Expected Sums of General Parking Functions

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Abstract

A $(u_1, u_2, \ldots)$-parking function of length n is a sequence $(x_1, x_2, \ldots, x_n)$ whose order statistics (the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ obtained by rearranging the original sequence in non-decreasing order) satisfy $x_{(i)} \le u_i$. The Gončarov polynomials $g_n(x; a_0, a_1, \ldots, a_{n-1})$ are polynomials biorthogonal to the linear functionals $\varepsilon(a_i)D^i$, where $\varepsilon(a)$ is evaluation at a and D is differentiation. In this paper, we give explicit formulas for the first and second moments of sums of $\mathbf{u}$-parking functions using Gončarov polynomials by solving a linear recursion based on a decomposition of the set of sequences of positive integers. We also give a combinatorial proof of one of the formulas for the expected sum. We specialize these formulas to the classical case when $u_i = a + (i-1)b$ and obtain, by transformations with Abel identities, different but equivalent formulas for expected sums. These formulas are used to verify the classical case of the conjecture that the expected sums are increasing functions of the gaps $u_{i+1} - u_i$. Finally, we give analogues of our results for real-valued parking functions.

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1 Introduction

This paper is a companion to our earlier papers [6, 7] in which we derive recursions and formulas for moments of sums of parking functions. We shall freely use notations and results from [6].

If $\mathbf{u}$ is a sequence $(u_1, u_2, \ldots)$ of non-decreasing integers, a $\mathbf{u}$-parking function of length n is a sequence $(x_1, x_2, \ldots, x_n)$ of positive integers whose order statistics (the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ obtained by rearranging the original sequence in non-decreasing order) satisfy $x_{(i)} \leq u_i$. The Gončarov polynomials $g_n(x; a_0, a_1, \ldots, a_{n-1})$ are polynomials defined by the biorthogonality relations

$$\varepsilon(a_i) D^i g_n(x; a_0, a_1, \ldots, a_{n-1}) = n! \delta_{in},$$

where $\varepsilon(a)$ is evaluation at a and D is differentiation. We will use the following facts about Gončarov polynomials, proved in [6].

Lemma 1.1. *(a) The differential relation.*

$$D g_n(x; a_0, a_1, \ldots, a_{n-1}) = n g_{n-1}(x; a_1, a_2, \ldots, a_{n-1}).$$

(b) The coefficient formula.

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(0; a_i, a_{i+1}, \ldots, a_{n-1}) x^i.$$

(c) The matrix relation. If $\mathcal{A}$ is the $(n+1) \times (n+1)$ lower triangular matrix

$$\left[\binom{i}{j} a_j^{i-j} \right]_{0 \leq i,j \leq n},$$

then

$$\mathcal{A}^{-1} = \left[\binom{i}{j} g_{i-j}(0; a_j, a_{j+1}, \ldots, a_{i-1}) \right]_{0 \leq i,j \leq n}.$$

In particular,

$$\mathcal{A}^{-1} \overrightarrow{x^i} = \overrightarrow{g_i(x; a_0, a_1, \ldots, a_{n-1})}.$$

The Gončarov polynomials are the natural basis of polynomials for enumerating parking functions. Let $P(u_1, u_2, \ldots, u_n)$ be the number of $\mathbf{u}$-parking functions of length n.

Lemma 1.2.

$$\begin{aligned} P(u_1, u_2, \ldots, u_n) &= g_n(0; -u_1, -u_2, \ldots, -u_n) \\ &= (-1)^n g_n(0; u_1, u_2, \ldots, u_n). \end{aligned}$$

The *expected sum $E_1(n; \mathbf{u})$ of a random $\mathbf{u}$-parking function of length n* is defined by

$$E_1(n; \mathbf{u}) = \frac{1}{P_n(\mathbf{u})} \sum_{(x_1, x_2, \ldots, x_n)} (x_1 + x_2 + \ldots + x_n)$$

where the sum ranges over all $\mathbf{u}$-parking functions of length n. We shall use the following special case of Theorem 7.1 in [6].

Theorem 1.3. *The expected sums $E_1(n;\mathbf{u})$ of $\mathbf{u}$-parking functions satisfy the following linear recursion:*

$$\frac{n(x+1)}{2} = \sum_{m=0}^{n} \binom{n}{m} \frac{(x-u_{m+1})^{n-m} P_m(\mathbf{u})}{x^n} \left(E_1(m;\mathbf{u}) + \frac{(n-m)(x+u_{m+1}+1)}{2} \right).$$

Proof. The left hand side, $n(1+x)/2$, is the expected sum of a random sequence in $\{1,2,\ldots,x\}^n$, the set of all integer sequences with terms from the discrete interval $\{1,2,\ldots,x\}$.

By Theorem 5.1 in [6], each sequence in $\{1,2.\ldots,x\}^n$ decomposes into a "maximum" $\mathbf{u}$-parking function of length m and a sequence in $\{u_{m+1}+1,\ldots,x\}^{n-m}$. For a fixed m-element subset $\{i_1,i_2,\ldots,i_m\}$ of the index set $\{1,2,\ldots,n\}$, the probability that a sequence in $\{1,2,\ldots,x\}^n$ decomposes into a $\mathbf{u}$-parking function of length-m indexed by $\{i_1,i_2,\ldots,i_m\}$ and a length-$(n-m)$ sequence in $\{u_{m+1}+1,\ldots,x\}^{n-m}$ indexed by the complement is

$$\frac{(x-u_{m+1})^{n-m} P_m(u_1,u_2,\ldots,u_m)}{x^n}$$

and the expected sum of such a sequence is

$$E_1(m;u_1,u_2,\ldots,u_m) + \frac{(n-m)(x+u_{m+1}+1)}{2}.$$

The theorem now follows by summing over conditional expected sums. □

Theorem 1.3 will be used in Section 2 to derive a formula for $E_1(n;\mathbf{u})$ in terms of Gončarov polynomials. When specialized to the "classical" case $u_i = a+(i-1)b$, this formula is different from the formula given in [7]. The two formulas, and formulas "interpolating" between them, are shown to be equivalent using Abel identities in Section 3. These formulas are used in Section 4 to prove the classical case of a monotonicity conjecture for $E_1(n;\mathbf{u})$. In Section 5, we give a formula for the second factorial moment of the sum of a random $\mathbf{u}$-parking function and indicate its connection with identities of Abel-type. Finally we state analogues of our results for real-valued parking functions in Section 6.

2 Expected sums of u-parking functions

Let $\mathbf{a}$ be the sequence (a_i) defined by

$$a_i = x - u_{i+1},$$

with $0 \le i < \infty$. Then the expected sum $E_1(n;\mathbf{u})$, as a function of $u_1,u_2,\ldots$ becomes a function of x and $a_0,a_1,a_2,\ldots$. Let

$$e_n^{(1)}(x;\mathbf{a}) = E_1(n;x-a_0,x-a_1,\ldots).$$

In terms of $e_n^{(1)}(x;\mathbf{a})$, the recursion in Theorem 1.3 becomes

$$\begin{aligned} \frac{nx^n(x+1)}{2} &= \sum_{m=0}^{n} \binom{n}{m} a_m^{n-m} g_m(x;\mathbf{a}) e_m^{(1)}(x;\mathbf{a}) \\ &\quad + \sum_{m=0}^{n} \binom{n}{m} a_m^{n-m} g_m(x;\mathbf{a})(n-m)\left(\frac{2x-a_m+1}{2}\right). \end{aligned} \tag{2.1}$$

Solving recursion (2.1), we obtain the following formula for $E_1(n;\mathbf{u})$ in terms of Gončarov polynomials.

Theorem 2.1. *The sum* $g_n(x;\mathbf{a})e_n^{(1)}(x;\mathbf{a})$ *equals*

$$\frac{nx(1+x)}{2}g_{n-1}(x;a_1,a_2,\ldots,a_{n-1})$$
$$- \quad \frac{n}{2}\sum_{i=0}^{n-1}\binom{n-1}{i}a_i(2x-a_i+1)g_{n-i-1}(a_i;a_{i+1},a_{i+2},\ldots,a_{n-1})g_i(x;a_0,a_1,\ldots,a_{i-1}).$$

Proof. We transform the recursion into a matrix equation using the notation: if $f_i(x), i = 0,1,2,\ldots,n$, is a sequence of polynomials, then $\overrightarrow{f_i(x)}$ is the column vector

$$(f_0(x), f_1(x), \ldots, f_n(x))^T.$$

In this notation the recurrence (2.1) can be rewritten as

$$\frac{x(1+x)}{2}\overrightarrow{ix^{i-1}} = \mathcal{A}\ \overrightarrow{g_i(x;\mathbf{a})e_i^{(1)}(x;\mathbf{a})} + \mathcal{B}\ \overrightarrow{\left(\frac{2x-a_i+1}{2}\right)g_i(x;\mathbf{a})}.$$

where $\mathcal{A}$ is the matrix in 1.1, and $\mathcal{B}$ is the $(n+1)\times(n+1)$ matrix

$$\left[i\binom{i-1}{j}a_j^{i-j}\right]_{0\le i,j,\le n}.$$

Here we use the convention that the binomial coefficient $\binom{i}{j}$ is zero if $j > i$. Applying the inverse of $\mathcal{A}$ to both sides, we obtain

$$\frac{x(1+x)}{2}\mathcal{A}^{-1}\ \overrightarrow{ix^{i-1}} = \overrightarrow{g_i(x;\mathbf{a})e_i^{(1)}(x;\mathbf{a})} + \mathcal{A}^{-1}\mathcal{B}\ \overrightarrow{\left(\frac{2x-a_i+1}{2}\right)g_i(x;\mathbf{a})}. \qquad (2.2)$$

Observing that ix^{i-1} is the derivative of x^i, we conclude, by Lemma 1.1 that the left hand side of Eq. (2.2) equals

$$\frac{x(1+x)}{2}\overrightarrow{ig_{i-1}(x;a_1,a_2,\ldots,a_{i-1})},$$

where we use the convention (consistent with the differential relation) that Gončarov polynomials with negative indices are identically zero.

To simplify the right hand side, consider the matrix $\mathcal{A}^{-1}\mathcal{B}$. Since both $\mathcal{A}$ and $\mathcal{B}$ are lower triangular and the diagonal entries of $\mathcal{B}$ are zero, $\mathcal{A}^{-1}\mathcal{B}$ is lower triangular with zero diagonal, that is, its ij-entry is zero if $i \le j$. Suppose that $i > j$. Then the ij-th entry of $\mathcal{A}^{-1}\mathcal{B}$ equals

$$\sum_{k=0}^{n}\binom{i}{k}g_{i-k}(0;a_k,\ldots,a_{i-1})k\binom{k-1}{j}a_j^{k-j}$$
$$= \quad (i-j)\binom{i}{j}a_j\sum_{t=0}^{n-j-1}\binom{i-j-1}{t}g_{i-j-1-t}(0;a_{j+1+t},\ldots,a_{i-1})a_j^t.$$

By the coefficient formula in Lemma 1.1,

$$g_{i-j}(x;a_j,\ldots,a_{i-1}) = \sum_{t=0}^{i-j}\binom{i-j}{t}g_t(0;a_{i-t},\ldots,a_{i-1})x^{i-j-t}.$$

Taking the derivative of both sides, we obtain

$$
\begin{aligned}
Dg_{i-j}(x; a_j, \ldots, a_{i-1}) &= (i-j) \sum_{t=0}^{i-j-1} \binom{i-j-1}{t} g_t(0; a_{i-t}, \ldots, a_{i-1}) x^{i-j-t-1} \\
&= (i-j) \sum_{t=0}^{i-j-1} \binom{i-j-1}{t} g_{i-j-1-t}(0; a_{j+1+t}, \ldots, a_{i-1}) x^t.
\end{aligned}
$$

We conclude that the ijth entry of $\mathcal{A}^{-1}\mathcal{B}$ equals

$$\binom{i}{j} a_j Dg_{i-j}(a_j; a_j, a_{j+1}, \ldots, a_{i-1}).$$

By the differential relation in Lemma 1.1,

$$Dg_{i-j}(x; a_j, a_{j+1}, \ldots, a_{i-1}) = (i-j) g_{i-j-1}(x; a_{j+1}, a_{j+2}, \ldots, a_{i-1})$$

Hence, an alternate way to write the ijth entry of $\mathcal{A}^{-1}\mathcal{B}$ is

$$i \binom{i-1}{j} a_j g_{i-j-1}(a_j; a_{j+1}, a_{j+2}, \ldots, a_{i-1}).$$

Putting all the above into Eq. (2.2), we obtain the theorem. □

Setting $x = 0$ and $a_i = -u_{i+1}$ and using Lemma 1.2, we obtain the following corollary.

Corollary 2.2. *The expected sum $E_1(n; \mathbf{u})$ equals*

$$n \sum_{i=1}^{n} \binom{n-1}{i-1} \binom{u_i+1}{2} \frac{P_{n-i}(u_{i+1}-u_i, u_{i+2}-u_i, \ldots, u_n-u_i) P_{i-1}(u_1, u_2, \ldots, u_{i-1})}{P_n(u_1, u_2, \ldots, u_n)}.$$

The formula in Corollary 2.2 is a sum of positive terms and suggests the following combinatorial proof. Observe that any permutation of a $\mathbf{u}$-parking function is a $\mathbf{u}$-parking function. Hence,

$$
\begin{aligned}
P_n(\mathbf{u}) E_1(n; \mathbf{u}) &= \sum_{(x_1, x_2, \ldots, x_n)} (x_1 + x_2 + \ldots + x_n) \\
&= n \sum_{(x_1, x_2, \ldots, x_n)} x_n,
\end{aligned}
$$

where the sums range over all $\mathbf{u}$-parking functions.

For any $\mathbf{u}$-parking function $(x_1, x_2, \ldots, x_n)$, let β be the largest integer j such that $(x_1, x_2, \ldots, x_{n-1}, j)$ is a $\mathbf{u}$-parking function. (The parameter β was first used by Kreweras [4]. In the language of parking, β is the space occupied by the last car.) Clearly, $\beta = u_i$ for some i. Let $\mathcal{P}_i$ be the collection of $\mathbf{u}$-parking functions having the parameter β equal to u_i. Consider the sum

$$\sum_{(x_1, x_2, \ldots, x_n) \in \mathcal{P}_i} x_n. \tag{2.3}$$

If $(x_1, x_2, \ldots, x_n)$ is a parking function in $\mathcal{P}_i$, then $1 \leq x_n \leq u_i$. Among the first $n-1$ terms $x_1, x_2, \ldots, x_{n-1}$, there are $i-1$ many "smaller" terms that are less than u_i. The subsequence formed by these smaller terms is a $\mathbf{u}$-parking function of length $i-1$. Hence, there are $P_{i-1}(u_1, \ldots, u_{i-1})$ many ways to choose the subsequence of smaller terms.

The remaining $n-i$ "bigger" terms take values from u_i+1 to u_n and their order statistics satisfy $x_{(1)} \le u_{i+1}, x_{(2)} \le u_{i+2}, \ldots, x_{(n-i)} \le u_n$. Thus, the subsequence formed by those $n-i$ terms can be written as the sum

$$(u_i + y_1, u_i + y_2, \ldots, u_i + y_{n-i}),$$

where $(y_1, y_2, \ldots, y_{n-i})$ is a $(u_{i+1}-u_i, u_{i+2}-u_i, \ldots, u_n-u_i)$-parking function. From this, we conclude that there are $P_{n-i}(u_{i+1}-u_i, u_{i+2}-u_i, \ldots, u_n-u_i)$ many ways to choose the subsequence of bigger terms.

Because there are $\binom{n-1}{i-1}$ ways to merge the two subsequences into a $\mathbf{u}$-parking function of length n, the sum (2.3) equals

$$(1+2+\cdots+u_i)\binom{n-1}{i-1}P_{i-1}(u_1,\ldots,u_{i-1})P_{n-i}(u_{i+1}-u_i, u_{i+2}-u_i, \ldots, u_n-u_i).$$

Summing over j, we obtain the formula in Corollary 2.2.

The combinatorial argument also yields the following recursion for the number of parking functions:

$$P_n(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n}\binom{n-1}{i-1}u_i P_{n-i}(u_{i+1}-u_i, u_{i+2}-u_i, \ldots, u_n-u_i)P_{i-1}(u_1, u_2, \ldots, u_{i-1}).$$

3 The classical case and Abel identities

The classical case of parking functions is when $u_i = a + (i-1)b$. We shall write $E_1(n;a,b)$ instead of $E_1(n;a,a+b,\ldots,a+(n-1)b)$. Noting that $P_i(a, a+b, \ldots, a+(i-1)b) = a(a+ib)^{i-1}$ and $P_j(b,b,\ldots,b) = b^j$, we obtain from Corollary 2.2 the following formula

$$E_1(n;a,b) = \frac{n}{2}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{(a+ib+1)b^{n-i-1}(n-i)^{n-i-2}(a+ib)^i}{(a+nb)^{n-1}}. \tag{3.1}$$

This formula is one of many reasonable formulas for $E_1(n;a,b)$. In this section, we shall derive three other formulas for $E_1(n;a,b)$ using Abel identities.

Theorem 3.1.

$$E_1(n;a,b) = \frac{n(a+1)}{2} + \frac{n}{2}\sum_{i=1}^{n-1}\binom{n-1}{i-1}\frac{b^{n-i}(n-i)^{n-i-1}(a+ib)^i}{(a+nb)^{n-1}} \tag{3.2}$$

$$E_1(n;a,b) = \frac{n(a+1)}{2} + \frac{1}{2}\sum_{j=2}^{n}\frac{n!}{(n-j)!}\frac{b^{j-1}(a+(j-1)b)}{(a+nb)^{j-1}} \tag{3.3}$$

$$E_1(n;a,b) = \frac{n(a+1)}{2} + b\binom{n}{2} - \frac{1}{2}\sum_{i=2}^{n}\binom{n}{i}\frac{i!b^i}{(a+nb)^{i-1}} \tag{3.4}$$

When $a=b=1$, Eq. (3.4) specializes to the formula obtained in [1] and [2] for expected sums of ordinary parking functions.

We prepare for the proof of Theorem 3.1 with a list of the Abel identities we need. An old but still standard reference for Abel identities is [10]. One natural way to view these identities is as special cases of identities between Gončarov polynomials using the fact (Eq. (3.1) in [6]) that

$$g_n(x; a, a+b, \ldots, a+(n-1)b) = (x-a)(x-a-nb)^{n-1}.$$

The simplest Abel identity is Abel's generalization of the binomial theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} (y+ib)^{n-i} x(x-ib)^{i-1}. \tag{3.5}$$

This can be obtained by setting $x := x+y$ and $a := y$ into the linear recursion of Gončarov polynomials:

$$x^n = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x; a_0, a_1, \ldots, a_{i-1}).$$

Setting $x := x+y+nb$, $y := y+nb$, and $b := -b$, and changing indices from i to $n-i$, we obtain *Hurwitz's versions* of Abel's binomial theorem.

$$(x+y+nb)^n = \sum_{i=0}^{n} \binom{n}{i} (y+ib)^i x(x+(n-i)b)^{n-i-1}. \tag{3.6}$$

Differentiating both sides of Eq. (3.6), we obtain

$$n(x+y+nb)^{n-1} = \sum_{i=1}^{n} \binom{n}{i-1} (n-i+1)(y+ib)^{i-1} x(x+(n-i)b)^{n-i-1}.$$

Taking the case $n-1$ of this identity and setting $x := b$ and $y := a$, we obtain one of the identities we will need later.

Lemma 3.2.

$$(n-1)(a+nb)^{n-2} = \sum_{i=1}^{n-1} \binom{n-1}{i-1} b^{n-i-1}(n-i)^{n-i-1}(a+ib)^{i-1}.$$

We will also need the following slightly modified version of the identity called $A_n(x, y; 1, -1)$ in [10]:

$$\sum_{i=0}^{n} \binom{n}{i} (x+ib)^{i+1} y(y+(n-i)b)^{n-i-1} = \sum_{i=0}^{n} \frac{n!}{i!}(x+y+nb)^i b^{n-i}(x+(n-i)b). \tag{3.8}$$

Eq. (3.8) can be easily proved by observing that the left hand side is an expansion in the Abel polynomials $y(y+mb)^{m-1}$ in y with nodes at $-mb, m = 0, 1, \ldots$. Hence, the identity follows from the following computation, where D_y is differentiation with respect to y.

$$\begin{aligned} & \varepsilon(-(n-i)b) D_y^{n-i} \left(\sum_{j=0}^{n} \frac{n!}{j!}(x+y+nb)^j b^{n-j}(x+(n-j)b) \right) \\ = \quad & n! \sum_{k=0}^{i} \frac{b^{i-k}(x+ib)^k(x+(i-k)b)}{k!}. \end{aligned}$$

The zeroth term in the sum is $b^i(x+ib)$. When $k > 0$, we can rewrite the kth term as

$$\frac{b^{i-k}(x+ib)^{k+1}}{k!} - \frac{b^{i-(k-1)}(x+ib)^k}{(k-1)!}.$$

Hence, the sum telescopes and the right hand side equals

$$\frac{n!(x+ib)^{i+1}}{i!}$$

and Eq. (3.8) is proved.

Two special cases of Eq. (3.8) will be needed. The first is obtained by setting $x := a + b$ and $y := b$ in the case $n - 2$ of Eq. (3.8):

Lemma 3.3.

$$\sum_{i=0}^{n-2} \binom{n-2}{j} (a+(j+1)b)^{j+1} b^{n-j-3} (n-1-i)^{n-j-3} = \sum_{i=0}^{n-2} \frac{(n-2)!}{j!} (a+nb)^j b^{n-j-3} (a+(n-j-1)b).$$

The second is obtained by setting $x := a$ and $y := 0$ and changing indices from i to $n - i$ in Eq. (3.8).

Lemma 3.4.

$$(a+nb)^{n+1} = \sum_{i=0}^{n} \frac{n!}{(n-i)!} (a+nb)^{n-i} b^i (a+ib). \tag{3.9}$$

Returning to the proof of Theorem 3.1, we substitute $x = a$, $a_i = -(i-1)b$,

$$\begin{aligned} g_n(a; 0, -b, -2b, \ldots, -(n-1)b) &= a(a+nb)^{n-1}, \\ g_{n-1}(a; -b, -2b, \ldots, -(n-1)b) &= (a+b)(a+nb)^{n-2}, \\ g_{n-i-1}(-ib; -(i+1)b, \ldots, -(n-1)b) &= b^{n-i-1}(n-i)^{n-i-2} \end{aligned}$$

into the formula in Theorem 2.1 to obtain

$$\begin{aligned} & a(a+bn)^{n-1} e_n^{(1)}(a; 0, -b, -2b, \ldots, -(n-1)b) \\ = \; & \frac{na(a+1)}{2} (a+b)(a+bn)^{n-2} \\ & + n \sum_{i=0}^{n-1} \binom{n-1}{i} i b^{n-i} (n-i)^{n-i-2} \left(\frac{2a+ib+1}{2} \right) a(a+ib)^{i-1}. \end{aligned}$$

The sum in this expression can be simplified slightly (by manipulating binomial coefficients) to

$$n \sum_{i=1}^{n-1} \binom{n-1}{i-1} b^{n-i} (n-i)^{n-i-1} \left(\frac{2a+ib+1}{2} \right) a(a+ib)^{i-1}. \tag{3.10}$$

We break up this sum into two parts by writing

$$\frac{2a+ib+1}{2} = \frac{a+1}{2} + \frac{a+ib}{2}.$$

The first part is the sum

$$\frac{nab(a+1)}{2} \sum_{i=1}^{n-1} \binom{n-1}{i-1} b^{n-i-1} (n-i)^{n-i-1} (a+ib)^{i-1}.$$

which, by Lemma 3.2, equals

$$\binom{n}{2} ab(a+1)(a+nb)^{n-2}.$$

When this quantity is added to $na(a+1)(a+b)(a+nb)^{n-2}/2$, we get

$$na(a+1)(a+nb)^{n-1}/2.$$

This yields Eq. (3.2) in Theorem 3.1.

Next, we use Lemma 3.3 to conclude that the sum

$$\frac{na}{2}\sum_{i=1}^{n-1}\binom{n-1}{i-1}b^{n-i}(n-i)^{n-i-1}(a+ib)^{i},$$

the second part of the sum (3.10), equals

$$\frac{ab^2n(n-1)}{2}\sum_{j=0}^{n-2}\frac{(n-2)!}{j!}(a+nb)^{j}b^{n-j-3}(a+(n-j-1)b).$$

Changing indices from j to $n-j$ and regrouping terms, this becomes

$$\frac{a}{2}\sum_{j=2}^{n}\frac{n!}{(n-j)!}(a+nb)^{n-j}b^{j-1}(a+(j-1)b).$$

This yields Eq. (3.3) in Theorem 3.1.

Finally, we extract the first two terms in the sum in (3.9) and move them to the left. When we simplify the left hand side, we find that there is a factor of b. Dividing both sides by b, we obtain

$$n(n-1)b(a+nb)^{n-1}=\sum_{j=2}^{n}\frac{n!}{(n-j)!}(a+nb)^{n-j}b^{j-1}(a+jb). \tag{3.11}$$

Dividing both sides of Eq. (3.11) by $a(a+nb)^{n-1}$ and substituting the result into Eq. (3.3), we arrive at Eq. (3.4). This completes the proof of Theorem 3.1.

4 Order properties of expected sums

In this section, we shall consider the following conjecture.

Conjecture 4.1. *The expected sum $E_1(n;u_1,u_2,\ldots,u_n)$ is an increasing function of n and the gaps $u_{i+1}-u_i$.*

There are two factors to consider when n and the gaps are increased. On the positive side, the parking functions are allowed to take on higher values. On the negative side, there are more parking functions, and since parking functions cannot take on too many higher values, the sample might consist mostly of parking functions with smaller sums. The conjecture says that the positive factor predominates.

We begin with a simple general result supporting this conjecture.

Proposition 4.2. *If γ is a rational number greater than 1 such that γu_i are integers, then*

$$E_1(n;u_1,u_2,\ldots,u_n)<E_1(n;\gamma u_1,\gamma u_2,\ldots,\gamma u_n).$$

Proof. By Corollary 2.2, $E_1(n;\mathbf{u})P_n(\mathbf{u})$ is the sum $F(\mathbf{u})+G(\mathbf{u})$ of two homogeneous functions in the variables $u_1,u_2,\ldots,u_n$, where $F(\mathbf{u})$ has total degree $n+1$ and $G(\mathbf{u})$ has total degree n. Hence, by homogeneity of $P_n(\mathbf{u})$,

$$
\begin{aligned}
E_1(n;\gamma u_1,\gamma u_2,\ldots,\gamma u_n) &= \frac{\gamma^{n+1}F(\mathbf{u})+\gamma^n G(\mathbf{u})}{\gamma^n P_n(\mathbf{u})}\\
&= \frac{(\gamma-1)F(\mathbf{u})}{P_n(\mathbf{u})}+E_1(n;u_1,u_2,\ldots,u_n).
\end{aligned}
$$

Since $\gamma > 1$, the proposition follows. □

For the classical case, when $u_i = a+(i-1)b$, Conjecture 4.1 states that the expected sums $E_1(n;a,b)$ are increasing functions of n and a, b. We shall prove a theorem which verifies this special case.

Lemma 4.3. *If* $0 < a < c$,

$$E_1(n;a,b) < E_1(n;c,b).$$

Proof. Use Eq. (3.4) and observe that $-(c+nb)^{-1} > -(a+nb)^{-1}$. □

Lemma 4.4.

$$E_1(n;a,b) < E_1(n+1;a,b).$$

Proof. Rewrite Eq. (3.4) in the form

$$E_1(n;a,b) = \frac{n(a+1)}{2} + b\binom{n}{2} - \frac{b}{2}\sum_{i=2}^{n}\frac{(n)_i}{(\gamma+n)^{i-1}},$$

where $(n)_i$ is a falling factorial and $\gamma = a/b$. Then, the forward difference

$$E_1(n+1;a,b) - E_1(n;a,b)$$

equals

$$\frac{a+1}{2} + bn - \frac{b}{2}\sum_{i=2}^{n}\left[\frac{(n+1)_i}{(n+1+\gamma)^{i-1}} - \frac{(n)_i}{(n+\gamma)^{i-1}}\right] - \frac{b}{2}\frac{(n+1)!}{(n+1+\gamma)^n}.$$

By an elementary induction argument, one can show that for γ a positive real number and $i = 2, 3, \ldots, n$,

$$\frac{(n+1)_i}{(n+1+\gamma)^{i-1}} - \frac{(n)_i}{(n+\gamma)^{i-1}} < 1.$$

From this inequality, we conclude that

$$
\begin{aligned}
E_1(n+1;a,b) - E_1(n;a,b) &> bn + \frac{a+1}{2} - \frac{b(n-1)}{2} - \frac{b}{2}\\
&= \frac{bn+a+1}{2} > 0.
\end{aligned}
$$

□

Lemma 4.5. *If c is an integer strictly greater than b, then*

$$E_1(n;a,b) < E_1(n;a,c).$$

Proof. Rewrite Eq. (3.3) in the form

$$E_1(n;a,b) = \frac{n(a+1)}{2} + \frac{1}{2}\sum_{j=2}^{n} \frac{n!}{(n-j)!}\left[\frac{ab^{j-1}}{(a+nb)^{j-1}} + \frac{(j-1)b^j}{(a+nb)^{j-1}}\right].$$

The lemma now follows from the easy inequality: if $a > 0$ and $c > b$, then $b/(a+nb) < c/(a+nc)$. ☐

The preceding three lemmas imply the following theorem.

Theorem 4.6. *The expected sum $E_1(n;a,b)$ is a strictly increasing function of n, a and b.*

5 Second factorial moments

Using a somewhat more complicated version of the proof of Theorem 2.1 we obtain the following result.

Theorem 5.1. *The second falling factorial moment of the sum of a random* **u***-parking function of length n can be computed as*

$$\begin{aligned}
& P(u_1,u_2,\ldots,u_n)E_2(n;u_1,u_2,\ldots,u_n) \\
= \; & n\sum_{i=0}^{n-1}\binom{n-1}{i}u_{i+1}P_{n-i-1}(u_{i+2}-u_{i+1},u_{i+3}-u_{i+1},\ldots,u_n-u_{i+1}) \\
& \cdot\left[(u_{i+1}+1)P_i(u_1,\ldots,u_i)E_1(i;u_1,\ldots,u_i) + \left(\frac{u_{i+1}^2-1}{3}\right)P_i(u_1,\ldots,u_i)\right] \\
& -\frac{n(n-1)}{4}\sum_{i=0}^{n-2}\binom{n-2}{i}u_{i+1}^2(u_{i+1}+1)^2 \\
& \cdot P_{n-i-2}(u_{i+3}-u_{i+1},u_{i+4}-u_{i+1},\ldots,u_n-u_{i+1})P_i(u_1,\ldots,u_i).
\end{aligned}$$

With sufficient patience and motivation, formulas for any given higher moments can be calculated with the same method. Just as for expected sums, there are many different but equivalent formulas for a fixed kth moment. These different formulas imply the existence of higher-order identities, similar to the Abel identities given in Section 3.

6 Real-valued parking functions

Let **u** be a non-decreasing sequence of non-negative real numbers. A *real-valued parking function of length n* is a sequence $(x_1,x_2,\ldots,x_n)$ of non-negative real numbers whose sequence of order statistics satisfies $x_{(i)} \le u_i$. Using the same method, one can get the following analogue of the decomposition of integral parking functions [6].

Theorem 6.1. *There is a bijection between the set $[0,x]^n$ of all length-n sequences with terms in the continuous interval $[0,x]$ and the disjoint union of Cartesian products*

$$\bigcup_{\{i_1,i_2,\ldots,i_m\}} \mathrm{Park}(i_1,i_2,\ldots,i_m)\times(u_{m+1},x]^{n-m},$$

where $\mathrm{Park}(i_1,i_2,\ldots,i_m)$ is the set of real-valued length-m **u***-parking functions indexed by $\{i_1,i_2,\ldots,i_m\}$ and $(u_{m+1},x]^{n-m}$ is the set of length-$(m-n)$ sequences with terms in the continuous half-open interval $(u_{m+1},x]$ indexed by the complement of $\{i_1,i_2,\ldots,i_m\}$.*

Let $\bar{P}_n(\mathbf{u})$ be the probability that a random sequence $(X_1, X_2, \ldots, X_n)$ with the terms X_i chosen independently with uniform distribution from $[0, x]$ is a real-valued $\mathbf{u}$-parking function. Then, by Theorem 6.1, $\bar{P}_n(\mathbf{u})$ satisfies the following linear recursion:

$$1 = \sum_{m=0}^{n} \binom{n}{m} \frac{(x - u_{m+1})^{n-m}}{x^{n-m}} \bar{P}_m(u_1, u_2, \ldots, u_m).$$

Comparing this recursion with the recursion in Corollary 5.3 in [6], we obtain the following theorem.

Theorem 6.2.

$$\bar{P}_n(u_1, u_2, \ldots, u_n) = \frac{P_n(u_1, u_2, \ldots, u_n)}{x^n}.$$

This theorem has appeared earlier in the paper [9]. Pitman and Stanley proved this theorem using another decomposition for $\mathbf{u}$-parking functions (which works for real numbers u_i also). Pitman and Stanley's decomposition was also described in Section 6 of [6]. Theorem 6.2 implies that

$$\bar{P}_n(u_1, u_2, \ldots, u_n) = \frac{(-1)^n g_n(0; u_1, u_2, \ldots, u_n)}{x^n}.$$

This is an often rediscovered result with a simple proof using conditioning, integration, and a simple fact from the theory of order statistics. It seems to have been first discovered for "reversed" parking function in determinantal form, by Steck [11]. See also [5, 8].

From Theorem 6.1, we obtain the following recursion for the expected sums $\bar{E}_1(n; \mathbf{u})$ of random length-n real-valued $\mathbf{u}$-parking functions:

$$\frac{nx}{2} = \sum_{m=0}^{n} \binom{n}{m} \frac{(x - u_{m+1})^{n-m} P_m(\mathbf{u})}{x^n} \left(\bar{E}_1(m; \mathbf{u}) + \frac{(n-m)(x + u_{m+1})}{2} \right).$$

From this, we deduce that $\bar{E}_1(n; \mathbf{u})$ equals

$$\frac{n}{2} \sum_{j=1}^{n} \binom{n-1}{j-1} u_j^2 \frac{P_{n-j}(u_{j+1} - u_j, u_{j+2} - u_j, \ldots, u_n - u_j) P_{j-1}(u_1, u_2, \ldots, u_{j-1})}{P_n(u_1, u_2, \ldots, u_n)}.$$

With a little work, one also obtains the following analogue of the expected-sum formula for the classical case:

$$\bar{E}_1(n; a, a+b, \ldots, a+(n-1)b) = \frac{na}{2} + b\binom{n}{2} - \frac{1}{2} \sum_{i=2}^{n} \binom{n}{i} \frac{i! b^i}{(a+nb)^{i-1}}.$$

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