q+1)^{p-1}$.

Just as with $\boldsymbol{U}$-parking functions, $(p,q)$-parking functions can be generalized to higher dimensions, which is also considered in [7]. Let $p_1, p_2, \ldots, p_k$ be positive integers with $n = p_1 + \cdots + p_k$ and $q_i = n - p_i$ for $i \in [k]$. A $(p_1, \ldots, p_k)$-*sequence* is a $k$-tuple $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ of integer sequences of respective lengths $p_1, \ldots, p_k$ such that $0 \leq \boldsymbol{a}_j^{(i)} \leq q_i$ for all $i \in [k]$, $j \in [p_i]$. Consider the following scenario: there are $p_i$ cars of color $r_i$ for each $i \in [k]$ wishing to park along a one-way street with $n$ parking spots in such a way that the $j$-th driver ($j \in [p_i]$) of an $r_i$-color car desires to have $\boldsymbol{a}_j^{(i)}$ cars of different colors parked before him. Then $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ is a $(p_1, \ldots, p_k)$-*parking function* if and only if there is a parking arrangement that accommodates the desires of all $n$ drivers. We say the $(p_1, \ldots, p_k)$-parking function $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ is *increasing* if $\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)}$ are all non-decreasing sequences. In [7], $(p_1, \ldots, p_k)$-parking functions are defined via a permutation that serves as a *certificate*, though our above definition in terms of parking cars is equivalent. As in the 2-dimensional case, we have a similar relation between the two notions of higher dimensional parking functions.

**Theorem 3.6.** A $(p_1, \ldots, p_k)$-sequence $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ is a $(p_1, \ldots, p_k)$-parking function if and only if it is a $k$-dimensional $\boldsymbol{U}$-parking function, where $\boldsymbol{U} = \{(u^{(1)}_{i_1,\ldots,i_k}, \ldots, u^{(k)}_{i_1,\ldots,i_k}) : 0 \leq i_1 \leq p_1, \ldots, 0 \leq i_k \leq p_k\}$ with $u^{(j)}_{i_1,\ldots,i_k} = 1 - i_j + \sum_{r=1}^{k} i_r$ for $j \in [k]$.

*Proof.* Because it is a direct extension of the proof of Theorem 3.1, we only give a sketch. Assume $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ is an increasing $(p_1, \ldots, p_k)$-parking function, meaning it corresponds to a successful parking arrangement of $n = p_1 + \cdots + p_k$ cars in total, where $p_i$ cars are of color $r_i$ for each $i \in [k]$. Construct a lattice path $P$ from the origin to the point $(p_1, \ldots, p_k)$ as follows: beginning with the car parked in the first spot and continuing down the line, if the next parked car has color $r_j$, the next step of $P$ will be from $(i_1, \ldots, i_j - 1, \ldots, i_k)$ to $(i_1, \ldots, i_j, \ldots, i_k)$. But this will necessarily be the $(i_j)$-th car of color $r_j$ to park, meaning $\boldsymbol{a}^{(j)}_{i_j} < i_1 + \cdots + i_k - i_j + 1 = u^{(j)}_{i_1,\ldots,i_k}$. Thus, $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ is a $k$-dimensional $\boldsymbol{U}$-parking function.

Now if $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ is a $k$-dimensional $\boldsymbol{U}$-parking function, then the order statistics of $(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)})$ are bounded by some lattice path, say $Q$, from the origin to $(p_1, \ldots, p_k)$ with respect to $\boldsymbol{U}$. We can associate $Q$ with a parking arrangement according to the above process, and this parking arrangement will satisfy the preferences of all $n$ drivers by construction. $\square$

It is shown in Proposition 19 of [7] that the number of $(p_1, \ldots, p_k)$-parking functions is

$$(n+1)^{k-1} \prod_{i=1}^{k} (n - p_i + 1)^{p_i - 1},$$

with the subset of increasing $(p_1, \ldots, p_k)$-parking functions having size

$$\frac{1}{n+1} \prod_{i=1}^{k} \binom{n+1}{p_i}.$$

Cori and Poulalhon noted that the ratio of increasing $(p_1, \ldots, p_k)$-parking functions among the increasing $(p_1, \ldots, p_k)$-sequences is equal to the ratio of $(p_1, \ldots, p_k)$-parking functions among the

$(p_1, \ldots, p_k)$-sequences. In addition, this ratio is computed as

$$\frac{(n+1)^k}{\prod_{i=1}^n (n-p_i+1)},$$

where $n = \sum_{i=1}^k p_i$.

We will now extend this result to $\boldsymbol{U}$-parking functions for any affine node-set $\boldsymbol{U}$, that is, $\boldsymbol{U} = \{u_{\mathbf{i}} \in \mathbb{N}^k : \mathbf{i} = (i_1, \ldots, i_k), 0 \le i_j \le p_j\}$, where

$$u_{\mathbf{i}} = A\mathbf{i} + \mathbf{s} \tag{1}$$

for some $k \times k$ $\mathbb{N}$-matrix $A = (a_{ij})$ and $\mathbf{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$.

Let $\mathcal{X}_{\mathbf{p}}$ be the set of all $k$-tuples $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)})$ of integer sequences with respective lengths $p_1, \ldots, p_k$ and such that

$$0 \le (\boldsymbol{x}^{(i)})_j < s_i + (A\mathbf{p})_i = s_i + \sum_{j=1}^k a_{ij} p_j, \qquad \forall i = 1, \ldots, k, \forall j = 1, \ldots, p_i,$$

and let $\mathcal{IX}_{\mathbf{p}}$ be the set of non-decreasing sequences in $\mathcal{X}_{\mathbf{p}}$. Then

$$|\mathcal{X}_{\mathbf{p}}| = \prod_{i=1}^k \left( s_i + \sum_{j=1}^k a_{ij} p_j \right)^{p_i} \quad \text{and} \quad |\mathcal{IX}_{\mathbf{p}}| = \prod_{i=1}^k \frac{\left( s_i + \sum_{j=1}^k a_{ij} p_j \right)^{(p_i)}}{p_i!}.$$

where $x^{(k)} = x(x+1)\cdots(x+k-1)$ is the $k$-th rising factorial.

**Theorem 3.7.** Let $\boldsymbol{U}$ be an affine set of node defined by Equation (1). Then the ratio of $\boldsymbol{U}$-parking functions in the set $\mathcal{X}_{\mathbf{p}}$ is the same as the ratio of increasing $\boldsymbol{U}$-parking functions in the set $\mathcal{IX}_{\mathbf{p}}$.

*Proof.* The number of $\boldsymbol{U}$-parking functions and the number of increasing $\boldsymbol{U}$-parking functions for an affine node-set $\boldsymbol{U}$ can be computed via the theory of differential and difference Gončarov polynomials. Explicitly, by Proposition 17 of [1] we have that

$$\left| \mathcal{IPF}_{\mathbf{p}}^{(k)}(\boldsymbol{U}) \right| = \frac{1}{p_1! \cdots p_k!} \det(K) \prod_{i=1}^k (s_i + \sum_j a_{ij} p_j + 1)^{(p_i - 1)}, \tag{2}$$

where $K = (k_{ij})$ is a $k \times k$ matrix given by

$$k_{ij} = \begin{cases} s_i + \sum_{j \ne i} a_{ij} p_j & \text{if } i = j \\ -p_i a_{ji} & \text{if } i \ne j. \end{cases} \tag{3}$$

On the other hand, $\boldsymbol{U}$-parking functions are defined by the multivariate Gončarov polynomials, as in [11]. Applying Theorem 6.1(i) of [15] with the differential operators, we have

$$\#\mathcal{PF}_{\mathbf{p}}^{(k)}(\boldsymbol{U}) = \det(K) \prod_{i=1}^k (s_i + \sum_j a_{ij} p_j)^{p_i - 1}, \tag{4}$$

where $K$ is given by (3). Combining Equations (2) and (4) we have

$$\frac{\left|\mathcal{IPF}^{(k)}_{\mathbf{p}}(\boldsymbol{U})\right|}{|\mathcal{IX}_{\mathbf{p}}|} = \frac{\left|\mathcal{PF}^{(k)}_{\mathbf{p}}(\boldsymbol{U})\right|}{|\mathcal{X}_{\mathbf{p}}|} = \frac{\det(K)}{\prod_i (s_i + \sum_j a_{ij} p_j)}. \tag{5}$$

□

For the special case $A = J - I$ where $J$ is the all-1 matrix, and $\mathbf{s} = (1, \ldots, 1)$, we recover Cori and Poulalhon's $(p_1, \ldots, p_k)$-parking functions. In this case, the ratio at the right-hand side of Equation (5) is exactly $(n+1)^k / \prod_{i=1}^k (n - p_i + 1)$.

## 4 Sum-enumerator for increasing $(p, q)$-parking functions

One of the most important statistics of classical parking functions is the sum of the terms. For bivariate $\boldsymbol{U}$-parking functions, the sum-enumerator can be obtained as a specialization of the bivariate Gončarov polynomials, as discussed in [11]. In this section we study the sum-enumerator of increasing $(p, q)$-parking functions. Consider the bivariate generating function

$$F_{p,q}(s,t) = \sum_{(\boldsymbol{a},\boldsymbol{b}) \text{ increasing}} s^{\sum_{i=1}^p a_i} \cdot t^{\sum_{j=1}^q b_j}. \tag{6}$$

where $(\boldsymbol{a}, \boldsymbol{b})$ ranges over all increasing $(p, q)$-parking functions, which by Theorem 3.1 are precisely the increasing bivariate $\boldsymbol{U}$-parking functions with $\boldsymbol{U} = \{(u_{i,j}, v_{i,j}) : 0 \le i \le p, 0 \le j \le q\}$ and $u_{i,j} = j + 1$, $v_{i,j} = i + 1$.

Clearly, $F_{p,q}(s,t) = F_{q,p}(t,s)$. If $p$ or $q$ is 0, set $F_{p,q}(s,t) = 1$. For small values of $p$ and $q$, we have

$$\begin{aligned}
F_{1.1}(s,t) &= 1 + t + s \\
F_{2,1}(s,t) &= 1 + t + s + t^2 + ts + s^2 \\
F_{3,1}(s,t) &= 1 + s + t + s^2 + t^2 + s^3 + t^3 + st + st^2 + s^2 t \\
F_{2,2}(s,t) &= 1 + s + t + 2s^2 + 2t^2 + t^3 + s^3 + t^4 + s^4 + st + 2st^2 + 2s^2 t + st^3 + s^3 t + 2s^2 t^2.
\end{aligned}$$

For an increasing $(p, q)$-parking function $(\boldsymbol{a}, \boldsymbol{b})$, let $(P_{\boldsymbol{a}}, P_{\boldsymbol{b}})$ be the corresponding weakly disjoint lattice paths from $(0, 0)$ to $(p, q)$, as defined in the previous section. Then $\sum_{i=1}^p a_i$ is the area enclosed by $P_{\boldsymbol{a}}$, $y = 0$, and $x = p$; and $\sum_{j=1}^q b_j$ is the area enclosed by $P_{\boldsymbol{b}}$, $x = 0$, and $y = q$. See the left grid in Figure 2.

With this lattice path representation, it is immediate that

$$F_{p,q}(s,t) = F_{q,p}(s,t) \qquad \text{and} \qquad F_{p,q}(s,t) = F_{p,q}(t,s). \tag{7}$$

In addition,

$$F_{p,q}(s,0) = \begin{bmatrix} p+q \\ p \end{bmatrix}_s.$$

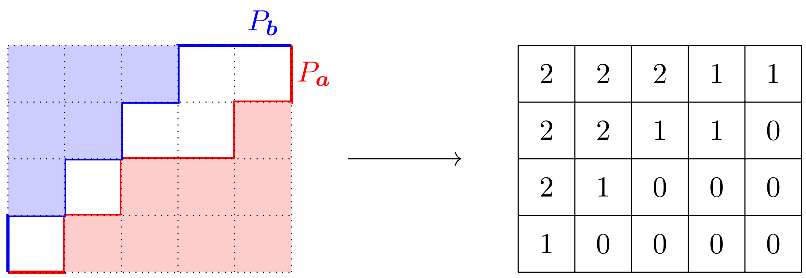


Figure 2: The areas determined by $P_{\boldsymbol{a}}$ and $P_{\boldsymbol{b}}$, and the corresponding plane partition.

This is because when $t = 0$, the nontrivial terms in the summation of $F_{p,q}(s,t)$ are those with $b_i = 0$ for all $i$, for which $P_{\boldsymbol{b}}$ is the unique path with $p$ $N$-steps followed by $q$ $E$-steps, and $P_{\boldsymbol{a}}$ can be any lattice path from $(0,0)$ to $(p,q)$. Now $F_{p,q}(s,0)$ is the area-enumerator of all such $P_{\boldsymbol{a}}$ and is given by the Gaussian coefficient.

### 4.1 A recurrence relation for $F_{p,q}(s,t)$

Next we give a recurrence for $F_{p,q}(s,t)$. The main tool we will use is a decomposition of the set of pairs of non-decreasing sequences, as is used in [1] to give a combinatorial interpretation of the increasing bivariate $\boldsymbol{U}$-parking functions for general $\boldsymbol{U}$. We adopt the same notation as in [1] to establish a recurrence for the sum-enumerator of increasing bivariate $\boldsymbol{U}$-parking functions, i.e. for the generating function

$$F_{p,q}(s,t;\boldsymbol{U}) = \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathcal{IPF}^{(2)}_{p,q}(\boldsymbol{U})} s^{\sum_{i=1}^{p} a_i} t^{\sum_{j=1}^{q} b_j}$$

where $\mathcal{IPF}^{(2)}_{p,q}(\boldsymbol{U})$ is the set of all increasing 2-dimensional $\boldsymbol{U}$-parking functions of size $(p,q)$. Then we specialize the recurrence to the case where $u_{i,j} = j+1$ and $v_{i,j} = i+1$, in accordance with Theorem 3.1.

The following construction is taken from [1]. Note that we adjust the notations from [11] such that for an integer sequence, the subscript starts with 1.

Let $\mathcal{I}(p,q)$ be the set of pairs of integer sequences $(\boldsymbol{a},\boldsymbol{b})$ such that $\boldsymbol{a} = a_1a_2\dots a_p$ satisfies $0 \le a_1 \le a_2 \le \cdots \le a_p < x$ and $\boldsymbol{b} = b_1b_2\dots b_q$ satisfies $0 \le b_1 \le b_2 \le \cdots \le b_q < y$, where $x, y$ are positive integers. It is well-known that

$$\sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathcal{I}(p,q)} s^{\sum_{i=1}^{p} a_i} t^{\sum_{j=1}^{q} b_j} = \begin{bmatrix} x+p-1 \\ p \end{bmatrix}_s \begin{bmatrix} y+q-1 \\ q \end{bmatrix}_t. \tag{8}$$

Let $P$ be a lattice path from $(0,0)$ to $(p,q)$. Denote by $\mathcal{I}_{p,q}(P;\boldsymbol{U})$ the subset of $\mathcal{I}(p,q)$ consisting of the pairs of sequences $(\boldsymbol{a},\boldsymbol{b})$ of respective lengths $p$ and $q$ that are bounded by $P$ with respect to $\boldsymbol{U}$.

For any pair of sequences $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{I}(p, q)$, we construct a subgraph $G(\boldsymbol{c})$ of the digraph $G_{p,q}(\boldsymbol{U})$ as follows.

- $O = (0, 0)$ is a vertex of $G(\boldsymbol{c})$.
- For any vertex $(i, j)$ of $G(\boldsymbol{c})$,
  - if $a_{i+1} < u_{i,j}$, then add the vertex $(i + 1, j)$ and the $E$-step $\{(i, j), (i + 1, j)\}$ to $G(\boldsymbol{c})$.
  - if $b_{j+1} < v_{i,j}$, then add the vertex $(i, j + 1)$ and the $N$-step $\{(i, j), (i, j + 1)\}$ to $G(\boldsymbol{c})$.

By definition $G(\boldsymbol{c})$ is a connected graph containing at least the vertex $O$. It is proved in [11] that the set of vertices of $G(\boldsymbol{c})$ has a unique maximal vertex $v(\boldsymbol{c}) \in [0, p] \times [0, q]$ under the order $\preceq$, where $(i, j) \preceq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$, for $(i, j), (i', j') \in \mathbb{N}^2$.

Let $(i, j) \preceq (p, q)$. Define the set $K_{p,q}(i, j) = \{\boldsymbol{c} \in \mathcal{I}(p, q) : v(\boldsymbol{c}) = (i, j)\}$ and let $k_{p,q}(i, j) = |K_{p,q}(i, j)|$. Then $\mathcal{I}(p, q)$ can be decomposed into the disjoint union of sets $K_{p,q}(i, j)$ for $0 \leq i \leq p$ and $0 \leq j \leq q$, and

$$K_{p,q}(p, q) = \bigcup_{P:(0,0)\to(p,q)} \mathcal{I}_{p,q}(P; \boldsymbol{U}) = \mathcal{IPF}^{(2)}_{p,q}(\boldsymbol{U}).$$

Now a pair of sequences $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b})$ is in $K_{p,q}(i, j)$ if and only if there exists a lattice path $P' : (0, 0) \to (i, j)$ satisfying the following:

- The initial segments $\boldsymbol{a}' = a_1 \dots a_i$ and $\boldsymbol{b}' = b_1 \dots b_j$ are bounded by $P'$ with respect to $\boldsymbol{U}$. That is, $(\boldsymbol{a}', \boldsymbol{b}')$ is in $K_{i,j}(i, j)$. There are $k_{i,j}(i, j)$ such pairs of initial segments.
- The integer sequence $a_{i+1} \dots a_p$ satisfies $u_{i,j} \leq a_{i+1} \leq \cdots \leq a_p < x$.
- The integer sequence $b_{j+1} \dots b_q$ satisfies $v_{i,j} \leq b_{j+1} \leq \cdots \leq b_q < y$.

It follows that

$$\begin{aligned}
&\sum_{(\boldsymbol{a},\boldsymbol{b})\in K_{p,q}(i,j)} s^{\sum_{i=1}^{p} a_i} t^{\sum_{j=1}^{q} b_j} \\
= \;& F_{i,j}(s, t; \boldsymbol{U}) \cdot s^{(p-i)u_{i,j}} \begin{bmatrix} x - u_{i,j} + p - i - 1 \\ p - i \end{bmatrix}_s \cdot t^{(q-j)v_{i,j}} \begin{bmatrix} y - v_{i,j} + q - j - 1 \\ q - j \end{bmatrix}_t .
\end{aligned}$$

Therefore, we reach the following equation that can be used as a recurrence for $F_{p,q}(s, t; \boldsymbol{U})$.

**Theorem 4.1.**

$$\begin{aligned}
&\begin{bmatrix} x + p - 1 \\ p \end{bmatrix}_s \begin{bmatrix} y + q - 1 \\ q \end{bmatrix}_t \\
= \;& \sum_{i=0}^{p} \sum_{j=0}^{q} F_{i,j}(s, t; \boldsymbol{U}) \cdot s^{(p-i)u_{i,j}} \begin{bmatrix} x - u_{i,j} + p - i - 1 \\ p - i \end{bmatrix}_s \cdot t^{(q-j)v_{i,j}} \begin{bmatrix} y - v_{i,j} + q - j - 1 \\ q - j \end{bmatrix}_t .
\end{aligned}$$

Setting $u_{i,j} = j + 1$, $v_{i,j} = i + 1$ will give a recurrence for the sum-enumerator of $(p, q)$-parking functions, where $x, y$ are any integers satisfying $x > q + 1$ and $y > p + 1$.

**Corollary 4.2** (**Recurrence Relation for** $F_{p,q}(s,t)$**).** For any non-negative integers $p,q$ and integers $x > q+1$ and $y > p+1$, we have

$$\begin{aligned}&\begin{bmatrix} x+p-1 \\ p \end{bmatrix}_s \begin{bmatrix} y+q-1 \\ q \end{bmatrix}_t \\ = \ & \sum_{i=0}^{p}\sum_{j=0}^{q} F_{i,j}(s,t)\cdot s^{(p-i)(j+1)} \begin{bmatrix} x+p-i-j-2 \\ p-i \end{bmatrix}_s \cdot t^{(q-j)(i+1)} \begin{bmatrix} y+q-i-j-2 \\ q-j \end{bmatrix}_t.\end{aligned}$$

When $q=0$, we have $F_{p,0}(s,t)=1$, and so Corollary 4.2 becomes

$$\begin{bmatrix} x+p-1 \\ p \end{bmatrix}_s = \sum_{i=0}^{p} s^{p-i} \begin{bmatrix} x+p-i-2 \\ p-i \end{bmatrix}_s = \sum_{i=0}^{q} s^{i} \begin{bmatrix} x+i-2 \\ i \end{bmatrix}_s.$$

Letting $k = x-2$ and replacing the variable $s$ with $q$, the above equation becomes a $q$-analog of the Hockey Stick Identity:

$$\begin{bmatrix} k+p+1 \\ p \end{bmatrix}_q = \sum_{i=0}^{p} q^{i} \begin{bmatrix} i+k \\ k \end{bmatrix}_q.$$

**Remark.** For computational purposes, in Corollary 4.2 we can take $x = q+2$ and $y = p+2$. So the recurrence becomes

$$\begin{aligned}&\begin{bmatrix} p+q+1 \\ p \end{bmatrix}_s \begin{bmatrix} p+q+1 \\ q \end{bmatrix}_t \\ = \ & \sum_{i=0}^{p}\sum_{j=0}^{q} s^{(p-i)(j+1)} \begin{bmatrix} p+q-i-j \\ p-i \end{bmatrix}_s t^{(q-j)(i+1)} \begin{bmatrix} p+q-i-j \\ q-j \end{bmatrix}_t F_{i,j}(s,t). \qquad (9)\end{aligned}$$

For example, one can derive from (9) that $F_{q,1}(s,t) = \sum_{i+j\le m} s^i t^j$, which can be easily checked using the definition of the sum-enumerator.

## 4.2 Plane partitions in the box $p \times q \times 2$

There is a one-to-one correspondence between pairs of weakly disjoint lattice paths in the $p \times q$ grid and plane partitions in the box $p \times q \times 2$: put 2 in each square to the upper-left of the lattice path $P_{\boldsymbol{b}}$, 0 in each square to the lower-right of the lattice path $P_{\boldsymbol{a}}$, and 1 in the remaining squares. See the right grid in Figure 2.

For a plane partition $\pi$ in the box $p \times q \times 2$, each column can be viewed as a partition $\lambda$ inside the rectangle $2 \times q$, and $\pi$ can be viewed as a multichain $\hat{0} = \lambda^{(0)} \le \lambda^{(1)} \le \cdots \le \lambda^{(p)} \le \lambda^{(p+1)} = \hat{1}$, where $\lambda^{(i)}$ is the $(p+1-i)$-th column in $\pi$ and $\hat{0} = \emptyset = (0^q)$, $\hat{1} = (2^q)$.

Let $L_q$ be the lattice of all partitions inside the rectangle $2 \times q$, ordered by inclusion. Then $L_q$ is a distributive lattice that is isomorphic to $J(P_q)$, the order ideals of the poset $P_q = \mathbf{2} \times \mathbf{q}$, where $\mathbf{q}$ is the $q$-element chain.

The above argument leads to the following proposition.

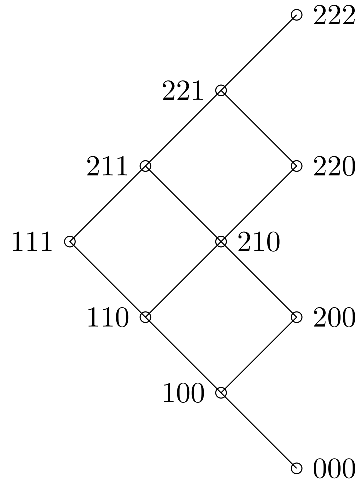


Figure 3: The lattice $L_3$.

**Proposition 4.3.** The Narayana number $N(1+p+q,p)$ counts the number of multichains $\hat{0} = I_0 \leq I_1 \leq \cdots \leq I_{p+1} = \hat{1}$ of length $p+1$ in $L_q$.

From Proposition 3.5.1 of [18], we have

**Corollary 4.4.** The Narayana number $N(1+p+q,p)$ also counts the following:

1. order-preserving maps $\sigma : P_q \to \mathbf{p+1}$;
2. the set of order ideals $J(P_q \times \mathbf{p})$.

Next we use the multichain representation of plane partitions in Proposition 4.3 to derive a Catalan-type recurrence for the generating function of $F_{p,q}(s,t)$. In the following we will just consider multichains of the form $\lambda^{(1)} \leq \cdots \leq \lambda^{(p)}$ which do not necessarily start with $\hat{0}$ or end with $\hat{1}$. For a plane partition $\pi$ in the box $p \times q \times 2$ whose content is $0^{r_0}1^{r_1}2^{r_2}$, let

$$wt(\pi) = a^{r_0} b^{r_1} c^{r_2},$$

and define

$$S_{p,q}(a,b,c) = \sum_{\pi \in p \times q \times 2} wt(\pi). \tag{10}$$

Note that we may work interchangeably with $S_{p,q}(a,b,c)$ and $F_{p,q}(s,t)$, for $F_{p,q}(s,t) = S_{p,q}(s,1,t)$ and $S_{p,q}(a,b,c) = b^{pq} F_{p.q}(a/b, c/b)$. It is more convenient to work with $S_{p,q}(a,b,c)$ in the lattice $L_q$.

Fixing $q$, we will compute the generating function $\mathcal{S}_q(a,b,c;z)$ defined by

$$\mathcal{S}_q(a,b,c;z) = \sum_{p=0}^{\infty} S_{p,q}(a,b,c) z^p. \tag{11}$$

First, it is clear that $\mathcal{S}_0(a,b,c;z) = 1/(1-z)$.

For a plane partition $\pi$ with columns $\lambda^{(p)}, \dots, \lambda^{(1)}$, we have

$$wt(\pi) = \prod_{i=1}^{m} wt(\lambda^{(i)}).$$

For $q = 1$, the poset $L_1$ is a chain consisting of the three elements $\emptyset, (1), (2)$ with weights $a, b, c$ respectively. Thus, a multichain $\lambda^{(1)} \le \cdots \le \lambda^{(p)}$ is just a sequence with $\emptyset$ followed by some (1)'s and then (2)'s. Hence,

$$\mathcal{S}_1(a,b,c;z) = \frac{1}{1-az} \cdot \frac{1}{1-bz} \cdot \frac{1}{1-cz}. \tag{12}$$

In general, we can compute $\mathcal{S}_q(a,b,c;z)$ by a summation over the set of chains in $L_q$ as follows. Let $K$ be a chain of $L_q$ and let $\mathcal{L}(K)$ be the set of multichains whose support is $K$. Then

$$\sum_{C \in \mathcal{L}(K)} wt(C) z^{|C|} = \prod_{\lambda^{(i)} \in K} \frac{wt(\lambda^{(i)})z}{1 - wt(\lambda^{(i)})z},$$

where $|C|$ is the number of elements in the multichain $C$. Consequently,

$$\mathcal{S}_q(a,b,c;z) = \sum_{K} \prod_{\lambda^{(i)} \in K} \frac{wt(\lambda^{(i)})z}{1 - wt(\lambda^{(i)})z}, \tag{13}$$

where $K$ ranges over all chains in the lattice $L_q$.

For example, for $q = 2$ we have

$$\mathcal{S}_2(a,b,c;z) = \frac{1}{(1-a^2z)(1-abz)(1-bcz)(1-c^2z)} \left( \frac{1}{1-b^2z} + \frac{1}{1-acz} - 1 \right). \tag{14}$$

Formula (13) shows that $\mathcal{S}_q(a,b,c;z)$ is a rational function, although its usage to compute $\mathcal{S}_q(a,b,c;z)$ is impractical due to the large possible number of chains in $L_q$. Next we present a recurrence for $\mathcal{S}_q(a,b,c;z)$ that generalizes the Catalan recurrence

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$$

and provides a faster way to compute $\mathcal{S}_q(a,b,c;z)$.

**Theorem 4.5.** The generating functions $\mathcal{S}_q(a,b,c;z)$ satisfy the following recurrence: for $q \ge 2$,

$$\begin{aligned} \mathcal{S}_q(a,b,c;z) = &\frac{1}{1-c^qz} \left( \frac{\mathcal{S}_{q-1}(a,b,c;bz)}{1-a^qz} \right. \\ &\left. + \sum_{i=1}^{q-1} a^{q-i} c^i z \mathcal{S}_i(a,b,c;a^{q-i}z) \mathcal{S}_{q-i-1}(a,b,c;bc^iz) \right) \end{aligned} \tag{15}$$

*Proof.* The idea is to decompose all multichains in a proper way. Let $\mu^{(i)}$ be the partition $(2^i 0^{q-i})$. For each $i = 1, 2, \ldots, q-1$, let $\mathcal{M}_i$ be the set of multichains that contain $\mu^{(i)}$ but not $\mu^{(j)}$ for any $j$ such that $i < j < q$.

Let $L_q^-(\mu^{(i)}) = \{\lambda \in L_q : \ \lambda \leq \mu^{(i)}\}$ and $L_q^+(\mu^{(i)}) = \{\lambda \in L_q : \ \lambda \geq \mu^{(i)}\}$. We compute the weighted sum of the multichains from $\mathcal{M}_i$. Any such a multichain consists of a multichain over the poset $L_q^-(\mu^{(i)})$, at least one occurrence of $\mu^{(i)}$, and then a multichain over the sub-poset $L_q^+(\mu^{(i)}) - \{\mu^{(i)}, \ldots, \mu^{(q-1)}\}$.

The first observation is that $L_q^-(\mu^{(i)}) \simeq L_i$ but the weight of an element in $L_q^-(\mu^{(i)})$ has one more factor of $a^{q-i}$ than its corresponding term in $L_i$. Hence, multichains in $L_q^-(\mu^{(i)})$ contribute a factor of $\mathcal{S}_i(a, b, c; a^{q-i}z)$. The second observation is that the sub-poset $L_q^+(\mu^{(i)}) - \{\mu^{(i)}, \ldots, \mu^{(q-1)}, \mu^{(q)}\}$ is isomorphic to $L_{q-i-1}$; actually, the former is obtained from the latter by adjoining $i+1$ parts of $(2^i 1)$, each having weight $bc^i$, to each partition. In addition, $\mu^{(q)}$ is larger than any other element and can occur any time in a multichain. Hence the multichains in $L_q^+(\mu^{(i)}) - \{\mu^{(i)}, \ldots, \mu^{(q-1)}\}$ contribute a factor of

$$\mathcal{S}_{q-i-1}(a, b, c; bc^i z)/(1 - c^q z).$$

Combining the two observations and summing over $i = 1, 2, \ldots, q-1$, while additionally noting that the weight of $\mu^{(i)}$ in $L_q$ is $a^{q-i}c^i$, we obtain the summation part in Equation (15).

One more case remains: the multichain does not contain any $\mu^{(i)}$ for $i = 1, 2, \ldots, q-1$. But $L_q - \{\mu^{(i)} : i = 1, \ldots, q-1\}$ can be viewed as obtained from $L_{q-1}$ by joining to each partition a new part of size 1, and then adding $\hat{0} = \emptyset$ and $\hat{1} = (2^q)$. Multichains in this case contribute the term

$$\frac{\mathcal{S}_{q-1}(a, b, c; bz)}{(1 - a^q z)(1 - c^q z)}.$$

$\square$

For $q = 2$, Theorem 4.5 gives Equation (14) again. For $q = 3$, we have

$$\begin{aligned}
\mathcal{S}_3(a, b, c; z) = \quad & \tfrac{1}{(1-a^3z)(1-c^3z)(1-a^2bz)(1-bc^2z)} \Big( \tfrac{ac^2z}{(1-abcz)(1-ac^2z)(1-ab^2z)} \\
& + \tfrac{ac^2z}{(1-a^2cz)(1-abcz)(1-ac^2z)} - \tfrac{ac^2z}{(1-abcz)(1-ac^2z)} \\
& + \tfrac{a^2cz}{(1-a^2cz)(1-abcz)(1-b^2cz)} + \tfrac{1}{(1-ab^2z)(1-b^2cz)(1-b^3z)} \\
& + \tfrac{1}{(1-ab^2z)(1-b^2cz)(1-abcz)} - \tfrac{1}{(1-ab^2z)(1-b^2cz)} \Big)
\end{aligned}$$

An interesting case is when $a = b = c = 1$. Let $\mathcal{S}_q(z) := \mathcal{S}_q(1, 1, 1; z)$. Then from Equation (15), we have

$$\mathcal{S}_q(z) = \frac{1}{(1-z)^2}\mathcal{S}_{q-1}(z) + \sum_{i=1}^{q-1} \frac{z}{1-z}\mathcal{S}_i(z)\mathcal{S}_{q-i-1}(z).$$

The initial cases are $\mathcal{S}_0(z) = (1-z)^{-1}$ and $\mathcal{S}_1(z) = (1-z)^{-3}$. If we let $\mathcal{S}_q(z) = T_q(z)/(1-z)^{2q+1}$, then $(T_q(z))$ is necessarily a sequence of polynomials satisfying

$$T_q(z) = T_{q-1}(z) + \sum_{i=1}^{q-1} zT_i(z)T_{q-i-1}(z), \tag{16}$$

for $q \geq 2$, with the initial conditions $T_0(z) = T_1(z) = 1$.

Define the $n$-th Narayana polynomial by letting

$$N_n(z) = \sum_{k=1}^{n} N(n,k)z^{k-1},$$

where $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is the Narayana number. For $n = 0$, set $N_0(z) = 1$. We remark that this definition of the Narayana polynomial is slightly different than the usual one appearing in the literature, which has an extra factor of $z$. For example, see [21]. Nevertheless, the $N_n(z)$ defined here is the rank-generating function of the lattice of noncrossing partitions of $[n]$, which plays an important role in the next result.

**Proposition 4.6.** Let $q$ be a positive integer. Then $T_q(z) = N_q(z)$ and hence

$$\mathcal{S}_q(z) = \frac{N_q(z)}{(1-z)^{2q+1}}.$$

*Proof.* Clearly $N_1(z) = 1$. We use the combinatorial interpretation that $N(q,k)$ counts the number of noncrossing partitions of $[q] = \{1,\ldots,q\}$ with $k$ blocks and verify that $N_q(z)$ satisfies the recurrence relation given by Equation (16).

For a noncrossing partition $\pi \in NC([q])$ with $k$ blocks, assume that $q$ is in the block whose minimal element is $i+1$. If $i = 0$, then $\pi$ is obtained from a noncrossing partition of $[q-1]$ with $k$ blocks by adjoining $q$ to the block that contains 1. For $i = 1,\ldots,q-1$, $\pi$ can be viewed as the disjoint union of two noncrossing partitions, one on $[i]$ and the other on $\{i+1,\ldots,q\}$ such that $i+1$ and $q$ are in the same block. Moreover, the total number of blocks is $k$. Putting them together and noting that in $N_q(z)$, $N(q,k)$ is the coefficient of $z^{k-1}$, we obtain the recurrence

$$N_q(z) = N_{q-1}(z) + \sum_{i=1}^{q-1} zN_i(z)N_{q-i-1}(z).$$

□

Since $\mathcal{S}_q(z) = \sum_{p\geq 0} F_{p,q}(1,1)z^p$ and $F_{p,q}(1,1) = N(1+p+q,1+p)$, we obtain the following equation of the Narayana numbers.

**Corollary 4.7.**

$$\sum_{p\geq 0} N(1+p+q,1+p)z^p = \frac{N_q(z)}{(1-z)^{2q+1}}.$$

Finally, we give a formula for the bivariate generating function

$$\mathcal{S}(x;z) \coloneqq \sum_{q\geq 0} \mathcal{S}_q(z)x^q.$$

It is known [20] that the bivariate generating function for the Narayana numbers

$$F(x,z) = \sum_{q=1}^{\infty}\sum_{k=1}^{q} N(q,k)x^q z^k \tag{17}$$

has the algebraic formula

$$F(x,z) = \frac{1-(z+1)x-\sqrt{1-2(z+1)x+(z-1)^2x^2}}{2x}.$$

Proposition 4.6 implies the equation

$$\mathcal{S}(x;z) = \frac{1}{1-z}\left(1+\sum_{q=1}^{\infty}\sum_{k=1}^{q} N(q,k)z^{k-1}\left(\frac{x}{(1-z)^2}\right)^q\right).$$

Using Equation (17) we obtain the following corollary.

**Corollary 4.8.**

$$\mathcal{S}(x;z) = \frac{1-z-x-\sqrt{1+x^2+z^2-2z-2x-2xz}}{2xz}.$$

### 4.3 A $q$-analog of the Narayana numbers

In this subsection we prove the following $q$-analog of the Narayana numbers.

**Theorem 4.9.** When $s=1/t$, $F_{p,q}(s,t)$ yields a $q$-analog of the Narayana numbers via the equation

$$t^{pq}F_{p,q}(1/t,t) = \frac{1}{[1+p+q]_t}\begin{bmatrix}1+p+q\\ p\end{bmatrix}_t\begin{bmatrix}1+p+q\\ q\end{bmatrix}_t$$

*Proof.* For an increasing $(p,q)$-parking function $(\boldsymbol{a},\boldsymbol{b})$, let $(P_{\boldsymbol{a}},P_{\boldsymbol{b}})$ be the corresponding pair of weakly disjoint lattice paths from $(0,0)$ to $(p,q)$. By moving $P_{\boldsymbol{a}}$ one unit to the right and adding an $E$-step at the beginning, and moving $P_{\boldsymbol{b}}$ one unit up and adding an $E$-step at the end, the pair $(P_{\boldsymbol{a}},P_{\boldsymbol{b}})$ becomes a pair of vertex disjoint lattice paths from the vertex set $(A_1,A_2)$ to $(B_1,B_2)$, where $A_1=(0,0)$, $A_2=(0,1)$, $B_1=(p+1,q)$, and $B_2=(p+1,q+1)$. This is a one-to-one correspondence. Hence we can use the Lemma of Gessel-Viennot to count the number of vertex disjoint path systems.

We adopt a weighted version of the lattice graph with $E$- and $N$- steps that is used to describe a combinatorial definition of Schur functions and the Jacobi-Trudi identity. See, for example,

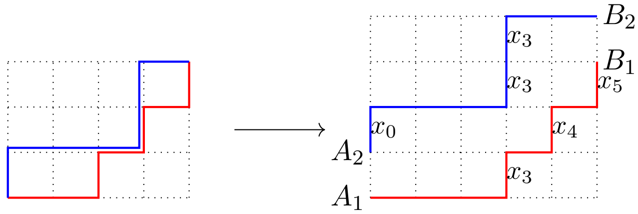


Figure 4: The corresponding vertex disjoint path system with weights for $(P_{\boldsymbol{a}}, P_{\boldsymbol{b}})$.

Theorem 7.16.1 of [19] or Theorem 8.8 in [2]. Consider the lattice graph with $E$- and $N$- steps, where all $E$-steps have weight 1, and an $N$-step at the line $x = k$ has weight $x_k$. The weight of a path $P$ is the product $\prod_{e \in P} weight(e)$. For a path system $\mathcal{P} = (P_1, P_2)$ where $P_i$ is from $A_i$ to $B_i$, we have

$$weight(\mathcal{P}) = \prod_{e \in P_1} weight(e) \prod_{f \in P_2} weight(f).$$

Applying the Lemma of Gessel-Viennot to the vertex sets $(A_1, A_2)$ and $(B_1, B_2)$, we have

$$\sum_{\mathcal{P} \text{ is vertex disjoint}} weight(\mathcal{P}) = \det M, \tag{18}$$

where the path matrix $M$ is a $2 \times 2$ matrix with entries $m_{i,j} = h_{q-i+j}(x_0, \ldots, x_{p+1})$.

By the Jacobi-Trudi identity,

$$\det M = s_\lambda(x_0, \ldots, x_{p+1}), \tag{19}$$

where $\lambda = (q, q)$.

We need the specialization $s_\lambda(1, t, \ldots, t^n)$, which is given by the Hook-Content Formula (see Theorem 7.21.2 of [19]).

**The Hook-Content Formula**. For any partition $\lambda$ and $n > 0$,

$$s_\lambda(1, q, \ldots, q^{n-1}) = q^{\sum (i-1)\lambda_i} \prod_{u \in \lambda} \frac{[n + c(u)]}{[h(u)]}.$$

where for a cell $u = (i, j)$ in an integer partition $\lambda$, the hook length $h(u)$ of $\lambda$ at $u$ is defined by $h(u) = \lambda_i + \lambda'_j - i - j + 1$, where $\lambda'$ is the conjugate of $\lambda$, and the content $c(u)$ is defined by $c(u) = j - i$.

Using the Hook-Content formula, we have

$$s_{q^2}(1, t, \ldots, t^{p+1}) = \frac{t^q}{[1 + p + q]_t} \begin{bmatrix} 1 + p + q \\ p \end{bmatrix}_t \begin{bmatrix} 1 + p + q \\ q \end{bmatrix}_t. \tag{20}$$

Next we connect the sum-enumerator $F_{p.q}(s, t)$ of increasing $(p, q)$-parking functions to $weight(\mathcal{P})$ for a vertex disjoint path system $\mathcal{P} = (P_1, P_2)$. Note that we require that $P_i$ is from $A_i$ to $B_i$ for $i = 1, 2$.

For a lattice path $P_i$ from $A_i$ to $B_i$, if $P_i$ has $N$-steps at $x = r_1, r_2, \ldots, r_n$, then the weight of $P_i$ is $x_{r_1} x_{r_2} \cdots x_{r_n}$. When $x_i = t^i$, the weight of $P_i$ becomes $t^{\sum_{i=1}^n r_i}$. Comparing with the definition of $F_{p,q}(s,t)$, we have that $weight(P_1) = t^{(p+1)q - \sum_i a_i}$ and $weight(P_2) = t^{\sum_j b_j}$. Therefore,

$$\sum_{\mathcal{P} \text{ is vertex disjoint}} weight(\mathcal{P}) = t^{(p+1)q} F_{p,q}(1/t, t). \tag{21}$$

Theorem 4.9 follows from Equations (18)–(21). □

## 5 Increasing $(p,q)$-parking functions and Catalan structures

In this final section, we focus on lattice paths in $\mathbb{Z}^2$ consisting of east and north steps. A Dyck path is a lattice path from $(0,0)$ to $(p,p)$ which may touch but never goes below the line $y = x$. Such a Dyck path is said to be of length $2p$.

In Theorem 2.3 of [6], a bijection is constructed between the set of pairs of weakly disjoint lattice paths from $(0,0)$ to $(p,p)$ and the set of noncrossing partitions of $[2p+1]$ having $p+1$ blocks. The following generalizes this bijection to lattice paths from $(0,0)$ to $(p,q)$ for arbitrary positive integers $p$ and $q$.

**Theorem 5.1.** Let $p, q$ be positive integers. Then there exists a bijection $\Phi$ between the set of pairs of weakly disjoint lattice paths from $(0,0)$ to $(p,q)$ and the set of noncrossing partitions of $[p+q+1]$ with $p+1$ blocks.

*Proof.* Let $P = e_1 e_2 \ldots e_{p+q}$ and $Q = d_1 d_2 \ldots d_{p+q}$ be weakly disjoint lattice paths from $(0,0)$ to $(p,q)$ consisting of $N = (0,1)$ and $E = (1,0)$ steps, where $Q$ is weakly below $P$. The bijection $\Phi$ defined below uniquely assigns to each such $(P,Q)$ a pair of degree sequences $(l_1, l_2, \ldots, l_{p+q+1})$ and $(r_1, r_2, \ldots, r_{p+q+1})$. Here, $l_i$ denotes the left-degree of vertex $i$ in the standard representation of a partition, meaning the number of vertices $j$ with $j < i$ connected to $i$; and similarly, $r_i$ denotes the right-degree of vertex $i$, or the number of vertices $j$ with $j > i$ connected to $i$. Initially, we set $l_1 = r_{p+q+1} = 0$; then for each $i = 1, \ldots, p+q$, $(r_i, l_{i+1})$ is determined as follows:

1. $(e_i, d_i) = (N, N)$ if and only if $(r_i, l_{i+1}) = (1,1)$.
2. $(e_i, d_i) = (N, E)$ if and only if $(r_i, l_{i+1}) = (1,0)$.
3. $(e_i, d_i) = (E, N)$ if and only if $(r_i, l_{i+1}) = (0,1)$.
4. $(e_i, d_i) = (E, E)$ if and only if $(r_i, l_{i+1}) = (0,0)$.

To uniquely form a noncrossing partition of $[p+q+1]$ from these left- and right-degree sequences, iteratively pair the leftmost vertex having nonzero left-degree with the nearest vertex having nonzero right-degree on its left. The result is the standard representation of a partition $R \coloneqq \Phi(\{P,Q\})$. To see why this iterative process of pairing vertices necessarily works, note that we can conclude from $Q$ being weakly below $P$ that $\sum_{i=1}^{k} r_i \geq \sum_{i=1}^{k+1} l_i$, which always guarantees the existence of a vertex having nonzero right-degree to the left of a chosen vertex.

It remains to check that $R$ is a noncrossing partition of $[p+q+1]$ with $p+1$ blocks. Note that our process for generating the degree sequences $(l_1, l_2, \ldots, l_{p+q+1})$ and $(r_1, r_2, \ldots, r_{p+q+1})$ from $(e_i, d_i)$ for each $i = 1, \ldots, p+q$ guarantees that the resulting standard representation has vertex set $[p+q+1]$. The total number of arcs is half the number of $N$-steps in both $P$ and $Q$, which is equal to $q$. Since there are $p+q+1$ vertices, this implies the resulting partition $R$ has $p+1$ blocks. Also, $R$ must be noncrossing based on our method of constructing the standard representation of $R$ from the degree sequences. $\square$

Additionally, an explicit correspondence between pairs of noncrossing lattice paths from $(0,0)$ to $(p,p)$ and Dyck paths from $(0,0)$ to $(2p+1, 2p+1)$ having $p+1$ $N$-steps at odd positions is presented in [6], which is based on the Labelle Merging Algorithm [14]. This correspondence is extended below to lattice paths from $(0,0)$ to $(p,q)$ for arbitrary positive integers $p$ and $q$.

**Theorem 5.2.** (Generalized Labelle Merging Algorithm) There exists a bijection $\Psi$ between the set of pairs of weakly disjoint lattice paths from $(0,0)$ to $(p,q)$ and the set of Dyck paths of length $2(p+q+1)$ having $p+1$ $N$-steps at odd positions.

*Proof.* Let $P = e_1 e_2 \ldots e_{p+q}$ and $Q = d_1 d_2 \ldots d_{p+q}$ be weakly disjoint lattice paths from $(0,0)$ to $(p,q)$, and assume $Q$ lies weakly below $P$. Set $Q' = d'_1 d'_2 \ldots d'_{p+q}$, where $N' = E$ and $E' = N$. Now define a new lattice path $L \coloneqq \Psi(P,Q)$ by letting

$$L = N e_1 d'_1 e_2 d'_2 \ldots e_{p+q} d'_{p+q} E.$$

Let $f$ be a function with $f(N) = 1$ and $f(E) = -1$. Since $Q$ lies weakly below $P$, we have $\sum_{i=1}^{r}(f(e_i)+f(d'_i)) = \sum_{i=1}^{r}(f(e_i)-f(d_i)) \geq 0$ for all $r = 1, 2, \ldots, p+q$. It follows that $\sum_{i=1}^{r}(f(e_i)+f(d'_i)) + f(e_{i+1}) \geq -1$ for all $r = 1, \ldots, p+q-1$. Note that $L$ begins with an $N$-step. Hence, $L$ is in fact a Dyck path from $(0,0)$ to $(p+q+1, p+q+1)$. Moreover, it has $p+1$ $N$-steps at odd positions, since $Q'$ contains $p$ $N$-steps.

To verify that $\Psi$ is a bijection, we define the reverse map. Let $L = \ell_1 \ell_2 \ldots \ell_{2p+2q+1} \ell_{2p+2q+2}$ be a Dyck path with $p+1$ $N$-steps at odd positions. Set $P = \ell_2 \ell_4 \ldots \ell_{2p+2q}$ and $Q = \ell'_3 \ell'_5 \ldots \ell'_{2p+2q+1}$, where $N' = E$ and $E' = N$. Since $L$ has $p+1$ $N$-steps at odd positions, $Q$ will have $p$ $E$-steps and $q$ $N$-steps. Since $L$ has length $2(p+q+1)$, $P$ must also have $p$ $E$-steps and $q$ $N$-steps. Now if there exists some $k$ such that $P$ and $Q$ cross at their $k$-th step (that is, after $k$ steps, $Q$ is above $P$), then

$$\#\{i : \ell_{2i} = N, 1 \leq i \leq k\} < \#\{i : \ell'_{2i+1} = N, 1 \leq i \leq k\} = \#\{i : \ell_{2i+1} = E, 1 \leq i \leq k\},$$

which implies that $L$ crosses the line $y = x$ by its $k$-th step, a contradiction. So $P$ and $Q$ are noncrossing, and $\Psi$ is indeed a bijection. $\square$

In discussing pairs of weakly disjoint lattice paths, an object which naturally arises is the parallelogram polyomino, which is a polyomino in a $p \times q$ rectangular grid that is bounded by two $(N,E)$-lattice paths from $(0,0)$ to $(p,q)$ which touch only at their starting and ending points. Aval et al. [3] present a bijection **ptd** between the set of parallelogram polyominoes in a $p \times q$ grid and the set of prime Dyck paths of length $2(p+q)$, that is, Dyck paths of length $2(p+q)$ that touch the diagonal $y = x$ only at their starting and their ending points. We will briefly review that bijection here. Consider a parallelogram polyomino bounded by the two $(N, E)$-lattice paths $P = e_1e_2 \ldots e_{p+q}$ and $Q = d_1d_2 \ldots d_{p+q}$ from $(0, 0)$ to $(p, q)$, where $Q$ is weakly below $P$. By necessity, $P$ and $Q$ touch only at their starting and ending points. Define $Q' = d'_1 d'_2 \ldots d'_{p+q}$, where $N' = E$ and $E' = N$. Now form the Dyck path $C = \mathbf{ptd}(P, Q) = e_1d'_1 e_2d'_2 \ldots e_{p+q}d'_{p+q}$. This construction is easily reversed to yield a bijective correspondence between the set of all parallelogram polyominoes in an $p \times q$ rectangular grid and the set of all prime Dyck paths from $(0, 0)$ to $(p + q, p + q)$ having exactly $p$ $N$-steps in even positions.

The Generalized Labelle Merging Algorithm in Theorem 5.2 and the bijection **ptd** in [3] for constructing a Dyck path from a pair of lattice paths are very similar. The difference is that in Theorem 5.2, the pair $(P, Q)$ of lattice paths from $(0, 0)$ to $(p, q)$ can touch, and the image lattice path $L = \Psi(P, Q)$ from $(0, 0)$ to $(p + q + 1, p + q + 1)$ is guaranteed to be a Dyck path due to the initial $N$-step and final $E$-step imposed in the construction. Yet in [3], because $P$ and $Q$ bound a parallelogram polyomino and consequently only touch at start and end, the extra $N$- and $E$-steps are not necessary. Actually, starting from a pair of weakly disjoint lattice paths $(P, Q)$ where $Q$ is weakly below $P$, if we let $Q'$ be the path obtained through the addition of an initial $E$-step and final $N$-step to $Q$, and $P'$ be the path obtained by adding an initial $N$-step and final $E$-step to $P$, then the resulting figure is a parallelogram polyomino bounded by $(P', Q')$ in a $(p + 1) \times (q + 1)$ grid (see Figure 5). Furthermore, $\mathbf{ptd}(P', Q') = N\ L\ E$, where $L = \Psi(P, Q)$. The extra $N$- and $E$-steps guarantee that $\mathbf{ptd}(P', Q')$ is a prime Dyck path.

In the following, we will use a result in [3] to compute the areas bounded by the lattice paths. To avoid confusion, we will always indicate which construction we are using.

In [3], a method of encoding each parallelogram polyomino as an *area word* consisting of barred and unbarred natural numbers is presented, which in turn can be recognized from the polyomino's corresponding Dyck path under the map **ptd**. This encoding assigns unbarred natural numbers to all $E$-steps of the lower bounding lattice path and barred natural numbers to all $N$-steps of the upper bounding lattice path, with the area word a particular permutation of these barred and unbarred numbers. In fact, the sum of the numbers in the area word (ignoring the bars) is equal to the area of the parallelogram polyomino. For our purposes, we will only be concerned with finding the barred and unbarred numbers which compose the area word. These are determined as follows. Let $\mathcal{P}$ be a paralleogram polyomino in a $p \times q$ grid.

1. For each $E$-step of the lower lattice path, draw a line with a slope of $-1$ which starts at the right endpoint of this $E$-step and stops upon reaching the upper lattice path. Label the original $E$-step with the (unbarred) number of squares crossed by this line.

2. After doing this for every $E$-step of the lower lattice path, label each $N$-step of the upper lattice path with the (barred) number of squares to the right and in the interior of the parallelogram polyomino which were not crossed by any of the diagonal lines we drew.

Alternatively, given a (prime) Dyck path $C$ from $(0, 0)$ to $(p + q, p + q)$, we can find the area word of the associated parallelogram polyomino as follows:

1. Label the diagonal through $(0,1)$ and $(p+q-1,p+q)$ with $\bar{0}$, the diagonal through $(0,2)$ and $(p+q-2,p+q)$ with 1, the diagonal through $(0,3)$ and $(p+q-3,p+q)$ with $\bar{1}$, the diagonal through $(0,4)$ and $(p+q-4,p+q)$ with 2, the diagonal through $(0,5)$ and $(p+q-5,p+q)$ with $\bar{2}$, and so on.

2. Label each $N$-step of $C$ with the label of the diagonal line immediately above it.

3. Read the labels of the $N$-steps of $C$ in order to obtain the area word of the associated parallelogram polyomino.

Figure 5 gives an example of the labeling described. The left is a pair of weakly disjoint lattice paths $(P,Q)$, the middle is the corresponding parallelogram polyomino bounded by $(P',Q')$, and the right is the prime Dyck path $C' = \mathbf{ptd}(P',Q')$, together with the labeled diagonals. Reading the labels on the $N$-steps of $C'$ from bottom to top yields the area word $01\bar{1}21\bar{1}2\bar{2}$.

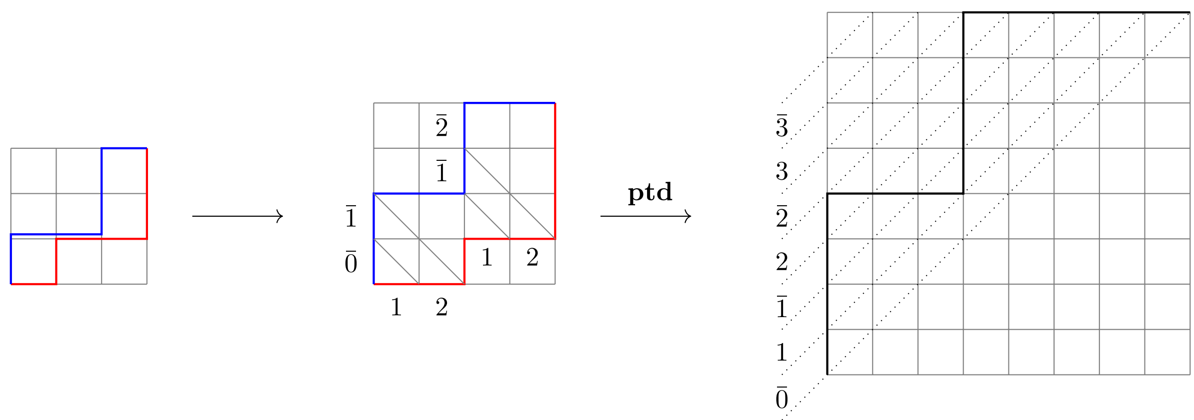


Figure 5: A pair of weakly disjoint paths $(P,Q)$, the parallelogram polyomino bounded by $(P',Q')$, and the prime Dyck path $\mathbf{ptd}(P',Q')$.

The following theorem uses this area word encoding of parallelogram polyominoes to establish a relation between the well studied area statistic of Dyck paths and the left- and right-areas associated with a pair of weakly disjoint lattice paths. The area statistic of Dyck paths, denoted by $\texttt{area}(C)$, is the number of whole squares between the Dyck path $C$ and the line $y = x$. Given a pair $(P,Q)$ of weakly disjoint lattice paths from $(0,0)$ to $(p,q)$ with $Q$ weakly below $P$, the left-area statistic $\texttt{area}_{\mathrm{L}}(P)$ of $P$ is the number of squares in the $p \times q$ rectangular grid to the left of $P$, and the right-area statistic $\texttt{area}_{\mathrm{R}}(Q)$ is the number of squares to the right of $Q$.

**Theorem 5.3.** Let $L_1$ and $L_2$ be weakly disjoint lattice paths from $(0,0)$ to $(p,p)$ with $L_2$ weakly below $L_1$. If $C$ is the Dyck path associated with the pair $(L_1,L_2)$ according to the Generalized Labelle Merging Algorithm, then

$$\texttt{area}(C) = (2p+1)p - 2(\texttt{area}_{\mathrm{L}}(L_1) + \texttt{area}_{\mathrm{R}}(L_2)).$$

*Proof.* Let $L_1'$ be the lattice path from $(0,0)$ to $(p+1,p+1)$ obtained by adding an initial $N$-step and a final $E$-step to $L_1$. Similarly, let $L_2'$ be the lattice path from $(0,0)$ to $(p+1,p+1)$ by adding an initial $E$-step and a final $N$-step to $L_2$. Let $\mathcal{P}$ be the parallelogram polyomino bounded by $L_1'$ and $L_2'$.

Consider the prime Dyck path $C' = \mathbf{ptd}(L_1', L_2')$ associated with $\mathcal{P}$, which goes from $(0,0)$ to $(2(p+1), 2(p+1))$. We have $C' = NCE$. Observe that we can compute $\texttt{area}(C')$ by labeling each $N$-step of $C'$ with the number of whole squares to its right up to the line $y = x$ and then adding all of these $p$ labels together. Equivalently, for each $i = 1, \ldots, 2p+1$, we can draw the diagonal line through $(0,i)$ and $(2p+2-i, 2p+2)$ and assign it the label $i-1$; then label each $N$-step of $C'$ with the label of the diagonal line immediately above it. Either of these methods will yield the same labels for the $N$-steps of $C'$. Clearly these labels will differ from the labels which are assigned to the $N$-steps of $C'$ according to the method in [3]. In particular, if an $N$-step is assigned the label $2k$ according to our method, it will be assigned the label $\bar{k}$ according to the method in [3]; and if an $N$-step is assigned the label $2k-1$ according to our method, it will be assigned the label $k$ according to the method in [3].

Let $a_1, \ldots, a_{p+1}$ denote the unbarred numbers and $b_1, \ldots, b_{p+1}$ denote the barred numbers which together comprise the area word of the parallelogram polyomino $\mathcal{P}$. Then according to the method of conversion of the labels described above,

$$\begin{aligned}
\texttt{area}(C') &= (2a_1 - 1) + (2a_2 - 1) + \cdots + (2a_{p+1} - 1) + 2b_1 + 2b_2 + \cdots + 2b_{p+1} \\
&= 2(a_1 + a_2 + \cdots a_{p+1} + b_1 + b_2 + \cdots + b_{p+1}) - (p+1) \\
&= 2[(p+1)^2 - (\texttt{area}_{\mathrm{L}}(L_1') + \texttt{area}_{\mathrm{R}}(L_2'))] - (p+1) \\
&= \binom{2p+2}{2} - 2(\texttt{area}_{\mathrm{L}}(L_1') + \texttt{area}_{\mathrm{R}}(L_2')).
\end{aligned}$$

Now since $\texttt{area}(C') = \binom{2p+2}{2} - \binom{2p+1}{2} + \texttt{area}(C)$, $\texttt{area}_{\mathrm{L}}(L_1') = \texttt{area}_{\mathrm{L}}(L_1)$, and $\texttt{area}_{\mathrm{R}}(L_2') = \texttt{area}_{\mathrm{R}}(L_2)$, we will have

$$\texttt{area}(C) = \binom{2p+1}{2} - 2(\texttt{area}_{\mathrm{L}}(L_1) + \texttt{area}_{\mathrm{R}}(L_2)),$$

so the theorem is proved. □

**Theorem 5.4.** Let $L_1$ and $L_2$ be a pair of weakly disjoint lattice paths from $(0,0)$ to $(p,q)$ with $L_1$ weakly above $L_2$, and let $C$ be the Dyck path generated from $L_1$ and $L_2$ by the Generalized Labelle Merging Algorithm. Then

$$\texttt{area}(C) = (2p+1)q - 2(\texttt{area}_{\mathrm{L}}(L_1) + \texttt{area}_{\mathrm{R}}(L_2)).$$

*Proof.* When $r = p - q > 0$, we can add $r$ $N$-steps to the end of both $L_1$ and $L_2$. Call these new lattice paths from $(0,0)$ to $(p,p)$ $L_1'$ and $L_2'$. The Dyck path $C'$ generated from $L_1'$ and $L_2'$ by the Generalized Labelle Merging Algorithm will differ from $C$ in that it contains $r$ extra $NE$ pairs right before the last $E$-step. Adopting the notation of $\texttt{area}_{\mathrm{L}}(C)$ for Dyck paths to denote the number of squares to the left of $C$ in the rectangular grid $(p+q+1)\times(p+q+1)$, observe that $\mathtt{area}_{\mathrm{L}}(C') - \mathtt{area}_{\mathrm{L}}(C)$ will be

$$(p+q)+(p+q+1)+\cdots+(p+q+r-1) = r(p+q)+\frac{(r-1)r}{2}.$$

Also, we have $\mathtt{area}_{\mathrm{L}}(L_1') = \mathtt{area}_{\mathrm{L}}(L_1)+pr$ and $\mathtt{area}_{\mathrm{R}}(L_2') = \mathtt{area}_{\mathrm{R}}(L_2)$. Thus, by Theorem 3,

$$\begin{aligned}
\mathtt{area}(C) &= \binom{p+q+1}{2} - \mathtt{area}_{\mathrm{L}}(C)\\
&= \binom{p+q+1}{2} - \mathtt{area}_{\mathrm{L}}(C') + r(p+q) + \frac{r(r-1)}{2}\\
&= 2p^2+q-\binom{2p+1}{2} + \mathtt{area}(C')\\
&= 2p^2+q-\binom{2p+1}{2} + (2p+1)p - 2(\mathtt{area}_{\mathrm{L}}(L_1') + \mathtt{area}_{\mathrm{R}}(L_2'))\\
&= 2p^2+q-2(\mathtt{area}_{\mathrm{L}}(L_1)+pr+\mathtt{area}_{\mathrm{R}}(L_2))\\
&= (2p+1)q-2(\mathtt{area}_{\mathrm{L}}(L_1)+\mathtt{area}_{\mathrm{R}}(L_2))
\end{aligned}$$

Similarly, when $r=q-p>0$, we can add $r$ $E$-steps to the end of both $L_1$ and $L_2$, creating new lattice paths $L_1'$ and $L_2'$ from $(0,0)$ to $(p,q)$. Since this will add $r$ $EN$ pairs to the end of $C$, the new Dyck path $C'$ generated from $L_1'$ and $L_2'$ will satisfy

$$\mathtt{area}_{\mathrm{L}}(C') = \mathtt{area}_{\mathrm{L}}(C) + (p+q+1)+\cdots+(p+q+r) = \mathtt{area}_{\mathrm{L}}(C) + r(p+q) + \frac{r(r+1)}{2}.$$

Also, $\mathtt{area}_{\mathrm{R}}(L_2') = \mathtt{area}_{\mathrm{R}}(L_2)+qr$ and $\mathtt{area}_{\mathrm{L}}(L_1') = \mathtt{area}_{\mathrm{L}}(L_1)$. Therefore,

$$\begin{aligned}
\mathtt{area}(C) &= \binom{p+q+1}{2} - \mathtt{area}_{\mathrm{L}}(C)\\
&= \binom{p+q+1}{2} - \mathtt{area}_{\mathrm{L}}(C') + r(p+q) + \frac{r(r+1)}{2}\\
&= 2q^2+q-\binom{2q+1}{2} + \mathtt{area}(C')\\
&= 2q^2+q-\binom{2q+1}{2} + (2q+1)q - 2(\mathtt{area}_{\mathrm{L}}(L_1') + \mathtt{area}_{\mathrm{R}}(L_2'))\\
&= 2q^2+q-2(\mathtt{area}_{\mathrm{L}}(L_1)+qr+\mathtt{area}_{\mathrm{R}}(L_2))\\
&= (2p+1)q-2(\mathtt{area}_{\mathrm{L}}(L_1)+\mathtt{area}_{\mathrm{R}}(L_2)),
\end{aligned}$$

and we are done. □

We conclude the paper by relating the above discussion on Dyck paths to the bivariate generating function $F_{p,q}(s,t)$. The relations are presented in Equations (23) and (25). Let $\mathcal{D}(p)$ denote the set of Dyck paths from $(0,0)$ to $(p,p)$, and let $\mathcal{LP}(p,q)$ denote the set of pairs of weakly disjoint lattice paths $(L_1,L_2)$ from $(0,0)$ to $(p,q)$, where $L_1$ lies weakly above $L_2$. To remain in accordance with our variable choices in the generating function $F_{p,q}(s,t)$, we will use the variable $t$ instead of $q$ in all below definitions of Catalan $q$-analogs.

Let $\mathcal{C}_p(t)$ denote the $q$-analog of the Catalan numbers studied by Carlitz and Riordan [5]:

$$\mathcal{C}_p(t) = \sum_{P \in \mathcal{D}(p)} t^{\mathtt{area}(P)}.$$

Using the Generalized Labelle Merging Algorithm and Theorems 5.3 and 5.4, we have

$$\sum_{P \in \mathcal{D}(K+1)} t^{\mathtt{area}(P)} = \sum_{\substack{p,q \in \mathbb{N} \\ p+q=K}} \sum_{(L_1,L_2) \in \mathcal{LP}(p,q)} t^{(2p+1)(q)-2(\mathtt{area}_{\mathrm{L}}(L_1)+\mathtt{area}_{\mathrm{R}}(L_2))}. \tag{22}$$

With some algebraic manipulation, we get the following relation.

**Proposition 5.5.**

$$\sum_{\substack{p,q \in \mathbb{N} \\ p+q=K}} t^{\binom{p-q+1}{2}} F_{p,q}(t^2,t^2) = t^{\binom{K+1}{2}} \mathcal{C}_{K+1}(1/t). \tag{23}$$

Let $\mathrm{Cat}_p(t)$ denote the MacMahon $q$-analog of the Catalan numbers [17]:

$$\mathrm{Cat}_p(t) = \frac{1}{[1+p]_t} \begin{bmatrix} 2p \\ p \end{bmatrix}_t,$$

and let $\mathcal{C}_p(q,t)$ denote the $(q,t)$-Catalan numbers introduced by Garcia and Haiman [9]. In fact, the former is a special case of the latter, since it was shown in [9] that

$$t^{\binom{p}{2}} \mathcal{C}_p(t,1/t) = \mathrm{Cat}_p(t). \tag{24}$$

From Theorem 4.9, Equation (24), and the identity (see [22])

$$\mathrm{Cat}_p(t) = \sum_{k=1}^{p} N(p,k;t),$$

where $N(p,k;t)$ is the $t$-Narayana number $\frac{t^{k(k-1)}}{[p]_t} \begin{bmatrix} p \\ k \end{bmatrix}_t \begin{bmatrix} p \\ k-1 \end{bmatrix}_t$, we have the following equation.

**Proposition 5.6.** For fixed $K \in \mathbb{N}$:

$$\sum_{\substack{p,q \in \mathbb{N} \\ p+q=K}} t^{p(p+q+1)} F_{p,q}(t,1/t) = \mathrm{Cat}_{K+1}(t) = t^{\binom{K+1}{2}} \mathcal{C}_{K+1}(t,1/t). \tag{25}$$

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