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Note

A major index for matchings and set partitions

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Abstract

We introduce a statistic $\mathrm{pmaj}(P)$ for partitions of $[n]$, and show that it is equidistributed with cr_2, the number of 2-crossings, over all partitions of $[n]$ with given sets of minimal block elements and maximal block elements. This generalizes the classical result of equidistribution for the permutation statistics inv and maj.
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1. Introduction

One of the classical results on permutations is the equidistribution of the statistics inv and maj. For a permutation $\pi = (a_1 a_2 \cdots a_n)$, a pair (a_i, a_j) is called an *inversion* if $i < j$ and $a_i > a_j$. The statistic $\mathrm{inv}(\pi)$ is defined as the number of inversions of π. The *descent set* $D(\pi)$ is defined as $\{i\colon a_i > a_{i+1}\}$, and its cardinality is denoted $\mathrm{des}(\pi)$. The sum of the elements of $D(\pi)$ is called the *major index* of π (also called the greater index) and denoted $\mathrm{maj}(\pi)$. Similarly one can define the notions of inversion, descent set, and major index for any word $w = w_1 w_2 \cdots w_n$ of not-

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necessarily distinct integers. It is a result of MacMahon [7] that inv and maj are equidistributed on any rearrangement class $\mathcal{R}(w)$. A statistic equidistributed with inv is called *Mahonian*.

There are many research articles devoted to Mahonian statistics and their generalizations. For example, see [2,5] for Mahonian statistics for words, [10,11] for Mahonian statistics and Laguerre polynomials, [9] for a major index statistic for set partitions, and very recently [6] for inv and maj for standard Young tableaux. Foata has given a nice bijective proof of the maj–inv equidistribution result on permutations [3]. Given a partition of $[n]$, there is a natural generalization of inversions, namely, 2-crossings, which can be viewed easily on a graphical representation of the partition. In this paper we introduce a new statistic, called the *p-major index* and denoted $\mathrm{pmaj}(P)$, on the set of partitions of $[n]$. We prove that for any $S, T \subseteq [n]$ with $|S| = |T|$, pmaj and cr_2, the number of 2-crossings, are equally distributed on the set $P_n(S,T)$. Here $P_n(S,T)$ is the set of partitions of $[n]$ for which S is the set of minimal block elements and T is the set of maximal block elements. Restricted to permutations, the pair $(\mathrm{cr}_2, \mathrm{pmaj})$ coincides with $(\mathrm{inv}, \mathrm{maj})$. Hence our result gives another generalization of MacMahon's equidistribution theorem.

In the next section we list necessary notions and state the main results. The proofs and some examples are given in Section 3. We prove the main result in two ways. The first one uses generating functions, and the second one is a bijection derived from the algebraic argument.

2. Definitions and the main results

A *partition* of $[n] = \{1, 2, \dots, n\}$ is a collection P of disjoint nonempty subsets of $[n]$, whose union is $[n]$. Each subset in P is called a *block* of P. A (*perfect*) *matching* of $[n]$ is a partition of $[n]$ in which each block contains exactly two elements. Denote by Π_n the set of all partitions of $[n]$. Following [1], we represent each partition $P \in \Pi_n$ by a graph $\mathcal{G}_P$ on the vertex set $[n]$ whose edge set consists of arcs connecting the elements of each block in numerical order. Such a graph is called the *standard representation* of the partition P. For example, the standard representation of 1457-26-3 has the arc set $\{(1,4), (4,5), (5,7), (2,6)\}$, see Fig. 1. We always write an arc e as a pair (i, j) with $i < j$, and say that i is the *left-hand endpoint* of e and j is the *right-hand endpoint* of e.

A partition $P \in \Pi_n$ is a matching if and only if in $\mathcal{G}_P$, each vertex is the endpoint of exactly one arc. In other words, each vertex is either a left-hand endpoint, or a right-hand endpoint. In particular, a permutation π of $[m]$ can be represented as a matching M_π of $[2m]$ with arcs connecting $m + 1 - \pi(i)$ and $i + m$ for $1 \leqslant i \leqslant m$. See Fig. 2 for example.

Two arcs $(i_1, j_1), (i_2, j_2)$ of $\mathcal{G}_P$ form a 2-crossing if $i_1 < i_2 < j_1 < j_2$. Let $\mathrm{cr}_2(P)$ denote the number of 2-crossings of P. A 2-crossing is a natural generalization of an inversion of a permutation. It is easily seen that under the correspondence $\pi \to M_\pi$, $\mathrm{cr}_2(M_\pi) = \mathrm{inv}(\pi)$.

Given $P \in \Pi_n$, define

$$\min(P) = \{\text{minimal block elements of } P\},$$

$$\max(P) = \{\text{maximal block elements of } P\}.$$

For example, for $P = 1457\text{-}26\text{-}3$, $\min(P) = \{1, 2, 3\}$ and $\max(P) = \{3, 6, 7\}$.

Fix $S, T \subseteq [n]$ with $|S| = |T|$. Let $P_n(S,T)$ be the set $\{P \in \Pi_n\colon \min(P) = S, \max(P) = T\}$. For any set $X \subseteq [n]$, let $X_{>i} = X \cap \{i+1, \dots, n\}$.

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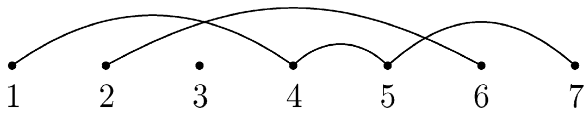

Fig. 1. The standard representation of partition $P = 1457\text{-}26\text{-}3$.

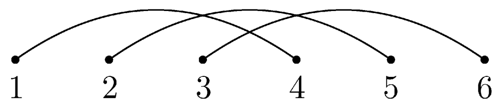

Fig. 2. The permutation $\pi = 321$ and the matching M_π.

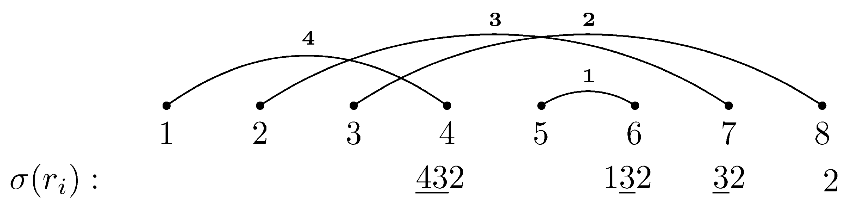

Fig. 3. The major index for partition 14-27-38-56 is 4.

Theorem 1. *Fix $S, T \subseteq [n]$ with $|S| = |T|$. Then*

$$\sum_{P \in P_n(S,T)} y^{\mathrm{cr}_2(P)} = \prod_{i \notin T} \bigl(1 + y + \cdots + y^{h(i)-1}\bigr),$$

where $h(i) = |T_{>i}| - |S_{>i}|$.

For a permutation $\pi = (a_1 a_2 \cdots a_n)$, the major index $\mathrm{maj}(\pi)$ can be computed as $\sum_{i=1}^{n} \mathrm{des}(a_i \cdots a_n)$. This motivates the following definition of the p-major index for set partitions. Given $P \in \Pi_n$, we start with the standard representation $\mathcal{G}_P$. First label the arcs of P by $1, 2, \ldots, k$ from right to left in order of their left-hand endpoints. That is, if the arcs are $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ with $i_1 > i_2 > \cdots > i_k$, then (i_r, j_r) has label r, for $1 \leqslant r \leqslant k$. Next we associate a sequence $\sigma(r)$ to each right-hand endpoint r. Assume that the right-hand endpoints are $r_1 > r_2 > \cdots > r_k$. (The set $\{r_1, \ldots, r_k\}$ is exactly $[n] \setminus S$.) The sequence $\sigma(r_i)$ is defined backward recursively: let $\sigma(r_1) = a$ if r_1 is the right-hand endpoint of the arc with label a. In general, after defining $\sigma(r_i)$, assume that the left-hand endpoints of the arcs labeled $a_1, \ldots, a_t$ are lying between r_{i+1} and r_i. Then $\sigma(r_{i+1})$ is obtained from $\sigma(r_i)$ by deleting entries $a_1, \ldots, a_t$ and adding b at the very beginning, where b is the label for the arc whose right-hand endpoint is r_{i+1}. Finally, define the statistic $\mathrm{pmaj}(P)$ by

$$\mathrm{pmaj}(P) := \sum_{r_i} \mathrm{des}\bigl(\sigma(r_i)\bigr).$$

Example 1. Let $P = 14\text{-}27\text{-}38\text{-}56$. Then $(r_1, r_2, r_3, r_4) = (8, 7, 6, 4)$. Fig. 3 shows how to compute $\mathrm{pmaj}(P)$. The sequences $\sigma(r_i)$ are $\sigma(8) = (2)$, $\sigma(7) = (32)$, $\sigma(6) = (132)$, and $\sigma(4) = (432)$. For each $\sigma(r_i)$, the elements in the descent set are underlined. The p-major index of P is $\mathrm{pmaj}(P) = 1 + 1 + 2 = 4$.

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Theorem 2. *Fix $S, T \subseteq [n]$ with $|S| = |T|$. Then*

$$\sum_{P \in P_n(S,T)} y^{\mathrm{pmaj}(P)} = \prod_{i \notin T} \left(1 + y + \cdots + y^{h(i)-1}\right), \tag{1}$$

where $h(i) = |T_{>i}| - |S_{>i}|$.

Combining Theorems 1 and 2, we have

Corollary 3. *For each $P_n(S, T)$,*

$$\sum_{P \in P_n(S,T)} y^{\mathrm{pmaj}(P)} = \sum_{P \in P_n(S,T)} y^{\mathrm{cr}_2(P)}.$$

That is, the two statistics cr_2 and pmaj have the same distribution over each set $P_n(S, T)$.

When $n = 2m$, $S = [m]$ and $T = [2m] \setminus [m]$, the map $\pi \to M_\pi$ gives a one-to-one correspondence between $P_n(S, T)$ and the set of permutations of $[m]$. It is easy to see that $\mathrm{pmaj}(M_\pi) = \mathrm{maj}(\pi)$. Hence the equidistribution of inv and maj for permutations is a special case of Corollary 3.

Another consequence of Theorems 1 and 2 is the symmetry of the number of partitions of $[n]$ with a given number of 2-crossings (respectively a given p-major index). Let $A(n, i; S, T)$ be the set of partitions in $P_n(S, T)$ such that $\mathrm{cr}_2(P) = i$, whose cardinality is $a(n, i; S, T)$.

Corollary 4. *Fix n and let $K = \sum_{i \notin T}(h(i) - 1)$. Then the sequence $\{a(n, i; S, T)\}_{i=0}^{K}$ is symmetric. That is,*

$$a(n, i; S, T) = a(n, K - i; S, T).$$

The same result holds if we replace $\mathrm{cr}_2(P)$ by $\mathrm{pmaj}(P)$ in defining $A(n, i; S, T)$ and $a(n, i; S, T)$.

3. Proofs for the main results

In this section we give the proofs for Theorems 1 and 2. Given a partition $P \in \Pi_n$, a vertex $i \in [n]$ in the standard representation $\mathcal{G}_P$ is one of the following types:

1. a left-hand endpoint if $i \in \min(P) \setminus \max(P)$,
2. a right-hand endpoint if $i \in \max(P) \setminus \min(P)$,
3. an isolated point if $i \in \min(P) \cap \max(P)$,
4. a left-hand endpoint and a right-hand endpoint if $i \notin \min(P) \cup \max(P)$.

In particular, $[n] \setminus \max(P)$ is the set of points which are the left-hand endpoints of some arcs, and $[n] \setminus \min(P)$ is the set of right-hand endpoints. Fixing $\min(P) = S$ and $\max(P) = T$ is equivalent to fixing the type of each vertex in $[n]$. Since the standard representation uniquely determines the partition, we can identify a partition $P \in \Pi_n$ with the set of arcs of $\mathcal{G}_P$. Hence the set $P_n(S, T)$ is in one-to-one correspondence with the set of matchings between $[n] \setminus T$ and $[n] \setminus S$ such that $i < j$ whenever $i \in [n] \setminus T$, $j \in [n] \setminus S$ and i is matched to j. In the following such a matching is referred as a *good* matching. Denote by $M_n(S, T)$ the set of good matchings from $[n] \setminus T$ to $[n] \setminus S$.

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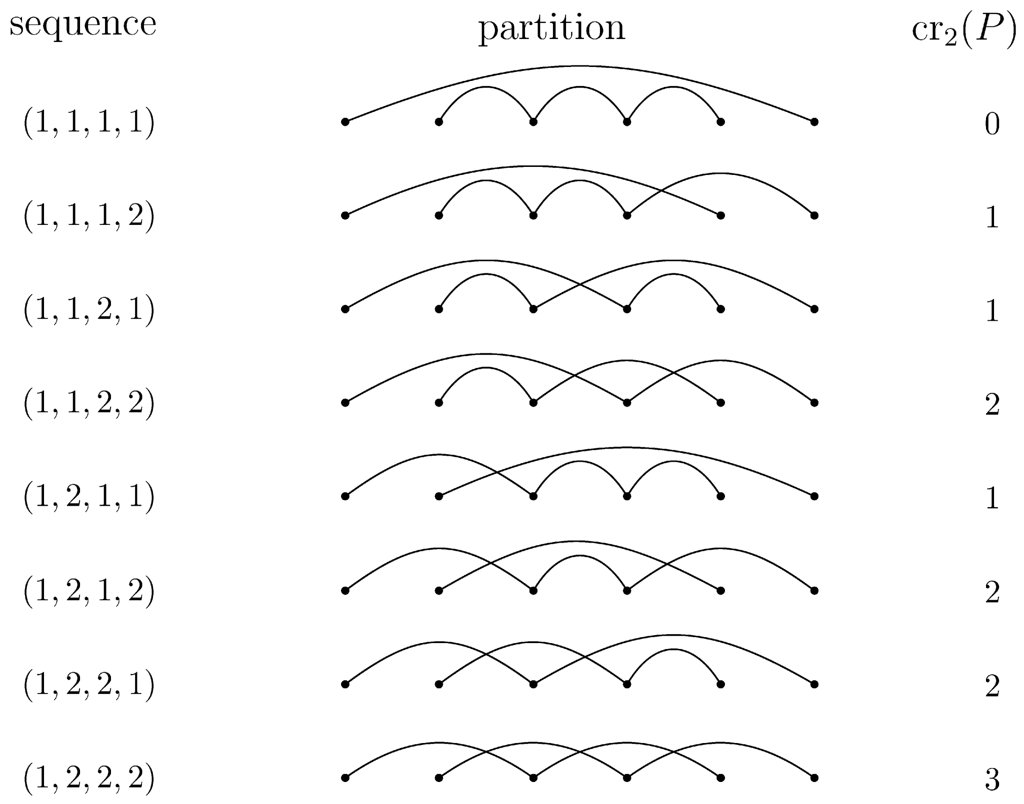

Fig. 4. $\mathrm{cr}_2(P)$ for $P \in P_6(\{1,2\},\{5,6\})$.

Proof of Theorem 1. Assume $[n] \setminus T = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Let $S(H)$ be the set of sequences $\{(a_1, a_2, \ldots, a_k)\}$ where $1 \leqslant a_r \leqslant h(i_r)$ for each $1 \leqslant r \leqslant k$. We give a bijection between the sets $M_n(S,T)$ and $S(H)$. The construction is essentially due to M. de Sainte-Catherine [8].

Given a sequence $\alpha = (a_1, a_2, \ldots, a_k)$ in $S(H)$, we construct a matching from $[n] \setminus T = \{i_1, i_2, \ldots, i_k\}$ to $[n] \setminus S$ as follows. First, there are exactly $h(i_k)$ elements in $[n] \setminus S$ which are greater than i_k. List them in increasing order as $1, 2, \ldots, h(i_k)$. Match i_k to the a_kth element, and mark this element as *dead*.

In general, after matching elements $i_{j+1}, \ldots, i_k$ to some elements in $[n] \setminus S$, we process the element i_j. At this stage there are exactly $h(i_j)$ many elements in $[n] \setminus S$ which are greater than i_j and not dead. List them in increasing order by $1, 2, \ldots, h(i_j)$. Match i_j to the a_jth of them, and mark it as dead. Continuing the process until $j = 1$, we get a good matching $M(\alpha) \in M_n(S,T)$. The map $f : \alpha \to M(\alpha)$ gives the desired bijection between $S(H)$ and $M_n(S,T)$.

Let $P(\alpha)$ be the partition of $[n]$ for which the arc set of $\mathcal{G}_P$ is $M(\alpha)$. By the above construction, the number of 2-crossings formed by arcs (i_j, b) and (a, c) with $a < i_j < c < b$ is exactly $a_j - 1$. Hence $\mathrm{cr}_2(P(\alpha)) = \sum_{j=1}^{k} (a_j - 1)$ and

$$\sum_{P \in P_n(S,T)} y^{\mathrm{cr}_2(P)} = \sum_{(a_1,\ldots,a_k) \in S(H)} y^{\sum_{j=1}^{k}(a_j-1)} = \prod_{i \notin T} \bigl(1 + y + \cdots + y^{h(i)-1}\bigr). \qquad \square \qquad (2)$$

Example 2. Let $n = 6$, $S = \{1,2\}$, and $T = \{5,6\}$. Then $[n] \setminus T = \{1,2,3,4\}$, $[n] \setminus S = \{3,4,5,6\}$, and $h(1) = 1$, $h(2) = h(3) = h(4) = 2$. Fig. 4 shows the correspondence between $S(H)$ and $P_n(S,T)$. For simplicity we omit the vertex labeling.

To prove Theorem 2, we need a lemma on permutations.

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Lemma 5. *Let $\sigma = a_1 a_2 \cdots a_{n-1}$ be a permutation of $\{2, 3, \ldots, n\}$. Let $\sigma_0 = 1a_1 \cdots a_{n-1}$ and for $i > 0$ let σ_i be obtained from σ by inserting 1 right after a_i. Then among the n permutations $\sigma_0, \ldots, \sigma_{n-1}$, the major indices are all distinct and run from $\mathrm{maj}(\sigma)$ to $\mathrm{maj}(\sigma) + n - 1$ in some order.*

Proof. First note that $\mathrm{maj}(\sigma_0) = \mathrm{maj}(\sigma) + \mathrm{des}(\sigma)$. Assume that there are t_i descents of σ that are greater than i. Then

$$\mathrm{maj}(\sigma_i) = \begin{cases} \mathrm{maj}(\sigma) + t_i & \text{if } a_i > a_{i+1}, \\ \mathrm{maj}(\sigma) + i + t_i & \text{if } a_i < a_{i+1} \text{ or } i = n-1. \end{cases}$$

It can be checked that the major indices of $\sigma_0, \ldots, \sigma_{n-1}$ are all distinct and run from $\mathrm{maj}(\sigma)$ to $\mathrm{maj}(\sigma) + n - 1$ in some order. □

A similar version of the lemma, where one inserts n instead of 1, is used in [4] to get the generating function of the major index over all permutations of $[n]$,

$$\sum_{\pi \in S_n} y^{\mathrm{maj}(\pi)} = \frac{(1-y)(1-y^2)\cdots(1-y^n)}{(1-y)^n}. \tag{3}$$

This formula is the special case of Theorem 2 with $S = [n]$ and $T = [2n] \setminus [n]$.

Proof of Theorem 2. Consider the contribution of the arc with label 1 to the generating function $\sum_{P \in P_n(S,T)} y^{\mathrm{pmaj}(P)}$. Again we identify the set $P_n(S,T)$ with the set $M_n(S,T)$ of good matchings from $[n] \setminus T$ to $[n] \setminus S$.

Let $i_k = \max([n] \setminus T)$, which is the left-hand endpoint of the arc labeled by 1 in the definition of $\mathrm{pmaj}(P)$, for any $P \in P_n(S,T)$. Assume $T_{>i_k} = \{j_1, j_2, \ldots, j_{h(i_k)}\}$. Let $A = [n] \setminus (T \cup \{i_k\})$, and $B = [n] \setminus (S \cup \{j_{h(i_k)}\})$. For any good matching M between A and B let M_t $(1 \leqslant t \leqslant h(i_k))$ be the matching obtained from M by joining the pair (i_k, j_t), and replacing each pair (a, j_r), $r > t$, with (a, j_{r+1}). Consequently, the arc labeling of M_t can be obtained from that of M by labeling the arc (i_k, j_t) by 1, and adding 1 to the label of each arc of M. Assume $\sigma(j_1) = b_1 b_2 \cdots b_{h(i_k)-1}$ for M. Then by the definition of pmaj, we have

$$\mathrm{pmaj}(M_t) = \mathrm{pmaj}(M) + \mathrm{maj}\big(b'_1 \cdots b'_{t-1} 1 b'_t \cdots b'_{h(i_k)-1}\big) - \mathrm{maj}(b_1 \cdots b_{h(i_k)-1}),$$

where $b'_i = b_i + 1$. By Lemma 5, the values of

$$\mathrm{maj}\big(b'_1 \cdots b'_{t-1} 1 b'_t \cdots b'_{h(i_k)-1}\big) - \mathrm{maj}(b_1 \cdots b_{h(i_k)-1})$$

are all distinct and run over the set $\{0, 1, \ldots, h(i_k) - 1\}$. Hence

$$\sum_{P \in P_n(S,T)} y^{\mathrm{pmaj}(P)} = \big(1 + y + \cdots + y^{h(i_k)-1}\big) \sum_{P \in P_{n-1}(A,B)} y^{\mathrm{pmaj}(P)}.$$

Eq. (1) follows by induction. □

Example 3. The p-major indices for the partitions in Example 2 are given in Fig. 5. For simplicity we omit the vertex labeling, but put the sequence $\sigma(r)$ under each right-hand endpoint r.

The above algebraic proof can be easily translated into a bijection g from $M_n(S,T)$ to $S(H)$, the set of integer sequences $\{(a_1, \ldots, a_k)\colon 1 \leqslant a_r \leqslant h(i_r) \text{ for each } 1 \leqslant r \leqslant k\}$, so that for all

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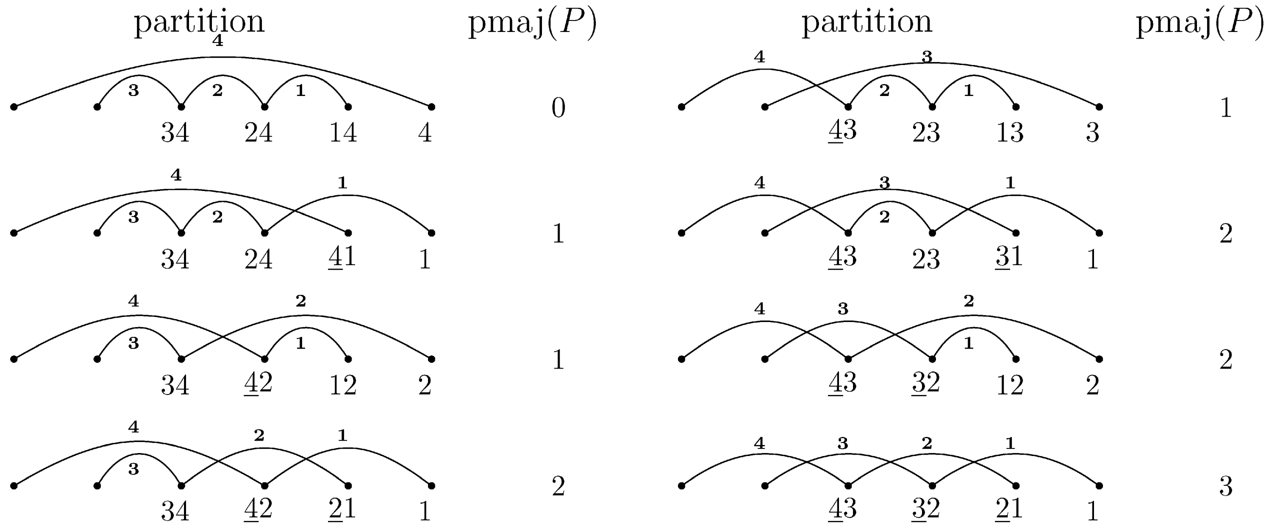

Fig. 5. $\mathrm{pmaj}(P)$ for $P \in P_6(\{1,2\},\{5,6\})$.

$M \in M_n(S,T)$, if $g(M) = (a_1, \ldots, a_k)$, then $\mathrm{pmaj}(M) = \sum_{j=1}^{k}(a_j - 1)$. Here we assume $[n] \setminus T = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$.

The map $g : M_n(S,T) \to S(H)$ is defined by induction on k, where $k = |[n] \setminus T|$, i.e., the number of pairs in any matching of $M_n(S,T)$. For $k = 1$, there is only one matching M in $M_n(S,T)$, and $S(H) = \{(1)\}$. Let $g(M) = (1)$. In general, assume that $k \geqslant 2$ and M is a good matching in $M_n(S,T)$. Let (i_k, j_k) be the pair in M whose first entry i_k is $\max([n] \setminus T)$. Then $M_1 = M \setminus \{(i_k, j_k)\}$ is a good matching with $k-1$ pairs. By the inductive hypothesis, $g(M_1)$ is well-defined. Assume $g(M_1) = (a_1, \ldots, a_{k-1})$. Set $g(M) = (a_1, a_2, \ldots, a_{k-1}, a_k)$ where $a_k = \mathrm{pmaj}(M) - \mathrm{pmaj}(M_1) + 1$. By Lemma 5 and the proof of Theorem 2, g is a bijection from $M_n(S,T)$ to $S(H)$, and $\mathrm{pmaj}(M) = \sum_{j=1}^{k}(a_j - 1)$.

Theorem 6. *There is a bijection ψ from $P_n(S,T)$ to itself such that* $\mathrm{pmaj}(M) = \mathrm{cr}_2(\psi(M))$ *for each partition $M \in P_n(S,T)$.*

Proof. Again identify $P_n(S,T)$ with the set $M_n(S,T)$ of good matchings. In the proof of Theorem 1 we constructed a bijection $f : S(H) \to M_n(S,T)$ such that for any sequence $\alpha = (a_1, \ldots, a_k)$, $\mathrm{cr}_2(f(\alpha)) = \sum_{j=1}^{k}(a_j - 1)$. Composing the map g with f yields the desired bijection. □

Remark 1. The joint distribution of cr_2 and pmaj is in general not symmetric over $P_n(S,T)$. For example, let $n = 8$, $S = \{1,2,3,5\}$ and $T = \{4,6,7,8\}$. Then

$$\sum_{P \in P_8(S,T)} x^{\mathrm{cr}_2(P)} y^{\mathrm{pmaj}(P)} = x^5y^5 + x^4y^4 + 2x^3y^4 + 2x^4y^3 + x^3y^3 + 3x^2y^2 + 2xy + 1 + 2x^3y^2 + x^2y^3 + x^2y + xy^3.$$

Remark 2. We explain the combinatorial meaning of the quantities $\{h(i) = |T_{>i}| - |S_{>i}|: i \notin T\}$. The paper [1] gives a characterization of nonempty $P_n(S,T)$'s. Given a pair (S,T) where $S, T \subseteq [n]$ and $|S| = |T|$, associate to it a lattice path $L(S,T)$ with steps $(1,1)$, $(1,-1)$ and $(1,0)$: start from $(0,0)$, read the integers i from 1 to n one by one, and move two steps for each i.

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1. If $i \in S \cap T$, move $(1, 0)$ twice.
2. If $i \in S \setminus T$, move $(1, 0)$ and then $(1, 1)$.
3. If $i \in T \setminus S$, move $(1, -1)$ and then $(1, 0)$.
4. If $i \notin S \cup T$, move $(1, -1)$ and then $(1, 1)$.

This defines a lattice path $L(S, T)$ from $(0, 0)$ to $(2n, 0)$. Conversely, the path uniquely determines (S, T). Then $P_n(S, T)$ is nonempty if and only if the lattice path $L(S, T)$ is a Motzkin path, i.e., never goes below the x-axis.

For each element $i \in [n] \setminus T$, there is a unique upper step $(1, 1)$ in the lattice path $L(S, T)$. We say an upper step is *of height* y if it goes from $(x - 1, y - 1)$ to (x, y). Then the multiset $\{h(i) = |T_{>i}| - |S_{>i}|\colon i \notin T\}$ is exactly the same as the multiset $\{\text{height of } U\colon U \text{ is an upper step in } L(S, T)\}$.

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