A Major Index for Matchings and Set Partitions

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Abstract

We introduce a statistic $pmaj(P)$ for partitions of $[n]$, and show that it is equidistributed as cr_2, the number of 2-crossings over all partitions of $[n]$ with given sets of minimal block elements and maximal block elements. This generalizes the classical result of equidistribution for the permutation statistics inv and maj.

1 Introduction

One of the classical results on permutations is the equidistribution of the statistics inv and maj. For a permutation $\pi = (a_1 a_2 \cdots a_n)$, a pair (a_i, a_j) is called an inversion if $i < j$ and $a_i > a_j$. The statistic $inv(\pi)$ is defined as the number of inversions of π. The *descent set* $D(\pi)$ is defined as $\{i : a_i > a_{i+1}\}$, whose cardinality is $des(\pi)$. The sum of the elements of $D(\pi)$ is called the *major index* of π (also called the greater index) and denoted $maj(\pi)$. Similarly one can define the notions of inversion, descent set, and the major index for any word $w = w_1 w_2 \cdots w_n$ of not-necessarily distinct integers. It is a result of MacMahon [8] that inv and maj are equidistributed on any rearrangement class $\mathcal{R}(w)$. A statistic equidistributed with inv is called *Mahonian*.

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There are many research articles devoted to Mahonian statistics and their generalizations. For example, see [2, 6] for Mahonian statistics for words, [11, 12] for Mahonian statistics and Laguerre polynomials, [10] for a major statistic for set partitions, and very recently [7] for *inv* and *maj* for standard Young tableaux. Given a partition of $[n]$, there is a natural generalization of inversions, namely, 2-crossings, which can be viewed easily on a graphical representation of the partition. In this paper we introduce a new statistic, called the *p-major index* and denoted $pmaj(P)$, on the set of partitions of $[n]$. We prove that for any $S, T \subseteq [n]$ with $|S| = |T|$, *pmaj* and cr_2, the number of 2-crossings, are equally distributed on the set $P_n(S,T)$. Here $P_n(S,T)$ is the set of partitions of $[n]$ for which S is the set of minimal block elements, and T is the set of maximal block elements. Restricted to permutations, the pair $(cr_2, pmaj)$ coincides with (inv, maj). Hence our result gives another generalization of MacMahon's equidistribution theorem.

In the next section we list necessary notions and state the main results. An algebraic proof and some examples are given in Section 3. In Section 4 we present a bijective proof which generalizes Foata's second fundamental transformation [3, 4].

2 Definitions and the main results

A *partition* of $[n] = \{1, 2, \ldots, n\}$ is a collection P of disjoint nonempty subsets of $[n]$, whose union is $[n]$. Each subset in P is called a *block* of P. A *(perfect) matching* of $[n]$ is a partition of $[n]$ in which each block contains exactly two elements. Denote by Π_n the set of all partitions of $[n]$. Following [1], we represent each partition $P \in \Pi_n$ by a graph $\mathcal{G}_P$ on the vertex set $[n]$ whose edge set consists of arcs connecting the elements of each block in numerical order. Such a graph is called the *standard representation* of the partition P. For example, the standard representation of 1457-26-3 has the arc set $\{(1,4),(4,5),(5,7),(2,6)\}$. We always write an arc e as a pair (i,j) with $i < j$, and say that i is the *lefthand endpoint* of e and j is the *righthand endpoint* of e.

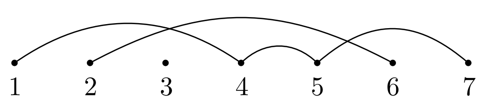

Figure 1: The standard representation of partition $P = 1457 - 26 - 3$.

A partition $P \in \Pi_n$ is a matching if and only if in $\mathcal{G}_P$, each vertex is the endpoint of exactly one arc. In other words, each vertex is either a lefthand endpoint, or a righthand endpoint. In particular, a permutation π of $[m]$ can be represented as a matching M_π of $[2m]$ with arcs connecting $m+1-\pi(i)$ and $i+m$ for $1 \le i \le m$. See Figure 2 for an example.

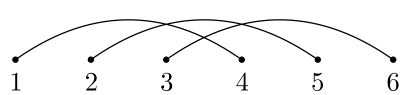

Figure 2: The permutation $\pi = 321$ and the matching M_π.

Two arcs $(i_1, j_1), (i_2, j_2)$ of $\mathcal{G}_P$ form a 2-crossing if $i_1 < i_2 < j_1 < j_2$. Let $cr_2(P)$ denote the number of 2-crossings of P. A 2-crossing is a natural generalization of the inversion of a permutation. It is easily seen that under the correspondence $\pi \to M_\pi$, $cr_2(M_\pi) = inv(\pi)$.

Given $P \in \Pi_n$, define

$$\min(P) = \{\text{minimal block elements of } P\},$$
$$\max(P) = \{\text{maximal block elements of } P\}.$$

For example, for $P = 1457\text{-}26\text{-}3$, $\min(P) = \{1,2,3\}$ and $\max(P) = \{3,6,7\}$.

Fix $S, T \subseteq [n]$ with $|S| = |T|$. Let $P_n(S,T)$ be the set $\{P \in \Pi_n : \min(P) = S, \max(P) = T\}$. For any set $X \subseteq [n]$, let $X_i = X \cap \{i+1, \ldots, n\}$.

Theorem 1. *Fix $S, T \subseteq [n]$ with $|S| = |T|$. Then*

$$\sum_{P \in P_n(S,T)} y^{cr_2(P)} = \prod_{i \notin T} (1 + y + \cdots + y^{h(i)-1}), \tag{1}$$

where $h(i) = |T_i| - |S_i|$.

For a permutation $\pi = (a_1 a_2 \cdots a_n)$, the major index $maj(\pi)$ can be computed as $\sum_{i=1}^n des(a_i \cdots a_n)$. This motivates the following definition of the p-major index for set partitions. Given $P \in \Pi_n$, we start with the standard representation $\mathcal{G}_P$. First label the arcs of P by $1, 2, \ldots, k$ from right to left in order of their lefthand endpoints. That is, if the arcs are $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ with $i_1 > i_2 > \cdots > i_k$, then (i_r, j_r) has label r, for $1 \leq r \leq k$. Next we associate a sequence $\sigma(r)$ to each righthand endpoint r. Assume that the righthand endpoints are $r_1 > r_2 > \cdots > r_k$. (The set $\{r_1, \ldots, r_k\}$ is exactly $[n] \setminus S$.) The sequence $\sigma(r_i)$ is defined backward recursively: let $\sigma(r_1) = a$ if r_1 is the righthand endpoint of the arc with label a. In general, after defining $\sigma(r_i)$, assume that the lefthand endpoints of the arcs labeled $a_1, \ldots, a_t$ are lying between r_{i+1} and r_i. Then $\sigma(r_{i+1})$ is obtained from $\sigma(r_i)$ by deleting entries $a_1, \ldots, a_t$ and adding b at the very beginning, where b is the label for the arc whose righthand endpoint is r_{i+1}. Finally, set the statistic $pmaj(P)$ as

$$pmaj(P) := \sum_{r_i} des(\sigma(r_i)).$$

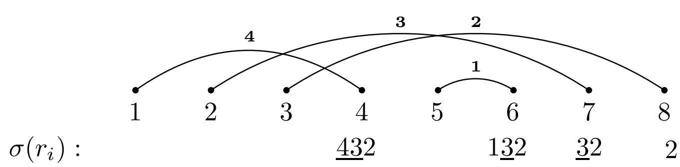

Figure 3: The major index for partition $14 - 27 - 38 - 56$ is 4.

Example 1. Let $P = 14-27-38-56$. Then $(r_1, r_2, r_3, r_4) = (8, 7, 6, 4)$. Figure 3 shows how to compute $pmaj(P)$. The sequences $\sigma(r_i)$ are $\sigma(8) = (2)$, $\sigma(7) = (32)$, $\sigma(6) = (132)$, and $\sigma(4) = (432)$. For each $\sigma(r_i)$, the elements in the descent set are underlined. The p-major index of P is $pmaj(P) = 1 + 1 + 2 = 4$.

Theorem 2. *Fix $S, T \subseteq [n]$ with $|S| = |T|$. Then*

$$\sum_{P \in P_n(S,T)} y^{pmaj(P)} = \prod_{i \notin T} (1 + y + \cdots + y^{h(i)-1}), \tag{2}$$

where $h(i) = |T_i| - |S_i|$.

Combining Theorems 1 and 2, we have

Corollary 3. *For each $P_n(S,T)$,*

$$\sum_{P \in P_n(S,T)} y^{pmaj(P)} = \sum_{P \in P_n(S,T)} y^{cr_2(P)}.$$

That is, the two statistics cr_2 and $pmaj$ have the same distribution over each set $P_n(S,T)$.

When $n = 2m$, $S = [m]$ and $T = [2m] \setminus [m]$, the map $\pi \to M_\pi$ gives a one-to-one correspondence between $P_n(S,T)$ and the set of permutations of $[m]$. It is easy to see that $pmaj(M_\pi) = maj(\pi)$. Hence the equidistribution of inv and maj for permutations is a special case of Corollary 3.

Another consequence of Theorems 1 and 2 is the symmetry of the number of partitions of $[n]$ with a given number of 2-crossings (resp. a given p-major index). Let $A(n, i; S, T)$ be the set of partitions in $P_n(S,T)$ such that $cr_2(P) = i$, whose cardinality is $a(n, i; S, T)$.

Corollary 4. *Fix n and let $K = \sum_{i \notin T} (h(i) - 1)$. Then the sequence $\{a(n, i; S, T)\}_{i=0}^{K}$ is symmetric. That is,*

$$a(n, i; S, T) = a(n, K - i; S, T).$$

The same result holds if we replace $cr_2(P)$ by $pmaj(P)$ in defining $A(n, i; S, T)$ and $a(n, i; S, T)$.

3 Proofs for the main results

In this section we give the proofs for Theorems 1 and 2. Given a partition $P \in \Pi_n$, a vertex $i \in [n]$ in the standard representation $\mathcal{G}_P$ is one of the following types:

1. a lefthand endpoint if $i \in \min(P) \setminus \max(P)$,

2. a righthand endpoint if $i \in \max(P) \setminus \min(P)$,

3. an isolated point if $i \in \min(P) \cap \max(P)$,

4. a lefthand endpoint and a righthand endpoint if $i \notin \min(P) \cup \max(P)$.

In particular, $[n] \setminus \max(P)$ is the set of points which are the lefthand endpoints of some arcs, and $[n] \setminus \min(P)$ is the set of righthand endpoints. Fixing $\min(P) = S$ and $\max(P) = T$ is equivalent to fixing the type of each vertex in $[n]$. Since the standard representation uniquely determines the partition, we can identify a partition $P \in \Pi_n$ with the set of arcs of $\mathcal{G}_P$. Hence the set $P_n(S,T)$ is in one-to-one correspondence with the set of matchings between $[n] \setminus T$ and $[n] \setminus S$ such that $i < j$ whenever $i \in [n] \setminus T$, $j \in [n] \setminus S$ and i is matched to j. In the following such a matching is referred as a *good* matching. Denote by $M_n(S,T)$ the set of good matchings from $[n] \setminus T$ to $[n] \setminus S$.

Proof of Theorem 1.

Assume $[n] \setminus T = \{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Let $S(H)$ be the set of sequences $\{(a_1, a_2, \dots, a_k)\}$ where $1 \le a_r \le h(i_r)$ for each $1 \le r \le k$. We give a bijection between the sets $M_n(S,T)$ and $S(H)$. The construction is essentially due to M. de Sainte-Catherine [9].

Given a sequence $\alpha = (a_1, a_2, \dots, a_k)$ in $S(H)$, we construct a matching from $[n] \setminus T = \{i_1, i_2, \dots, i_k\}$ to $[n] \setminus S$ as follows. First, there are exactly $h(i_k)$ many elements in $[n] \setminus S$ which are greater than i_k. List them in increasing order by $1, 2, \dots, h(i_k)$. Match i_k to the a_k-th element, and mark this element as *dead*.

In general, after matching elements $i_{j+1}, \dots, i_k$ to some elements in $[n] \setminus S$, we process the element i_j. At this stage there are exactly $h(i_j)$ many elements in $[n] \setminus S$ which are greater than i_j and not dead. List them in increasing order by $1, 2, \dots, h(i_j)$. Match i_j to the a_j-th of them, and mark it as dead. Continuing the process until $j = 1$, we get a good matching $M(\alpha) \in M_n(S,T)$. The map $\alpha \to M(\alpha)$ gives the desired bijection between $S(H)$ and $M_n(S,T)$.

Let $P(\alpha)$ be the partition of $[n]$ for which the arc set of $\mathcal{G}_P$ is $M(\alpha)$. By the above construction, the number of 2-crossings formed by arcs (i_j, b) and (a, c) with $a < i_j < c < b$ is exactly $a_j - 1$. Hence $cr_2(P(\alpha)) = \sum_{j=1}^{k} (a_j - 1)$ and

$$\sum_{P \in P_n(S,T)} y^{cr_2(P)} = \sum_{(a_1, \dots, a_k) \in S(H)} y^{\sum_{j=1}^{k}(a_j - 1)} = \prod_{i \notin T} (1 + y + \cdots + y^{h(i)-1}). \tag{3}$$

∎

Example 2. Let $n = 6$, $S = \{1, 2\}$, and $T = \{5, 6\}$. Then $[n] \setminus T = \{1, 2, 3, 4\}$, $[n] \setminus S = \{3, 4, 5, 6\}$, and $h(1) = 1$, $h(2) = h(3) = h(4) = 2$. Figure 4 shows the correspondence between $S(H)$ and $P_n(S,T)$. For simplicity we omit the vertex labeling.

To prove Theorem 2, we need a lemma on permutations.

Lemma 5. *Let $\sigma = a_1 a_2 \cdots a_{n-1}$ be a permutation of $\{2, 3, \dots, n\}$. Let $\sigma_0 = 1 a_1 \cdots a_{n-1}$ and σ_i be obtained from σ by inserting 1 right after a_i. Then of the n permutations $\sigma_0, \dots, \sigma_{n-1}$, the major indices are all distinct and run from $maj(\sigma)$ to $maj(\sigma) + n - 1$ in some order.*

Proof. First note that $maj(\sigma_0) = maj(\sigma) + des(\sigma)$. Assume that there are t_i descents of σ that are greater than i. Then

$$maj(\sigma_i) = \begin{cases} maj(\sigma) + t_i & \text{if } a_i > a_{i+1}, \\ maj(\sigma) + i + t_i & \text{if } a_i < a_{i+1} \text{ or } i = n - 1. \end{cases}$$

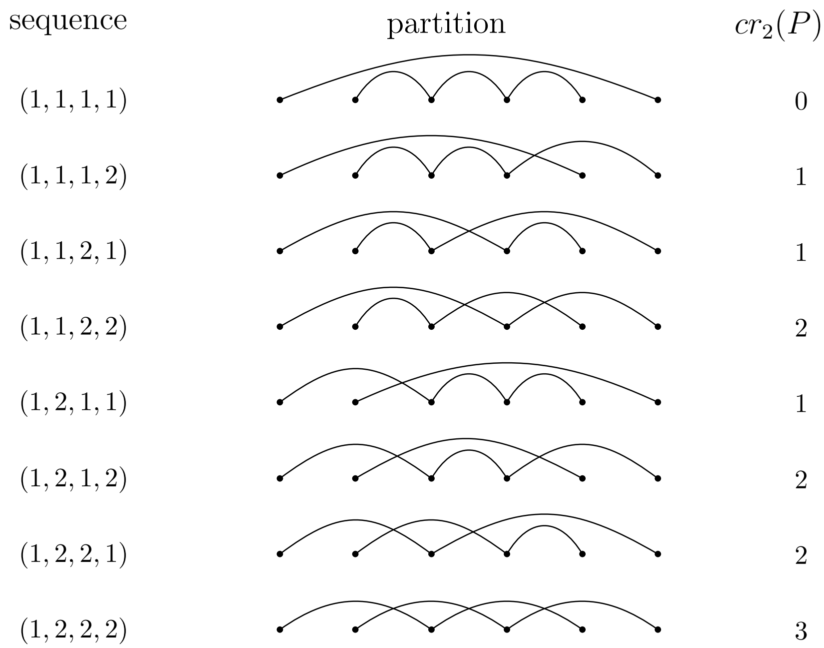

Figure 4: The $cr_2(P)$ for $P \in P_6(\{1,2\},\{5,6\})$.

It can be checked that the major indices of $\sigma_0, \ldots, \sigma_{n-1}$ are all distinct and run from $maj(\sigma)$ to $maj(\sigma)+n-1$ in some order. ∎

A similar version of the lemma, where one inserts n instead of 1, is used in [5] to get the generating function of the major index over all permutations of $[n]$.

$$\sum_{\pi \in S_n} y^{maj(\pi)} = \frac{(1-y)(1-y^2)\cdots(1-y^n)}{(1-y)^n}. \tag{4}$$

This formula is the special case of Theorem 2 with $S=[n]$ and $T=[2n]\setminus[n]$.

Proof of Theorem 2. Consider the contribution of the arc with label 1 to the generating function $\sum_{P\in P_n(S,T)} y^{pmaj(P)}$. Again we identify the set $P_n(S,T)$ with the set $M_n(S,T)$ of good matchings from $[n]\setminus T$ to $[n]\setminus S$.

Let $i_k = \max([n]\setminus T)$, which is the lefthand endpoint of the arc labeled by 1 in the definition of $pmaj(P)$, for any $P \in P_n(S,T)$. Assume $T_{i_k}^{(r)} = \{j_1, j_2, \ldots, j_{h(i_k)}\}$. Let $A = [n]\setminus(T\cup\{i_k\})$, and $B=[n]\setminus(S\cup\{j_{h(i_k)}\})$. For any good matching M between A and B let M_t $(1\le t\le h(i_k))$ be the matching obtained from M by joining the pair (i_k, j_t), and replacing each pair $(a, j_r), r>t$ with (a, j_{r+1}). Consequently, the arc labeling of M_t can be obtained from that of M by labeling the arc (i_k, j_t) by 1, and adding 1 to the label of each arc of M. Assume $\sigma(j_1) = b_1 b_2 \ldots b_{h(i_k)-1}$ for M. Then by the definition of $pmaj$, we have

$$pmaj(M_t) = pmaj(M) + maj(b'_1 \cdots b'_{t-1} 1 b'_t \cdots b'_{h(i_k)-1}) - maj(b_1 \cdots b_{h(i_k)-1}),$$

where $b'_i = b_i + 1$. By Lemma 5, the values of $maj(b'_1 \cdots b'_{t-1} 1 b'_t \cdots b'_{h(i_k)-1}) - maj(b_1 \cdots b_{h(i_k)-1})$ are all distinct and run over the set $\{0,1,\ldots,h(i_k)-1\}$. Hence

$$\sum_{P\in P_n(S,T)} y^{pmaj(P)} = (1+y+\cdots+y^{h(i_k)-1}) \sum_{P\in P_{n-1}(A,B)} y^{pmaj(P)}.$$

Equation (2) follows by induction. ∎

Example 3. The p-major indices for the partitions in Example 2 are given in Figure 5. For simplicity we omit the vertex labeling, but put the sequence $\sigma(r)$ under each righthand endpoint r.

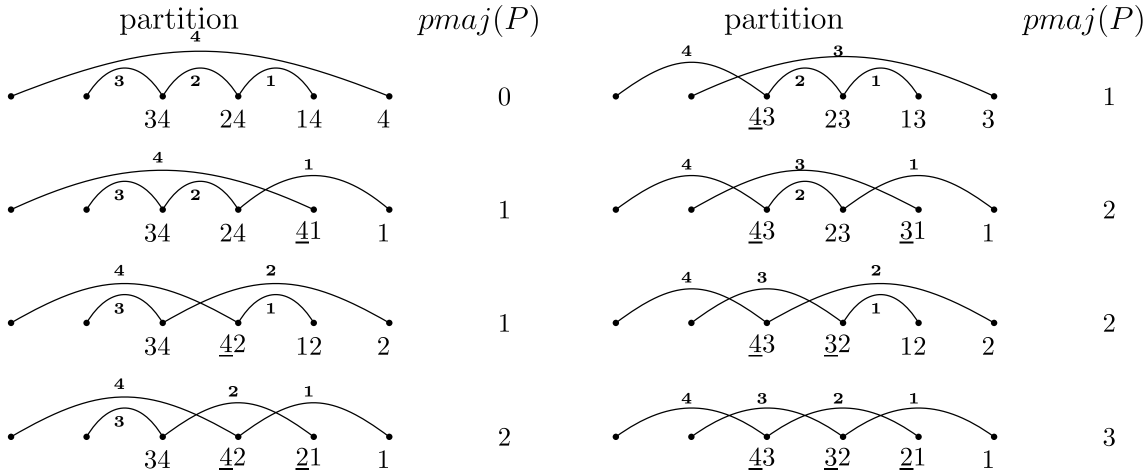

Figure 5: The $pmaj(P)$ for $P \in P_6(\{1,2\},\{5,6\})$.

Remark 1. The joint distribution of cr_2 and $pmaj$ is in general not symmetric over $P_n(S,T)$. For example, let $n = 8$, $S = \{1,2,3,5\}$ and $T = \{4,6,7,8\}$. Then

$$\sum_{P \in P_8(S,T)} x^{cr_2(P)} y^{pmaj(P)} = x^5y^5 + x^4y^4 + 2x^3y^4 + 2x^4y^3$$
$$+ x^3y^3 + 3x^2y^2 + 2xy + 1 + 2x^3y^2 + x^2y^3 + x^2y + xy^3.$$

Remark 2. We explain the combinatorial meaning of the quantities $\{h(i) = |T_i| - |S_i| : i \notin T\}$. The paper [1] gives a characterization of nonempty $P_n(S,T)$'s. Given a pair (S,T) where $S, T \subseteq [n]$ and $|S| = |T|$, associate to it a lattice path $L(S,T)$ with steps $(1,1)$, $(1,-1)$ and $(1,0)$: start from $(0,0)$, read the integers i from 1 to n one by one, and move two steps for each i.

1. If $i \in S \cap T$, move $(1,0)$ twice.
2. If $i \in S \setminus T$, move $(1,0)$ and then $(1,1)$.
3. If $i \in T \setminus S$, move $(1,-1)$ and then $(1,0)$.
4. If $i \notin S \cup T$, move $(1,-1)$ and then $(1,1)$.

This defines a lattice path $L(S,T)$ from $(0,0)$ to $(2n,0)$. Conversely, the path uniquely determines (S,T). Then $P_n(S,T)$ is nonempty if and only if the lattice path $L(S,T)$ is a Motzkin path, i.e., never goes below the x-axis.

For each element $i \in [n] \setminus T$, there is a unique upper step $(1,1)$ in the lattice path $L(S,T)$. We say an upper step is *of height* y if it goes from $(x-1, y-1)$ to (x,y). Then the multiset $\{h(i) = |T_i| - |S_i| : i \notin T\}$ is exactly the same as the multiset $\{$height of U: U is an upper step in $L(S,T)\}$.

4 The generalized Foata bijection

In this section we construct a bijection ϕ from $P_n(S,T)$ to itself such that $pmaj(P) = cr_2(\phi(P))$ for any set partition P. This provides a generalization for Foata's second fundamental transformation [3, 4] which is used to prove the equidistribution of the permutation statistics *inv* and *maj*.

Given a partition P, for each endpoint $i \notin S \cup T$, we may replace i with two neighboring endpoints, i.e., a righthand endpoint i^0 on the left and a lefthand endpoint i^1 on the right, such that the arc ending at i is incident to i^0 and the arc starting from i is incident to i^1. After dealing with each endpoint not in $S \cup T$ and removing all isolated points from P, we obtain a matching $M(P)$. Clearly $pmaj(P) = pmaj(M(P))$ and $cr_2(P) = cr_2(M(P))$. See Figure 6 for an example. Then it is sufficient to describe the bijection ϕ on matchings, as for any set partition P, one can obtain $\phi(P)$ from $\phi(M(P))$ by identifying i^0 and i^1 as one endpoint and adding back all isolated vertices.

P:

$M(P)$:

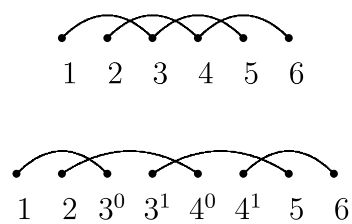

Figure 6: A set partition P and the corresponding matching $M(P)$.

We define some notations. Let M be a matching of $[2m]$. Suppose the arc incident to $2m$ is $(i, 2m)$. We say an arc (a,b) of M is *large* if $a < i$, and *small* if $a > i$. It can also be described in terms of the edge labeling. Recall that if $(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$ are the arcs of M where $i_1 > i_2 > \cdots > i_m$, then the arc (i_k, j_k) has label k. Suppose the arc $(i, 2m)$ has label b. Then an arc is large if its label is larger than b, and small if its label is smaller than b.

Given a matching M on $[2m]$ with the arc $(i, 2m)$, let $N = M \setminus (i, 2m)$. Let $\mathcal{R}$ be the set of righthand endpoints of N lying between i and $2m$. We divide the set $\mathcal{R}$ into three disjoint subsets. Define the *critical large arc* e_L to be the large arc with the biggest righthand endpoint in $\mathcal{R}$, and the *critical small arc* e_S to be the small arc with the smallest lefthand endpoint which crosses e_L. Assume $e_S = (e_S^l, e_S^r)$. Then we set

$$\begin{aligned}
\mathcal{R}_0(N) &= \{j \in \mathcal{R} : e_S^r < j < 2m\}, \\
\mathcal{R}_1(N) &= \{j \in \mathcal{R} : e_S^l < j \leq e_S^r\}, \\
\mathcal{R}_2(N) &= \{j \in \mathcal{R} : i < j < e_S^l\}.
\end{aligned}$$

If there exists no critical large arc e_L, then $\mathcal{R}_1(N)$ and $\mathcal{R}_2(N)$ are empty, and all endpoints belong to $\mathcal{R}_0(N)$. If $e_L = (e_L^l, e_L^r)$ exists but no small arc crosses it, then $\mathcal{R}_2(N)$ is composed of all righthand endpoints j with $i < j \le e_L^r$, $\mathcal{R}_1(N)$ is empty, and $\mathcal{R}_0(N)$ is composed of the remaining endpoints in $\mathcal{R}$.

Example 4. Let M be the matching with arcs $(1,14)$, $(2,7)$, $(3,16)$, $(4,6)$, $(5,9)$, $(8,12)$, $(10,15)$ and $(11,13)$. Let $N = M \setminus (3,16)$. In N the arcs $(1,14)$ and $(2,7)$ are large arcs. $(4,6), (5,9), (8,12), (10,15)$ and $(11,13)$ are small arcs. The critical large arc is $e_L = (1,14)$, and the critical small arc is $e_S = (10,15)$. The set $\mathcal{R}$ consists of vertices $\{6,7,9,12,13,14,15\}$, where $\mathcal{R}_0 = \emptyset$, $\mathcal{R}_1 = \{12,13,14,15\}$, and $\mathcal{R}_2 = \{6,7,9\}$. See Figure 7.

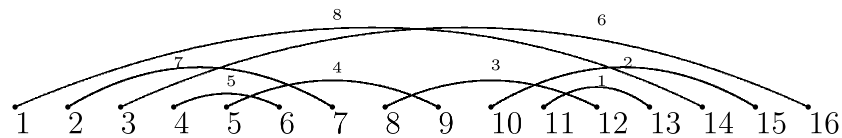

Figure 7: The construction of $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$.

In the following we describe the map ϕ on the set of matchings, that is, on $P_n(S,T)$ where $n = 2m$, $|S| = |T| = m$ and $S \cap T = \emptyset$. The map ϕ preserves the arc incident to $2m$, i.e., if $(i, 2m)$ is an arc of M, then it is also an arc of $\phi(M)$. We extend the map ϕ to matchings whose vertices are $a_1 < a_2 < \cdots < a_{2m}$ by identifying the vertex a_i with i.

The map ϕ is defined by induction on m. When $m = 1$, let ϕ be the identity map. Given a matching M of $2m$ for $m > 1$, let M_1 be the matching obtained from M by removing the arc $(i, 2m)$, and $M_1' = \phi(M_1)$. We construct a matching M_2 from M_1' by applying a series of operations on the arcs of M_1' whose righthand endpoints are in the set $\mathcal{R}$.

1. On $\mathcal{R}_0(M_1')$.

We fix all the arcs whose righthand endpoints are in $\mathcal{R}_0(M_1')$.

2. Algorithm I on $\mathcal{R}_1(M_1')$.

Let ptr_1, ptr_2 be two pointers. We apply the following algorithm on $\mathcal{R}_1(M_1')$ if $\mathcal{R}_1(M_1')$ is nonempty.

(A) Let ptr_1 point to e_S^r, and ptr_2 point to the next vertex in $\mathcal{R}_1(M_1')$ on the left of e_S^r.

(B) If ptr_2 is null, then go to (D). Otherwise, assume $ptr_1 = j_1$ and $ptr_2 = j_2$, where j_1, j_2 are righthand endpoints of the arcs (i_1, j_1) and (i_2, j_2).

- **(B1)** If (i_2, j_2) is a large arc, then change the two arcs $\{(i_1,j_1),(i_2,j_2)\}$ to $\{(i_1,j_2),(i_2,j_1)\}$. Move ptr_1 to j_2.
- **(B2)** If (i_2, j_2) is a small arc (i_2, j_2), then there are three cases to consider.
  - **(B2.1)** If there exists no righthand endpoint between i_1 and i_2, then move ptr_1 to j_2.
  - **(B2.2)** If $i_1 < i_2$ and there are some righthand endpoints between i_1 and i_2, do nothing.
  - **(B2.3)** If $i_2 < i_1$ and there are some righthand endpoints between i_1 and i_2, then find the smallest j_3 such that $j_3 > j_1$ and (i_3, j_3) is a large arc. Change the three arcs $\{(i_1,j_1),(i_2,j_2),(i_3,j_3)\}$ to $\{(i_1,j_2),(i_2,j_3),(i_3,j_1)\}$, and move ptr_1 to j_2. See Figure 8 for an illustration, where j is a righthand endpoint.

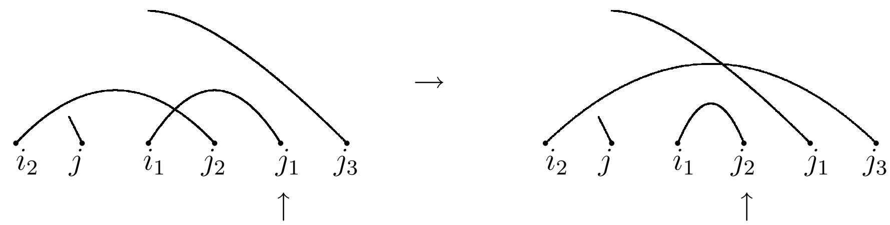

Figure 8: Step (B2.3), where the upper arrow represents the pointer ptr_1.

(C) Move ptr_2 to the next vertex in $\mathcal{R}_1(M_1')$ on the left of j_2. Go to (B).

(D) Stop.

3. Algorithm II on $\mathcal{R}_2(M_1')$.
We continue by processing vertices in $\mathcal{R}_2(M_1')$. Let the rightmost vertex of $\mathcal{R}_2(M_1')$ be t_0, which is the righthand endpoint of the arc (i_0, t_0).

- **(Pre-process)**: If (i_0, t_0) is a small arc, then find the large arc (i_1, t_1) on the right of (i_0, t_0) with smallest t_1. By the definition of $\mathcal{R}_1$ and $\mathcal{R}_2$, such a large arc exists. We change the two arcs $\{(i_0, t_0), (i_1, t_1)\}$ to $\{(i_0, t_1), (i_1, t_0)\}$. Now the vertex t_0 is connected to a large arc.

Remark 3. We call this step *borrowing*. Note that now the arc (i_0, t_1) is a small arc which crosses (i_1, t_0), but not cross any other large arc in $\mathcal{R}_1$, and t_1 is the smallest vertex in $\mathcal{R}_1$ with this property.

Let ptr_1, ptr_2 be two pointers. We apply the following operations on $\mathcal{R}_2(M_1')$.

(A′) Let ptr_1 point to t_0, and ptr_2 point to the next vertex in $\mathcal{R}$ on the left of t_0.

(B′) If ptr_2 is null, then go to (D′). Otherwise, assume $ptr_1 = j_1$ and $ptr_2 = j_2$, where j_1, j_2 are righthand endpoints of the arcs (i_1, j_1) and (i_2, j_2).

(B′1) If (i_2, j_2) is a large arc, then do nothing.

(B′2) If (i_2, j_2) is a small arc, then change the two arcs to (i_1, j_2) and (i_2, j_1).

(C′) Move ptr_1 to j_2, and move ptr_2 to the next vertex in $\mathcal{R}$ on the left of j_2. Go to (B′).

(D′) Stop.

Let M_2 be the matching obtained by applying Algorithms I and II on M_1'. Then $\phi(M) = M' = M_2 \cup \{(i, 2m)\}$. We give an example to illustrate the bijection in Figure 9.

To see that $\phi(M)$ is well-defined, we only need to check that Step (B2.3) is valid. Note that the algorithms do not change the relative positions of the large arcs. In Algorithm I, if $ptr_1 = j_1$, then (1) j_1 must be the righthand endpoint of a small arc (i_1, j_1), (2) i_1 is smaller than any vertex in $\mathcal{R}_1(M_1')$, and (3) any righthand endpoint between ptr_1 and ptr_2 must be

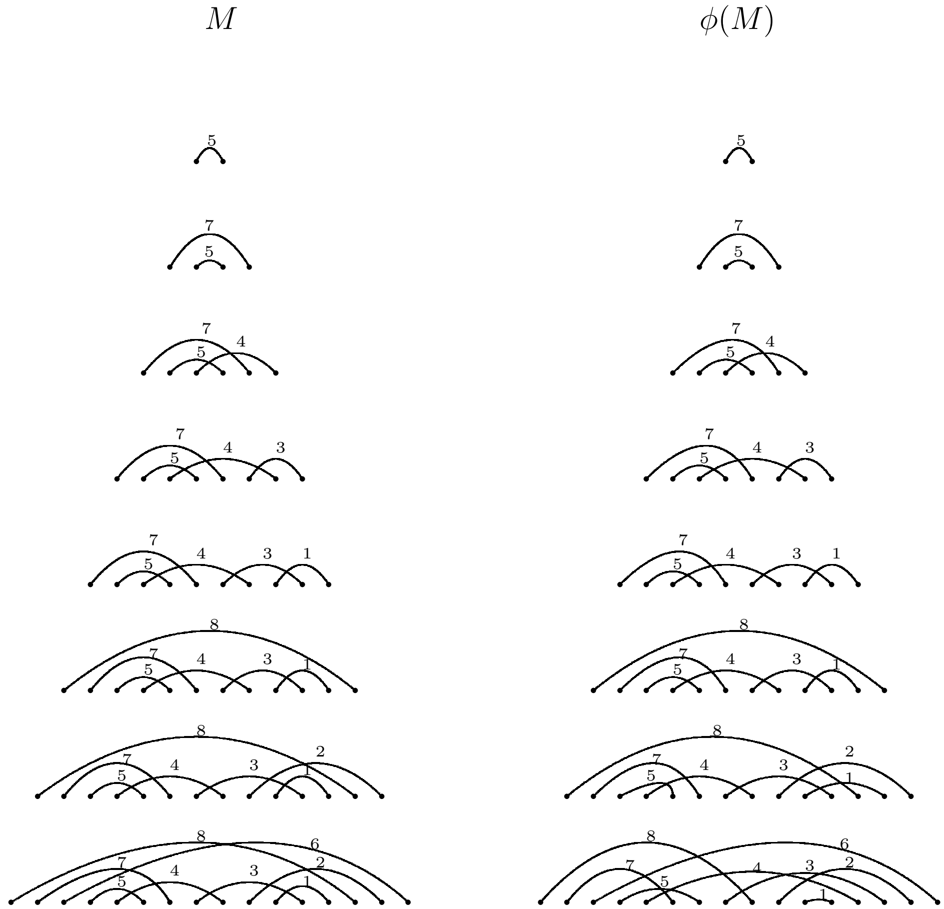

Figure 9: An example of the bijection ϕ

connected to a small arc. Thus if $ptr_1 = j_1$, $ptr_2 = j_2$ with arcs (i_1, j_1) and (i_2, j_2), where $i_2 < i_1$ and there are some righthand endpoints between i_1 and i_2, then the critical large arc must have been moved to the right of the pointers. Hence the large arc described in Step (B2.3) exists. This shows that the map ϕ is a well-defined map. It is also clear that ϕ preserves the set of lefthand endpoints, as well as the set of righthand endpoints.

From the above construction, we notice the following properties of Algorithms I, II, and the matching M_2.

Properties:

1. For each vertex $j \in \mathcal{R}_2(M_1')$, if j is the righthand endpoint of a small arc (i, j) in M_2, then there is a large arc in M_2 that crosses (i, j).

2. If $\mathcal{R}_2(M_1')$ is nonempty, then the smallest vertex in $\mathcal{R}_2(M_1')$ must connect to a large arc in M_2.

3. In Algorithm I, the pointer ptr_1 always points to the righthand endpoint of a small arc. In Algorithm II, ptr_1 always points to the righthand endpoint of a large arc.

4. In Algorithm I, if the pointer ptr_1 is connected to the small arc (i_1, j_1), then i_1 is smaller than any vertex in $\mathcal{R}_1(M'_1)$, and larger than any vertex in $\mathcal{R}_2(M'_1)$.

5. Assume that when Algorithm I stops, the pointer ptr_1 is at vertex j_1, which is the righthand endpoint of a small arc (i_1, j_1). Then j_1 must be the smallest vertex in $\mathcal{R}_1(M'_1)$. Algorithm II does not change the arc (i_1, j_1), and there is no arc (i_2, j_2) in M_2 such that $i_2 < i_1 < j_2 < j_1$.

Theorem 6. *The map ϕ, when restricted to $P_n(S, T)$, is a bijection.*

Proof. It is sufficient to describe how to invert Algorithms I and II in the definition of ϕ, on the set of matchings. Given a matching of $[2m]$ with the last arc $(i, 2m)$, let $N = M \setminus \{(i, 2m)\}$ and let $\mathcal{R}$ be the set of righthand endpoints of N lying between i and $2m$. First we need to decide the set $\mathcal{R}_2$. If there exists no large arc whose righthand endpoint lies between i and $2m$, then $\mathcal{R}_2 = \mathcal{R}_1 = \emptyset$, and $\mathcal{R}_0 = \mathcal{R}$. Suppose there are some large arcs in $\mathcal{R}$. Find the small arc $f = (f^l, f^r)$ with the smallest righthand endpoint f^r in $\mathcal{R}$ such that there is no arc (i_2, j_2) with $i_2 < f^l$ which crosses f. By Properties 1 and 5 above, $\mathcal{R}_2 = \{j \in \mathcal{R} : j < f^l\}$. If no such small arc exists, then $\mathcal{R}_2 = \mathcal{R}$.

Algorithm III: Inverse of Algorithm II.

To invert Algorithm II in $\mathcal{R}_2$, we apply the same steps as in Algorithm II except that

(1) Initially, let ptr_1 point to the smallest vertex in $\mathcal{R}_2$, and ptr_2 point to the next vertex in $\mathcal{R}_2$ on the right of ptr_1.

(2) We apply the Steps (B′)–(D′) on the vertices in $\mathcal{R}_2$ from left to right, i.e., in Step (C′) we need to move ptr_2 to the next vertex in $\mathcal{R}$ on the right of j_2.

Note that the pointer ptr_1 always points to the righthand endpoint of a large arc. When ptr_1 reaches the rightmost vertex j_1 of $\mathcal{R}_2$, with a current large arc (i_1, j_1), we need to determine whether this arc (i_1, j_1) is borrowed from $\mathcal{R}_1$. By Remark 3, it can be done as following.

- **(Invert borrowing)** Let (i_2, j_2) be the small arc with minimal j_2 such that $i_1 < i_2 < j_1 < j_2$.
  (a) If no such small arc exists, then there is no borrowing.
  (b) If there exists a righthand endpoint of a large arc between j_1 and j_2, then there is no borrowing.
  (c) If there exists no righthand endpoint of any large arc between j_1 and j_2, then the arc (i_1, j_1) is borrowed from $\mathcal{R}_1$. To invert, change the two arcs $\{(i_1, j_1), (i_2, j_2)\}$ to $\{(i_1, j_2), (i_2, j_1)\}$.

At this stage, if there is a small arc (i', j') with $i' < j_1 < j'$, then the arc (i', j') must cross some large arc whose righthand endpoint is in $\mathcal{R} \setminus \mathcal{R}_2$.

We continue by inverting Algorithm I in $\mathcal{R} \setminus \mathcal{R}_2$, when $\mathcal{R}_2 \neq \mathcal{R}$. In the following j always represents a righthand endpoint in $\mathcal{R} \setminus \mathcal{R}_2$.

Algorithm IV: Inverse of Algorithm I.

(IA) Let ptr_1 point to f^r, the smallest vertex in $\mathcal{R} \setminus \mathcal{R}_2$. Let ptr_2 point to the next righthand endpoint on the right of f^r.

(IB) If ptr_2 is null, then go to (ID). Otherwise, assume $ptr_1 = j_1$, $ptr_2 = j_2$, where j_1, j_2 are the righthand endpoints of arcs (i_1, j_1) and (i_2, j_2).

(IB1) If (i_2, j_2) is a large arc, we need to consider two cases.

(IB1.1) There exists a small arc (i_3, j_3) such that (i) no large arc lies between j_2 and j_3, (ii) $i_3 < i_1$ and $j_2 < j_3$, and (iii) there are some righthand endpoints between i_1 and i_3. In this case, we choose such a small arc (i_3, j_3) with minimal j_3, and then change the three arcs $\{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$ to $\{(i_1, j_2), (i_2, j_3), (i_3, j_1)\}$. Move ptr_1 to j_2.

(IB1.2) There exists no small arc (i_3, j_3) satisfying the conditions above. In this case, we change the two arcs $\{(i_1, j_1), (i_2, j_2)\}$ to $\{(i_1, j_2), (i_2, j_1)\}$. Move ptr_1 to j_2.

(IB2) If (i_2, j_2) is a small arc, there are also two cases.

(IB2.1) If there exists no righthand endpoint between i_1 and i_2, then move ptr_1 to j_2.

(IB2.2) If there exist some righthand endpoints between i_1 and i_2, then do nothing.

(IC) Let ptr_2 point to the next righthand endpoint on the right of j_2. Go to (IB).

(ID) Stop.

In running the above algorithm, if $ptr_1 = j_1$, $ptr_2 = j_2$ with the arcs (i_1, j_1) and (i_2, j_2), then the following properties hold:

1. (i_1, j_1) is a small arc. Any righthand vertex j of $\mathcal{R}$ with $j < i_1$ is in $\mathcal{R}_2$, and any righthand vertex j with $j > i_1$ is in $\mathcal{R} \setminus \mathcal{R}_2$.
2. If (i', j') is an arc with $j_1 < j' < j_2$, then there exists a righthand vertex j such that $i_1 < j < i'$.

The above steps enable us to get a matching N' such that when applying Algorithms I and II to N', one gets N. To see it, first assume (i_2, j_2) is a large arc.

(1) If there is an arc (i_3, j_3) as described in Step (IB1.1), then the current configuration can be obtained by applying Algorithm I, Step (B2.3) to arcs $\{(i_1, j_2), (i_2, j_3), (i_3, j_1)\}$ with $ptr_1 = j_2$ and $ptr_2 = j_3$. Step (IB1.1) reverses this operation.

(2) If there is no arc (i_3, j_3) as described in Step (IB1.1), then the current configuration can be obtained by applying Algorithm I, Step (B1) to $\{(i_1, j_2), (i_2, j_1)\}$. Step (IB1.2) reverses it.

In the case that (i_2, j_2) is a small arc,

(1) If there exists no righthand endpoint between i_1 and i_2, then the current configuration can be obtained by applying Algorithm I, Step (B2.1). Step (IB2.1) reverses this operation.

(2) If there exist some righthand endpoints between i_1 and i_2, then $i_1 < i_2$. Otherwise, assume $i_2 < i_1$, then any righthand endpoint j between i_1 and i_2 must be in $\mathcal{R}_2$. But at the stage when the Algorithm III and the step of Invert borrowing stop, there are some large arcs (i', j') crossing (i_2, j_2), and $j' \in \mathcal{R} \setminus \mathcal{R}_2$. By our construction, the arc (i_2, j_2) should have been destroyed by an application of Step (IB1.1). Contradiction! Hence $i_1 < i_2$, and the current configuration can be obtained by applying Algorithm I, Step (B2.2). Step (IB2.2) reverses it.

Since there is no large arc in $\mathcal{R}_0$, applying Algorithm IV in $\mathcal{R}\setminus\mathcal{R}_2 = \mathcal{R}_1 \cup \mathcal{R}_0$ will not change the arcs in $\mathcal{R}_0$.

Let N' be the matching obtained from N by applying Algorithms III and IV. The above argument shows that

$$N \xrightarrow{\text{Algorithms I and II}} N'.$$

It follows that the map ϕ is surjective. As the set $P_n(S,T)$ is finite, ϕ must be a bijection. ∎

We say an arc (i_1, j_1) of a matching M is *maximal* if there is no arc (i_2, j_2) in M such that $i_2 < i_1 < j_1 < j_2$. For a maximal arc $e = (i_1, j_1)$, let

$$t(e, M) = \min\{i' : \text{There is an arc } (i', j') \text{ s.t. } i_1 < i' < j_1 < j'\}.$$

If there is no such arc, let $t(e, M) = j_1 + 1$.

Lemma 7. *1. An arc (i_1, j_1) is maximal in M if and only if (i_1, k) is maximal in $M' = \phi(M)$, for some k.*

2. Let $e = (i_1, j_1)$ be a maximal arc in M, and $e' = (i_1, k)$ be the corresponding maximal edge in M'. Then $t(e, M) = t(e', M')$.

Proof. Let M be a matching of $[2m]$. We prove the lemma by induction on m. The case $m = 1$ is trivial. Assume the claim holds for all matchings on $2m-2$ linearly ordered vertices. Given a matching M of $[2m]$ with the arc $(i, 2m)$, let $M_1 = M \setminus \{(i, 2m)\}$, $M_1' = \phi(M_1)$, M_2 be the matching obtained from M_1' by applying Algorithms I and II, and $M' = \phi(M) = M_2 \cup \{(i, 2m)\}$. Clearly $e = (i, 2m)$ is a maximal arc in both M and M' with $t(e, M) = t(e, M') = 2m + 1$. Otherwise, an arc $e = (i_1, j_1)$ with $i_1 \neq i$ is a maximal arc of M (M') if and only if $i_1 < i$ and (i_1, j_1) is a maximal arc of M_1 (M_2). Denote by e, f, e' the arcs whose lefthand endpoint is i_1 in M_1, M_1' and M_2, respectively. Then by the inductive hypothesis, e is maximal in M_1 if and only if f is maximal in M_1', in which case $t(e, M_1) = t(f, M_1')$. Since the algorithms do not change the relative positions of large arcs, this happens if and only if e' is maximal in M_2.

Assume $e = (i_1, j_1)$ is maximal in M_1 and $x = t(e, M_1) = t(f, M_1')$.

(Case 1) If $x < i$, then $t(e, M) = x$. The equation $t(f, M_1') = x$ implies in M_1', among all arcs (i, j) crossing f with $i > i_1$, the one with the smallest lefthand endpoint is a large arc. This property is preserved by the algorithms, so $t(e', M_2) = x$, and hence $t(e', M') = x$.

(Case 2) If $x > i$, then $t(e, M) = i$ as the last arc $(i, 2m)$ crosses any large arc. $t(f, M_1') = x$ implies that there is no large arc in M_1' that crosses f. Hence there is no large arc in M_2 that crosses e'. So we again have $t(e', M') = i$.

In both cases, $t(e, M) = t(e', M')$. ∎

Theorem 8. *We have $pmaj(M) = cr_2(\phi(M))$ for all $M \in P_n(S, T)$.*

Proof. Again it is enough to prove the theorem for matchings of $[2m]$. Do induction on m. The theorem is clearly true for $m = 1$. Assume it is true for all matchings on $2m - 2$ linearly ordered vertices. Given a matching M of $[2m]$ with the arc $(i, 2m)$, let M_1, M_1', M_2 and M' be defined as in Lemma 7. By inductive hypothesis, $pmaj(M_1) = cr_2(M_1')$. Hence

$$\begin{aligned} pmaj(M) &= pmaj(M_1) + \#\{j : j \in \mathcal{R}_2(M_1)\} \\ &= cr_2(M_1') + \#\{j : j \in \mathcal{R}_2(M_1)\}. \end{aligned} \tag{5}$$

By our construction Algorithm I decreases the crossing number of M_1' by 1 for each righthand endpoint of large arcs in $\mathcal{R}_1(M_1')$, and Algorithm II increases the crossing number by 1 for each righthand endpoint of small arcs in $\mathcal{R}_2(M_1')$. Hence

$$\begin{aligned} cr_2(M_2) = cr_2(M_1') &- \#\{j \in \mathcal{R}_1(M_1') : j \text{ belongs to a large arc}\} \\ &+ \#\{j \in \mathcal{R}_2(M_1') : j \text{ belongs to a small arc}\}. \end{aligned}$$

Therefore

$$\begin{aligned} cr_2(M') = cr_2(M_2) &+ \#\{j : j \in \mathcal{R} : j \text{ belongs to a large arc}\} \\ = cr_2(M_1') &- \#\{j \in \mathcal{R}_1(M_1') : j \text{ belongs to a large arc}\} \\ &+ \#\{j \in \mathcal{R}_2(M_1') : j \text{ belongs to a small arc}\} \\ &+ \#\{j : j \in \mathcal{R} : j \text{ belongs to a large arc}\} \\ = cr_2(M_1') &+ \#\{j : j \in \mathcal{R}_2(M_1')\} \end{aligned} \tag{6}$$

Comparing Eqs. (5) and (6), we only need to show that

$$\mathcal{R}_2(M_1) = \mathcal{R}_2(M_1'). \tag{7}$$

But the critical large arc e_L of M_1, if exists, must be a maximal arc of M_1, and the left-end of the critical small arc e_S, if exists, must be $t(e_L, M_1)$. Hence the identity (7) follows from Lemma 7. ∎

Finally we explain how our construction extends Foata's second fundamental transformation on permutations, which can be described as follows. Let $w = w_1 w_2 \cdots w_n$ be a word on $\mathbb{N}$ and let $a \notin \{w_1, \ldots, w_n\}$. If $w_n < a$, the a-factorization of w is $w = v_1 b_1 \cdots v_p b_p$, where each b_i is a letter less than a, and each v_i is a word (possibly empty), all of whose letters are greater than a. Similarly, if $w_n > a$, the a-factorization of w is $w = v_1 b_1 \cdots v_p b_p$, where each b_i is a letter greater than a, and each v_i is a word (possibly empty), all of whose letters are less than a. In each case we define

$$\gamma_a(w) = b_1 v_1 \cdots b_p v_p.$$

With the above notation, let $a = w_n$ and let $w' = w_1 \cdots w_{n-1}$. The second fundamental transformation Φ is defined recursively by $\Phi(w) = w$ if w has length 1, and

$$\Phi(w) = \gamma_a(\Phi(w'))a,$$

if w has length $n > 1$. The map Φ has the property that $inv(\Phi(w)) = maj(w)$.

For a permutation π of length m, our bijection ϕ, when applied to the matching $M_\pi = \{(m + 1 - \pi(i), i + m) : 1 \leq i \leq m\}$, is essentially the same as Foata's transformation $\Phi(\pi)$. Note that the last arc of M_π corresponds to the last entry $\pi(m)$ of π, and the set $\mathcal{R}$ consists of all righthand endpoints except $2m$. Then

1. If $\pi(m-1) < \pi(m)$, then $\mathcal{R}_2 = \emptyset$. The map γ_a in Foata's transformation is equivalent to Algorithm I, where cases (B2.2) and (B2.3) will not happen.

2. If $\pi(m-1) > \pi(m)$, then $\mathcal{R}_0 = \mathcal{R}_1 = \emptyset$. The map γ_a in Foata's transformation is equivalent to Algorithm II.

References

[1] W.Y.C. Chen, E.Y.P. Deng, R.R.X. Du, R.P. Stanley and C.H. Yan, Crossings and nestings of matchings and partitions, to appear in *Transactions of the American Mathematical Society.*

[2] B. Clarke, A note on some Mahonian statistics, *Seminaire Lotharingien de Combiantoire* **53** (2005), Article B53a.

[3] D. Foata, On the Netto inversion number of a sequence, *Proc. Amer. Math. Soc.* **19** (1968), 236–240.

[4] D. Foata, Rearrangements of Words, in M. Lothaire, Combinatorics on Words, Cambridge University Press, Cambridge, 1997.

[5] H. Gupta, A new look at the permutation of the first n natural numbers, *Indian J. Pure Appl. Math.* **9** (1978), no. 6, 600–631.

[6] G.-N. Han, Une transformation fondamentale sure les réarrangements de mots, *Adv. Math.* **105** (1994), 26–41.

[7] J. Haglund and L. Stevens, An extension of the Foata map to standard Young tableaux, *Seminaire Lotharingien de Combiantoire* **56** (2006), Article B56c.

[8] P.A. MacMahon, Combinatory Analysis, vol. 1, Cambridge, Cambridge University Press, 1915. (Reprinted by Chelsea, New York, 1955).

[9] M. de Sainte-Catherine, Couplages et Pfaffiens en combinatoire, physique et informatique, Ph.D. Thesis, University of Bordeaux I, 1983.

[10] B. Sagan, A maj statistic for set partitions, *Europ. J. Combinatorics* (1991) **12**, 69–79.

[11] R. Simion and D. Stanton, Specializations of generalized Laguerre polynomials, *SIAM J. Math. Anal.* **25** (1994), no. 2, 712–719.

[12] R. Simion and D. Stanton, Octabasic Laguerre polynomials and permutation statistics, *J. Comput. Appl. Math.* **68** (1996), no. 1-2, 297–329.