Branching Processes with Negative Offspring Distributions

Ioana Dumitriu *
Joel Spencer †
Catherine Yan‡

1 Introduction

A branching process is a mathematical description of the growth of a population for which the individual produces offsprings according to stochastic laws. A typical process is of the following form. Considering a population of individuals developing from a single progenitor—the initial individual. The initial individual produces a random number of offsprings, each of them in turn produces a random number of offsprings; and so the process continues as long as there are live individuals in the population. An interesting question is to find the probability that the population survives, (or, extincts).

The branching process was proposed by Galton [5], and the probability of extinction was first obtained by Watson [16] by considering the probability generating function for the number of children in the nth generation. This mathematical model was known as the Galton-Watson branching process, and had been studied thoroughly in literature, for example, [15, 2, 6, 7, 10]. Some interesting details on the early history of branching processes can be found in [9].

Another model for the branching processes was based on the interpretation of the random walk $S_n - n$ and the branching processes in terms of queuing theory, which is due to Kendall [8]. Here $S_n = Z_1 + Z_2 + \cdots + Z_n$, where Z_i's are independent random variables with the identical distribution as the offsprings. In this model, the random family of a branching process is represented by a queue with a single server, and the service time for each customer is a unit period. The queue starts with one customer, corresponding to the initial individual in the branching process. Each individual in the random family represents a different customer, who arrives and waits for service in the queue. A customer j arrives during the service period of i if j is a child of i. Let Y_t be the number of customers in the queue at the end of the t-th service period. Then $Y_0 = 1$, and

$$Y_i = Y_{i-1} + Z_i - 1, \qquad (i \geq 1)$$

*Massachusetts Institute of Technology, Dept. of Math.
†Courant Institute of Mathematical Sciences (New York).
‡Texas A&M University, Dept. of Math. The author is supported by NSF grant DMS-0070574 and DMS 97-29992

where Z_i are independent random variables with the offspring distribution. Note that the total progeny is represented by the length of the first busy period of the server, i.e., smallest t such that $Y_t = 0$. The process continues forever iff $Y_i > 0$ for all i.

By the above correspondence, the probability of extinction given by the Galson-Watson model and the random walk model are identical. Let Z be the offspring distribution. A basic fact about branching processes states that when $E[Z] = c < 1$, with probability 1 the process dies out; but when $E[Z] = c > 1$, there is a positive probability α that the process will continue forever [15, 4]. Let $f(x)$ be the probability generating function for Z. In the case $c > 1$, α can be computed as $1 - s$ where s the probability of extinction, is the unique solution of $s = f(s)$ in the interval $[0, 1)$. In particular, if the offspring distribution is Poisson with mean $1 + \epsilon$, then the probability of survival is asymptotically 2ϵ (c.f., [13]). An important condition in solving such probability in Galton-Watson branching processes is that the possible values of the offspring distribution Z are non-negative integers.

In this paper we consider a random walk with a similar algebraic setting as the ones corresponding to branching processes. However, the step-size distribution can take negative values. Explicitly, let Z be a probability distribution with non-negative integral values, and $E[Z] = 1 - \epsilon$, where $0 < \epsilon < 1$. Define a random walk $\mathbf{Y} = Y_0, Y_1, \ldots$ by the following formula

$$\begin{cases} Y_0 = 1, \\ Y_i = Y_{i-1} + 1 - Z_i, \end{cases} \tag{1}$$

where $Z_1, Z_2 \ldots$ are independent random variables each with distribution Z. In particular, we consider the case where Z is distributed according to the Poisson law with mean $1 - \epsilon$, denoted by $Po(1 - \epsilon)$. Let T be the least t for which $Y_i \leq 0$. If no such t exists, we say that $T = +\infty$. Since the expected step-size $E[1 - Z] = \epsilon > 0$, with probability approaching 1, $Y_n > 0$ when $n \to \infty$. We will compute the probability that $Y_n > 0$ for all n, or equivalently, $\Pr[T = +\infty]$, in which case we say the random walk $\mathbf{Y}$ *escapes*. We also consider the asymptotics of $\Pr[T > N]$. Note that this problem is essentially different from the one defined by branching processes, since the values of the step-size random variable $1 - Z$ are mostly negative. Apparently all the methods used in literature to solve for the survival probability (e.g., [4]) fail here .

In this paper we find the exact value of the escaping probability for the random walk $\mathbf{Y}$. We present two proofs. The first is a pure probabilistic one, based on the linearity of expectations. The second is more analytic, using a theorem of Otter and Dwass in branching processes, which is equivalent to the Lagrange inversion formula for formal power series. The second proof applies to more general offspring distributions Z, provided that both the expected value and variance of Z exist. It is also used to obtain asymptotics of $\Pr[T > N]$. With suitable assumptions on Z, we prove that $\Pr[T = \infty | T > N)$ is asymptotically 1 if $N \gg \epsilon^{-2}$, when $\epsilon \to 0^+$. We also give a formula that generalizes the Otter-Dwass theorem, and use it to prove that if $Z = Po(1 - \epsilon)$, the $n \gg \epsilon^{-2}$

is both necessary and sufficient for $\Pr[T = \infty | T > N] \sim 1$.

2 The escaping probability

Theorem 1 *Fix $\epsilon > 0$. For the random walk $\mathbf{Y}$ defined by the formula (1) with $Z = Po(1-\epsilon)$, the escaping probability $\Pr[T = +\infty]$ is $e^{1-\epsilon}\epsilon$.*

1. The probabilistic proof. Let $p_i = Pr[Z = i]$ for $i \geq 0$. Considering a new random walk $\mathbf{Y}'$ defined by the same recurrence as (1) except that the beginning position is zero, i,e,

$$\begin{cases} Y'_0 = 0, \\ Y'_i = Y'_{i-1} + 1 - Z_i, \end{cases} \tag{2}$$

where $Z_1, Z_2 \ldots$ are independent random variables with the same distribution Z. Let T' be the least $t > 0$ for which $Y'_i \leq 0$. If no such t exists, we say that $T' = +\infty$. Denote by s' the escaping probability $\Pr[T' = +\infty]$ of $\mathbf{Y}'$. It is clear that if s is the escaping probability of the original random walk $\mathbf{Y}$, then $s' = p_0 s$ and hence $s = s'/p_0$.

Let $q' = 1 - s'$ be the probability of extinction for Y'. Let W_n be the indicator random variable for the event $Y'_n = 0$, i.e.,

$$W_n = \begin{cases} 1, & \text{if } Y'_n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and let $W = \sum_{i=1}^{\infty} W_n$. Then W is the number of t such that $Y'_t = 0$. (It is possible that $W = \infty$.)

A crucial property for the random walk $\mathbf{Y}'$ is that, in each step, $\mathbf{Y}'$ can raise at most one unit. Therefore, any random walk with $W = k$ (k is an integer and $k \geq 2$) consists of $k + 1$ part (see figure 1): The first k parts are random walks that begin at zero, drop to the x-axis, or drop below x-axis and then move back to zero. The last part is a random walk that begins at 0, and remains above the x-axis for all $t > 0$. It is clear that the last part corresponds to a random walk defined as $\mathbf{Y}'$ with $T' = \infty$. This happens with probability s'.

In $\mathbf{Y}'$, let $\tilde{T}$ be the least t such that $t > 0$ and $Y'_t = 0$. Hence

$$\begin{aligned} \Pr[W = k] &= \sum_{t_1, \ldots, t_k > 0} \Big(\prod_{i=1}^{k} \Pr[\tilde{T} = t_i] \Big) s' \\ &= s' \Big(\sum_{t>0} \Pr[\tilde{T} = t] \Big)^k \\ &= s' (\Pr[\tilde{T} < \infty])^k. \end{aligned}$$

If $\tilde{T} < \infty$, then $T' < \infty$. Conversely, we claim that if $T' < \infty$, then with probability one $\tilde{T} < \infty$. To prove the claim, first note that $T' < \infty$ but $\tilde{T} = \infty$ implies that there exists $N > 0$ such that

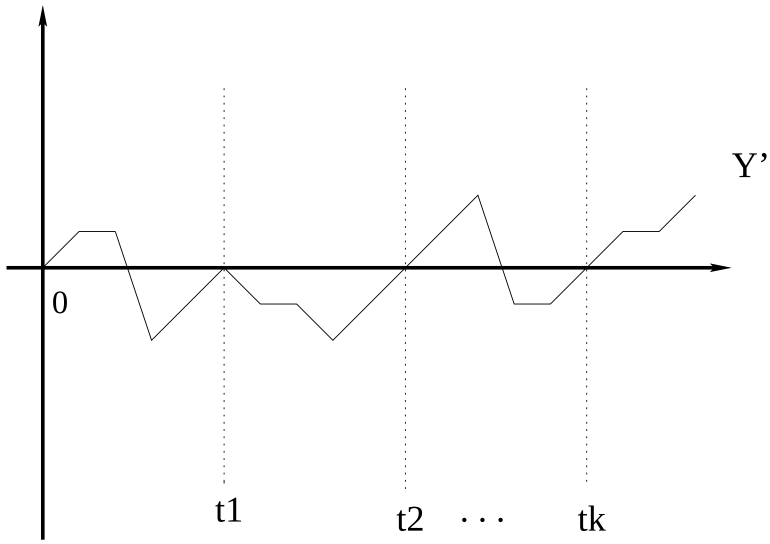

Figure 1:

for all $n \geq N$, $Y'_n < 0$. Let A_n, $n \in \mathbb{N}$ be the event that $Y'_n \geq 0$. Y'_n has the distribution $n - S_n$ where $S_n = Z_1 + Z_2 + \cdots + Z_n$, and $E[S_n] = nE[Z] = n(1-\epsilon)$. By Markov's inequality,

$$\Pr[S_n > n] \leq \frac{E[S_n]}{n} = 1 - \epsilon.$$

Hence $\Pr[A_n] = \Pr[Y'_n \geq 0] = \Pr[S_n \leq n] \geq \epsilon$. It follows that $\sum_{n=1}^{\infty} \Pr[A_n]$ diverges. By Borel-Cantelli Lemma, with probability one $Y'_n \geq 0$ for infinite many n. This proves the claim. Therefore $\Pr[\tilde{T} < \infty] = \Pr[T' < \infty] = q'$ and $\Pr[W = k] = q'^k s'$.

It follows that $\Pr[W \geq k] = q'^k$. Hence

$$E[W] = \sum_{k=1}^{\infty} \Pr[W \geq k] = \frac{q'}{1-q'} = \frac{1-s'}{s'}.$$

On the other hand, by the linearity of the expectation,

$$E[W] = \sum_{n=1}^{\infty} E[W_n] = \sum_{n=1}^{\infty} \Pr[Y'_n = 0].$$

In our problem, Z is the Poisson distribution with mean $1-\epsilon$. Thus $Y'_n = n - S_n$ where S_n has the Poisson distribution with mean $n(1-\epsilon)$. Therefore

$$E[W] = \sum_{n=1}^{\infty} \Pr[S_n = n] = \sum_{n=1}^{\infty} \frac{e^{-(1-\epsilon)n}(n-n\epsilon)^n}{n!} = \sum_{n=1}^{\infty} \frac{n^n}{n!} \left((1-\epsilon)e^{-(1-\epsilon)}\right)^n = te^{-t} R'(te^{-t}), \quad (3)$$

where $t = 1-\epsilon$ and $R(x) = (xe^{-x})^{<-1>}$ is the inverse function of $y = xe^{-x}$ [14][Chapter 5.3].

Let $y = te^{-t}$. From $R(te^{-t}) = t$, differentiating with respect to t, we have

$$R'(y)(e^{-t} - te^{-t}) = 1,$$

that is,

$$R'(y) = \frac{e^t}{1-t}.$$

Substitute into (3), we have

$$\frac{1-s'}{s'} = E[W] = \frac{t}{1-t} = \frac{1-\epsilon}{\epsilon}.$$

Therefore $s' = \epsilon$ and $s = \epsilon/p_0 = e^{1-\epsilon}\epsilon$. □

REMARK. Theorem 1 asserts that the escaping probability s' is ϵ for the random walk $\mathbf{Y}'$. Such a clean answer deserves an easy explanation. The following is an intuitive one:

In the infinite random walk $\mathbf{Y}'$, for any $i > 0$, let W_i be the number of t such that $Y'_t = i$. For an fixed i, almost surely the random walk will hit the line $y = i$, i.e., $\Pr[W_i \geq 1] = 1$. For any $k > 1$, similar to the argument in the preceding proof, we have $\Pr[W_i = k] = q'^{k-1}s'$, and $\Pr[W_i \geq k] = q'^{k-1}$. Therefore $E[W_i] = 1 + q' + q'^2 + \cdots = 1/s'$, consequently $E[W_1 + W_2 + \cdots + W_{N\epsilon}] = N\epsilon/s'$.

On the other hand,

$$E[W_1 + W_2 + \cdots + W_{N\epsilon}] = \sum_{j=0}^{\infty} \Pr[1 \leq Y'_j \leq N\epsilon].$$

where Y'_j has distribution $j - S_j$, $S_j = Z_1 + Z_2 + \cdots + Z_j \sim Po(j(1-\epsilon))$. For the Poisson distribution, we have the following large deviation result, (e.g.,see [1], Appendix A.15),

Lemma 1.1 *Let P have Poisson distribution with mean μ. For $\delta > 0$,*

$$\Pr[P \leq \mu(1-\delta)] \leq e^{-\delta^2\mu/2},$$

$$\Pr[P \geq \mu(1+\delta)] \leq [c_\delta]^\mu.$$

where $c_\delta = e^\delta(1+\delta)^{-(1+\delta)} < 1$ is a constant depending only on δ.

Using the lemma, for $\sqrt{N} < j < N - K\sqrt{N}$, $(K \gg 1)$,

$$\Pr[Y'_j > N\epsilon] = \Pr[S_j < j - N\epsilon] \leq e^{-\frac{1}{2}\frac{\epsilon^2(N-j)^2}{j(1-\epsilon)}} \leq e^{-\frac{1}{2}\frac{\epsilon^2}{(1-\epsilon)}\frac{K^2}{(1-K/\sqrt{N})}},$$

$$\Pr[Y'_j \leq 0] = \Pr[S_j - j \geq 0] \leq (c_\epsilon)^j$$

where $c_\epsilon < 1$ is a constant. Taking $K = N^{1/3}$,

$$\sum_{j=\sqrt{N}}^{N-N^{5/6}} \Pr[1 \leq Y'_j \leq N\epsilon] \sim N(1 + o(1)).$$

While for $j > N + K\sqrt{N}$, $K = N^{1/3}$,

$$\Pr[1 \leq Y'_j \leq N\epsilon] \leq \Pr[Y'_j \leq N\epsilon] = \Pr[S_j - j(1-\epsilon) \geq (j-N)\epsilon] \leq (c_\delta)^{j(1-\epsilon)}$$

where $\delta = \frac{\epsilon}{(1-\epsilon)} \frac{j-N}{j}$. From Lemma 1.1, c_δ is decreasing as δ increases, and $c_\delta \sim e^{-\delta^2/2}$ for δ small. So

$$c_\delta \le c_{\epsilon/(1-\epsilon)N^{-1/6}} \sim e^{-\frac{\epsilon^2}{2(1-\epsilon)^2}N^{-1/3}}.$$

Thus $\Pr[1 \le Y'_j \le N\epsilon] \le (e^{-\frac{\epsilon^2}{2(1-\epsilon)}N^{-1/3}})^j$ and

$$\sum_{j=N+N^{5/6}} \Pr[1 \le Y'_j \le N\epsilon] = O(e^{-\frac{\epsilon^2}{2(1-\epsilon)}N^{1/3}}) = o(1).$$

Therefore $E[W_1 + W_2 + \cdots + W_{N\epsilon}] = N(1+o(1)) + O(N^{1/2}) + O(N^{5/6}) + o(1) = N(1+o(1))$.

Comparing the two expressions for $E[W_1 + W_2 + \cdots + W_{N\epsilon}]$, we have $N\epsilon/s' \sim N$ when $N \to \infty$. That implies the escaping probability s' equals ϵ.

2. The analytic proof.

We compute the probability $\Pr[T > N]$ directly, using the Otter-Dwass theorem in the theory of branching processes. This proof applies to all integral distributions Z, provided that $E(Z)$ and $var(Z)$ exist.

In a branching process with offspring distribution Z, let $\#F_k$ be the total progeny in the process with k individuals to start with. The following theorem presents a remarkable formula for the distribution of $\#F_k$ restricted to positive integers. This formula was discovered by Otter [11] for $k = 1$, and extended to all $k \ge 1$ by Dwass [3].

Theorem. [Otter and Dwass] *For all $n, k = 1, 2, \ldots$*

$$\Pr[\#F_k = n] = \frac{k}{n}\Pr[S_n = n-k]. \tag{4}$$

where $S_n = Z_1 + Z_2 + \cdots + Z_n$.

To prove Theorem 1, as in the first proof, we consider the random walk $\mathbf{Y}'$ defined by

$$Y'_0 = 0, \quad Y'_i = Y'_{i-1} + 1 - Z_i,$$

where Z_i are independent with distribution Z, $E(Z) = 1-\epsilon > 0$, and $var(Z) = \sigma$. Again let T' be the least $t > 0$ such that $Y'_t \le 0$.

Lemma 1.2 *For $1 \le k \le n$,*

$$\Pr[T' > n, Y'_n = k] = \frac{k}{n}\Pr[S_n = n-k].$$

Proof of Lemma 1.2. For any walk $\{Y'_0, Y'_1, Y'_2, \ldots, Y'_n\}$ satisfying $Y'_0 = 0$, $Y'_i > 0$ for $i = 1, 2, \ldots, n-1$ and $Y'_n = k$, define the *reversed* walk $\mathbf{X} = \{X_0, X_1, \ldots, X_n\}$ as follows: $X_0 = k$, $X_i = Y'_{n-i}$. Then $X_i = Y'_{n-i} = Y'_{n-i-1} + 1 - Z_{n-i} = X_{i+1} + 1 - Z_{n-i}$. That is, $\mathbf{X}$ satisfies the recurrence $X_0 = k$, $X_{i+1} = X_i + Z_i - 1$ where Z_i are independent random variables with the identical distribution Z. In other words, $\{X_0, X_1, \ldots, X_n\}$ is a random walk corresponding to the Galton-Watson branching process with the offspring distribution Z and k individuals at the zero generation. The conditions $Y'_0 = 0$, $Y'_i > 0$ for $i = 1, 2, \ldots, n-1$ imply that $X_i > 0$ for $i = 1, 2, \ldots, n-1$, and $X_n = 0$. So this corresponds to the case that the total progeny in the branching process is exactly n, which happens with probability $\Pr[\#F_k = n]$. Now the lemma follows immediately from Otter-Dwass theorem.

For the random walk $\mathbf{Y}'$, $T' > n$ means $Y'_1 = 1$, (i.e., $Z_1 = 0$). Hence for the random walk $\mathbf{Y}$, $p_0 \Pr[T > n, Y_n = k] = \Pr[T' > n+1, Y'_{n+1} = k]$ where $p_0 = \Pr[Z = 0]$. Therefore

$$\Pr[T > n, Y_n = k] = \frac{1}{p_0} \frac{k}{n+1} \Pr[S_{n+1} = n+1-k]. \tag{5}$$

and

$$\Pr[T > n] = \frac{1}{p_0} \frac{1}{n+1} \sum_{k=1}^{n+1} k \Pr[S_{n+1} = n+1-k]. \tag{6}$$

We evaluate the sum in Eq. (6).

$$\begin{aligned}
& \frac{1}{n+1} \sum_{k=1}^{n+1} k \Pr[S_{n+1} = n+1-k] \\
= \quad & \frac{1}{n+1} \sum_{k=0}^{n} (n+1-k) \Pr[S_{n+1} = k] \\
= \quad & \frac{1}{n+1} (n+1) \Pr[S_{n+1} \le n] - \frac{1}{n+1} \sum_{k=0}^{n} k \Pr[S_{n+1} = k] \\
= \quad & \Pr[S_{n+1} \le n] - \frac{1}{n+1} (E[S_{n+1}] - \sum_{k>n} k \Pr[S_{n+1} = k]) \\
= \quad & 1 - \Pr[S_{n+1} \ge n+1] - (1-\epsilon) + \frac{1}{n+1} \sum_{k>n} k \Pr[S_{n+1} = k] \\
= \quad & \epsilon - \Pr[S_{n+1} \ge n+1] + \frac{1}{n+1} \sum_{k>n} k \Pr[S_{n+1} = k]
\end{aligned}$$

Note that $E[S_{n+1}] = (n+1)(1-\epsilon)$, and $var(S_{n+1}) = (n+1)\sigma$. By Chebyshev's inequality,

$$\Pr[S_{n+1} \ge n+1] = \Pr[S_{n+1} - (n+1)(1-\epsilon) \ge (n+1)\epsilon] \le \frac{\sigma}{(n+1)\epsilon^2} = o(1),$$

and

$$\begin{aligned}
\frac{1}{n+1} \sum_{k>n} k \Pr[S_{n+1} = k] \quad &= \quad \Pr[S_{n+1} \ge n+1] + \frac{1}{n+1} \sum_{k>n+1} \Pr[S_{n+1} \ge k] \\
&\le \quad \frac{\sigma}{(n+1)\epsilon^2} + \frac{1}{n+1} \sum_{t=1}^{\infty} \frac{(n+1)\sigma}{((n+1)\epsilon + t)^2} \\
&\sim \quad \frac{\sigma}{(n+1)\epsilon^2} + \sigma \int_1^{\infty} \frac{1}{((n+1)\epsilon + t)^2} dt \\
&= \quad O(n^{-1}) = o(1)
\end{aligned}$$

Hence

$$\Pr(T = \infty) = \lim_{n\to\infty} \Pr(T > n) = \epsilon/p_0.$$

For the case that Z is the Poisson distribution with mean $1-\epsilon$, $p_0 = e^{-(1-\epsilon)}$, and hence $\Pr[T = \infty] = e^{1-\epsilon}\epsilon$. □

Let $p_i = \Pr[Z = i]$. In the random walk $\mathbf{Y}$, for $1 \le k \le n$,

$$\Pr[T > n, Y_n = k] = \sum_{(z_1,z_2,\ldots,z_n)} \prod_{i=1}^{n} p_{z_i}$$

where $(z_1, z_2, \ldots, z_n)$ are non-negative integers so that $Y_i > 0$ for $i = 1, 2, \ldots, n-1$, and $Y_n = k$. In terms of z_i, these conditions are equivalent to

$$\begin{aligned} z_1 &\le 1, \\ z_1 + z_2 &\le 2, \\ \cdots &\le \cdots \\ z_1 + z_2 + \cdots + z_{n-1} &\le n-1 \\ z_1 + z_2 + \ldots + z_n &= n+1-k. \end{aligned} \tag{7}$$

When Z has the Poisson distribution with mean μ,

$$\begin{aligned} \Pr[T > n, Y_n = k] &= \sum_{(z_1,z_2,\ldots,z_n)} \prod_{i=1}^{N} e^{-\mu}\frac{\mu^{z_i}}{z_i!} = \sum_{(z_1,z_2,\ldots,z_n)} e^{-\mu n} \prod_{i=1}^{N} \frac{\mu^{z_i}}{z_i!} \\ &= e^{-\mu n}\frac{\mu^{n+1-k}}{(n+1-k)!} \sum_{(z_1,z_2,\ldots,z_n)} \binom{n+1-k}{z_1, z_2, \ldots, z_n}. \end{aligned} \tag{8}$$

where $(z_1, z_2, \ldots, z_n)$ satisfies the condition (7). By Eq. (5),

$$\Pr[T > n, Y_n = k] = e^{\mu}\frac{k}{n+1}\Pr[S_{n+1} = n+1-k] = e^{\mu}\frac{k}{n+1}e^{-(n+1)\mu}\frac{((n+1)\mu)^{n+1-k}}{(n+1-k)!} \tag{9}$$

Combining (8) and (9), we get

$$\sum_{(z_1,z_2,\ldots,z_n)} \binom{n+1-k}{z_1, z_2, \ldots, z_n} = k(n+1)^{n-k}, \tag{10}$$

where $(z_1, z_2, \ldots, z_n)$ satisfies condition (7).

The sum in the left-hand side of Eq. (10) has the following combinatorial interpretation. Put balls $B_1, B_2, \ldots, B_{n+1-k}$ into n different boxes. The sum counts the number of ways one can do it so that the number of balls in the first i boxes does not exceed i, for $i = 1, 2, \ldots, n$. Hence

Corollary 1 *Let $1 \le k \le n$. The number of ways to put $n+1-k$ distinct balls into n distinct boxes, so that the number of balls in the first i boxes does not exceed i, for all $i = 1, 2, \ldots, n$, is $k(n+1)^{n-k}$.*

3 The asymptotic formula

In this section we consider the asymptotics of the escaping probability of $\mathbf{Y}$ when $\epsilon \to 0^+$. Using the Otter-Dwass Theorem and Lemma 1.2, we estimate the escaping probability after a finite number of steps. More precisely, let $Z^{(\epsilon)}$ be a family of non-negative integral random variables such that as $\epsilon \to 0^+$, $\Pr[Z^{(\epsilon)} = 0] \to p_0 \in (0,1)$, $E[Z^{(\epsilon)}] = 1 - c\epsilon + o(\epsilon)$ for some positive constant c. Further we assume a uniform bound M such that $var(Z^{(\epsilon)}) \leq M$ for all positive ϵ in some neighborhood of the origin. Under these conditions,

Theorem 2 *Let the random walk* $\mathbf{Y}^{(\epsilon)}$ *be defined by*

$$Y_0^{(\epsilon)} = 1, \quad Y_i^{(\epsilon)} = Y_{i-1}^{(\epsilon)} + 1 - Z_i^{(\epsilon)},$$

where $Z_i^{(\epsilon)}$ *are independent random variables with identical distribution* $Z^{(\epsilon)}$. *Set* $S_n^{(\epsilon)} = Z_1^{(\epsilon)} + Z_2^{(\epsilon)} + \cdots + Z_n^{(\epsilon)}$, *and* $T^{(\epsilon)}$ *the least* t *such that* $Y_t^{(\epsilon)} = 0$. *then* $\Pr[T^{(\epsilon)} > N] \sim c\epsilon/p_0$ *if* $N \gg \epsilon^{-2}$.

Proof. From the analytic proof of Theorem 1, we have

$$\Pr[Z^{(\epsilon)} = 0] \Pr[T^{(\epsilon)} > N] = 1 - E[Z^{(\epsilon)}] + \frac{1}{N+1} \sum_{k > N+1} \Pr[S_{N+1}^{(\epsilon)} \geq k]$$

Let $K = N\epsilon^2$. If $N \gg \epsilon^{-2}$, then $K \to \infty$ as $\epsilon \to 0^+$. By Chebyshev's inequality,

$$\begin{aligned} \frac{1}{N+1} \sum_{k>N+1} \Pr[S_{N+1}^{(\epsilon)} \geq k] &\leq \frac{1}{N+1} \sum_{t \geq 1} \frac{(N+1)var(Z^{(\epsilon)})}{((N+1)(1 - E[Z^{(\epsilon)}]) + t)^2} \\ &\leq \frac{M}{N(1 - E[Z^{(\epsilon)}])} = \frac{M\epsilon}{K(c + o(1))} = o(\epsilon). \end{aligned}$$

Hence

$$\frac{\Pr[T^{(\epsilon)} > N]}{\epsilon} = \frac{c + o(1)}{p_0 + o(1)} \to \frac{c}{p_0}.$$

□

Remark. A typical example is given by $Z^{(\epsilon)} = Po(1-\epsilon)$, in which case $\Pr[Z^{(\epsilon)} = 0] = e^{-(1-\epsilon)} \to e^{-1}$, $E[Z^{(\epsilon)}] = 1 - \epsilon$, and $var(Z^{(\epsilon)}) = 1 - \epsilon \leq 1$ for $\epsilon \in [0,1]$. Taking, for example, $N = \epsilon^{-2} \ln\ln \epsilon^{-1}$ we have $\Pr[T^{(\epsilon)} > N] \sim e\epsilon$.

As an extension to the Otter-Dwass theorem, we have the following formula for the random walk $\mathbf{Y}$.

Theorem 3 *In the random walk* $\mathbf{Y}$ *with any offspring distribution* Z,

$$\Pr[T = n, Y_n = 0] = \frac{1}{n} \Pr[S_n = n+1].$$

Proof. Note that

$$\Pr[T = n, Y_n = 0] = \sum_{(z_1, z_2, \ldots, z_n)} \prod_{i=1}^{n} p_{z_i},$$

where z_i are non-negative integers such that $Y_i \geq 1$ for $i = 1, 2, \ldots, n-1$, and $Y_n = 0$. Equivalently, z_i satisfies the condition

$$\begin{aligned} z_1 &\leq 1 \\ z_1 + z_2 &\leq 2, \\ \cdots &\leq \cdots \\ z_1 + z_2 + \cdots + z_{n-1} &\leq n-1, \\ z_1 + z_2 + \cdots + z_n &= n+1. \end{aligned} \tag{11}$$

Also note

$$\Pr[S_n = n+1] = \sum_{(z_1, z_2, \ldots, z_n)} \prod_{i=1}^{n} p_{z_i},$$

where z_i are non-negative integers with $z_1 + \cdots + z_n = n+1$. Hence it is sufficient to show that for any sequence of non-negative integers $\alpha = (a_1, a_2, \ldots, a_n)$ with $a_1 + a_2 + \cdots + a_n = n+1$, there is exactly one cyclic shift $\alpha_i = (a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1})$ satisfying Condition (11). The proof is contained in the following two claims.

Claim 1. For any sequence α, there is at most one cyclic shift of α satisfying Condition (11). Prove by contradiction. Assume both $\alpha = (a_1, a_2, \ldots, a_n)$ and $\alpha_i = (a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1})$ satisfy Condition (11), where i is an index between 2 and n. Then $a_1 + a_2 + \cdots + a_{i-1} \leq i-1$, and $a_i + \cdots + a_n \leq n-i$. Hence $a_1 + a_2 + \cdots + a_n \leq n$, contradicting the fact that $a_1 + a_2 + \cdots + a_n = n+1$. Note that Claim 1 also implies that all the conjugate of α are distinct.

Claim 2. For any sequence $\alpha = (a_1, a_2, \ldots, a_n)$ of non-negative integers, there exists a cyclic shift α_i satisfying Condition (11).

Let $x_i = i - (a_1 + a_2 + \cdots + a_i)$ for $i = 1, 2, \ldots, n-1$. If all $x_i \geq 0$, then α satisfies Condition (11). Otherwise, let $x = \min\{x_i, i = 1, 2, \ldots, n-1\}$. Then $x < 0$. Find the smallest index i such that $x_i = x$. We prove that the cyclic shift $\alpha_{i+1} = (a_{i+1}, \ldots, a_n, a_1, \ldots, a_i)$ satisfies (11). Denote α_{i+1} by $(b_1, b_2, \ldots, b_n)$.

1. For $1 \leq k \leq n-1-i$, $b_1 + b_2 + \cdots + b_k = a_{i+1} + a_{i+2} + \cdots + a_{k+i}$. Since

$$(k+i) - (a_1 + a_2 + \cdots + a_{k+i}) = x_{k+i} \geq x_i = i - (a_1 + a_2 + \cdots + a_i),$$

$$b_1 + b_2 + \cdots + b_k \leq k.$$

2. For $k = n - i$,

$$b_1 + b_2 + \cdots + b_k = a_{i+1} + a_{i+2} + \cdots + a_n = (n+1) - (a_1 + a_2 + \cdots + a_i) = n + 1 - (i - x) =\leq n - i.$$

as $x < 0$.

3. For $n-i<k<n$,

$$b_1+b_2+\cdots+b_k = a_{i+1}+\cdots+a_n+a_1+\cdots+a_{k-(n-i)} = n+1-(i-x)+a_1+\cdots+a_{k-(n-i)}.$$

Since

$$k-(n-i)-(a_1+\cdots+a_{k-(n-i)}) \geq x+1,$$
$$b_1+b_2+\cdots+b_k \leq n+1-i+x+k-(n-i)-(x+1) = k.$$

This finishes the proof. □

Remark. In the above proof we didn't use the condition that Z_i are non-negative. Hence Theorem 3 holds for any integral distribution Z. On the other hand, Otter and Dwass Theorem doesn't have such generality.

Using Theorem 3, we have the following result for $Z = Po(1-\epsilon)$.

Theorem 4 *In the random walk* $\mathbf{Y}$ *with* $Z = Po(1-\epsilon)$,

$$\Pr[T>N] \sim e\epsilon \qquad \textit{iff} \qquad N \gg \epsilon^{-2}.$$

Or equivalently,

$$\Pr[T=\infty | T>N] \sim 1 \qquad \textit{iff} \qquad N \gg \epsilon^{-2}.$$

Proof. The sufficient part is proved in Theorem 2. For the necessary part, we need to show

$$\text{If} \qquad \sum_{i=N}^{\infty} \Pr(T=i) = o(\epsilon), \qquad \text{then} \qquad N \gg \epsilon^{-2},$$

where the summation does not contain the term $\Pr[T=+\infty]$. We show this by finding a lower bounds for $\sum_{i=N}^{\infty}\Pr[T=i]$. When $Z=Po(1-\epsilon)$, S_N has distribution $Po(N(1-\epsilon))$.

$$\Pr[T=N] \geq \Pr[T=N, Y_N=0] = \frac{1}{N}\Pr[S_N=N+1] = \frac{e^{-tN}t^{N+1}N^N}{(N+1)!} \sim \frac{t}{\sqrt{2\pi}}\frac{(te^{\epsilon})^N}{(N+1)^{3/2}},$$

where $t=1-\epsilon$. Note that $te^{\epsilon} \sim 1-\epsilon^{-2}/2$. So if $N=C\epsilon^{-2}$ for a constant C, then

$$\begin{aligned}
\sum_{i=N}^{\infty}\Pr[T=i] &\geq \frac{t}{\sqrt{2\pi}}\sum_{i=N}^{\infty}\frac{(te^{\epsilon})^i}{(i+1)^{3/2}} \\
&\geq \frac{t}{\sqrt{2\pi}}\sum_{i=N}^{2N}\frac{(te^{\epsilon})^i}{(i+1)^{3/2}} \\
&\geq \frac{t}{\sqrt{2\pi}}(te^{\epsilon})^{2N}\sum_{i=N}^{2N}(i+1)^{-3/2} \\
&\sim \frac{2t}{\sqrt{2\pi C}}e^{-C}(1-\frac{1}{\sqrt{2}})\epsilon.
\end{aligned}$$

That is, if $N=C\epsilon^{-2}$, then

$$\sum_{i=N}^{\infty}\Pr[T=i] > C_1\cdot\epsilon,$$

for a constant C_1. This proves that for $\sum_{i=N}^{\infty}\Pr(T=i)=o(\epsilon)$, it is necessary that $N\gg\epsilon^{-2}$. □

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