# Generalized Parking Functions, Tree Inversions and Multicolored Graphs

*Dedicated to Dominique Foata*

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## Abstract

A generalized $\mathbf{x}$-parking function associated to a positive integer vector of the form $(a, b, b, \ldots, b)$ is a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers whose non-decreasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies $b_i \leq a + (i-1)b$. The set of $\mathbf{x}$-parking functions has the same cardinality as the set of sequences of rooted $b$-forests on $[n]$. We construct a bijection between these two sets. We show that the sum enumerator of complements of $\mathbf{x}$-parking functions is identical to the inversion enumerator of sequences of rooted $b$-forests by generating function analysis. Combinatorial correspondences between the sequences of rooted forests and $\mathbf{x}$-parking functions are also given in terms of depth-first search and breadth-first search on multicolored graphs.

## 1 Introduction

The notion of parking function was introduced by Konheim and Weiss as a colorful way to study a hashing problem. In the paper [9], they proved that the number of parking functions of length $n$ is $(n+1)^{n-1}$. Later the subject has attracted the interest of many mathematicians, in particular, combinatorialists. A simple method of counting the number of parking functions was found by Pollak (see Riordan, [14]), for which an equivalent description was given by Stanley [19, 21] in group-theoretic terms. Knuth surveyed the early results on parking functions in his famous book, *The Art of Computer Programming, Sorting and Searching* [7, Sec. 6.4]. His description of parking functions was given in terms of a hashing algorithm, with an explicit "parking" description [7, Ex. 6.4.29-31].

A *parking function* of length $n$ may be defined as a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers whose non-decreasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies $b_i \leq i$. Note that the number of parking functions of length $n$, $(n+1)^{n-1}$, is equal to the number of labeled rooted trees on $\{0, 1, 2, \ldots, n\}$, or equivalently, the number of acyclic functions on $[n] = \{1, 2, \ldots, n\}$. Several bijections between the set of parking functions of length $n$ and the set of labeled rooted forests on $[n]$ are known. The first published one was due to Schützenberger [15] in 1968. Pollak (see Riordan [14, Sec. 3, 4], and Foata and Riordan ([1, Sec. 2]) constructed bijections in which a parking function is associated with a code which, by Prüfer's correspondence, corresponds to a tree. In the same paper ([1, Sec. 3]), Foata and Riordan also constructed another bijection using pairs $(\mathbf{r}, \pi)$,

where $\mathbf{r} \in \mathbb{N}^n$ is balanced, and $\pi$ is a permutation that is compatible with $\mathbf{r}$ (cf. Sec. 2). Françon [2] discussed the second construction of Foata and Riordan's and showed that it can be generalized to a much larger class of selection procedures. Kreweras [10] investigated the recurrence relations satisfied by the generating functions of parking functions and labeled trees. This recurrence led to a new bijection between these two objects, constructed via induction. Other bijections were also found, for example, by Knuth [7], Moszkowski [12], and Gilbey and Kalikow [6].

Parking functions are also related to other combinatorial structures. Stanley [21] used the set of ordinary parking functions to give an edge-labeling for maximal chains in the lattice of non-crossing partitions. He also revealed the relations between parking functions and hyperplane arrangements, interval orders, and plane partitions [19, 20]. Pitman & Stanley [13] discussed the connection between parking functions and empirical distributions, plane trees, polytopes and the associahedrons. They showed how to enumerate some generalized parking functions from results in uniform order statistics and empirical distributions. Gilbey and Kalikow [6] constructed bijections from the set of parking functions to allowable pairs of permutations of a priority queue. Parking functions has also been of interest to statisticians and probabilists.

Following [13], the notion of parking function can be generalized. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$. Define an $\mathbf{x}$-*parking function* to be a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers whose non-decreasing rearrangement $b_1 \le b_2 \le \cdots \le b_n$ satisfies $b_i \le x_1 + \cdots + x_i$. Thus an ordinary parking function corresponds to the case $\mathbf{x} = (1, 1, \ldots, 1)$. Let $P_n(\mathbf{x})$ denote the number of $\mathbf{x}$-parking functions. Clearly that $P_n(\mathbf{x}) = 0$ if $x_1 = 0$.

In general, it is difficult to write explicit formulas for the number of generalized parking functions. In [23] we found formulas for some special types of vectors $\mathbf{x}$ using easy combinatorial arguments. In the present paper, we concentrate on $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, b, \ldots, b) \in \mathbb{N}^n$. In this case, the number $P_n(\mathbf{x})$ is known (see, for example, Pitman and Stanley [13]):

**Theorem 1** *For* $\mathbf{x} = (a, b, b, \ldots, b)$, $P_n(\mathbf{x}) = a(a + nb)^{n-1}$.

This formula can be proved by a simple combinatorial argument generalizing the proof of Pollak for the ordinary parking functions [14, 1].

In Section 2, we generalize the relation between the set of ordinary parking functions and the set of acyclic functions (or equivalently, the set of labeled rooted forests) to $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, \ldots, b)$, which gives a bijective proof of Theorem 1. The proof is an extension of the second bijection of Foata and Riordan [1, Sec. 3]. Explicitly, we introduced a set $C_n$ which consists of pairs $(\mathbf{r}, \pi)$ where $\mathbf{r}$ is a vector of length $a + (n-1)b$ that is balanced, $\pi$ is a permutation of $[n]$ and is compatible with $\mathbf{r}$. We construct one-to-one correspondences of the set $C_n$ with both the set of $\mathbf{x}$-parking functions and the set of sequences of rooted $b$-forests on $[n]$.

Next we study the sum enumerator $\bar{P}_n^{(a,b)}(q)$ of complements of $\mathbf{x}$-parking functions. We prove that this enumerator is identical to the enumerator $I_n^{(a,b)}(q)$ of sequences of rooted $b$-forests by the number of their inversions. In doing so, we find recurrence relations satisfied by $\bar{P}_n^{(a,b)}(q)$ and $I_n^{(a,b)}(q)$, and introduce the concept of a multicolored $(a, b)$-graph whose excess edges and roots are enumerated by a polynomial $C_n^{(a,b)}(q)$. We show that $C_n^{(a,b)}(q)$ satisfies the same recurrence relations as both $\bar{P}_n^{(a,b)}(1 + q)$ and $I_n^{(a,b)}(1 + q)$, hence prove that $\bar{P}_n^{(a,b)}(q) = I_n^{(a,b)}(q)$.

Whenever a result is found by generating function analysis, a combinatorial explanation is expected. In the last section, we establish combinatorial correspondences between sequences of rooted forests, generalized parking functions, and labeled multicolored graphs. The techniques we

used are depth-first search and breadth-first search algorithms in labeled multicolored graphs. The depth-first search on labeled connected graphs was first studied by Gessel and Wang [3], and further explored by Gessel and Sagan [4]. It is extended naturally to labeled multicolored graphs. Gessel and Sagan also analyzed a neighbors-first search algorithm, which is similar to the breadth-first search we described. But our construction reveals the connection to the set of parking functions. Our construction is inspired by the work on graph enumeration and random graph evolution of Spencer [16].

## 2 Bijective proof for Theorem 1

It is well-known that the number of ordinary parking functions is $(n + 1)^{n-1}$. The number of labeled rooted forests on $[n]$ is also $(n + 1)^{n-1}$. In [1], Foata and Riordan constructed bijections between these two objects. In this section, we generalize their result to the $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, \ldots, b)$.

We will construct a bijective mapping between the sets $A_n$ and $B_n$, where $A_n$ is the set of $\mathbf{x}$-parking functions associated to $\mathbf{x} = (a, b, \ldots, b)$, ($a$, $b$ are positive integers). To describe $B_n$, we need some notations. First, a *rooted b-forest* on $[n]$ is a rooted forest on vertices $[n] = \{1, 2, \ldots, n\}$ with edges colored with the colors $\bar{0}, \bar{1}, \ldots, \overline{b-1}$. There is no further restriction on the possible coloring of the edges. Let $B_n$ be the set of all sequences $(S_1, S_2, \ldots, S_a)$ of length $a$ such that (1) each $S_i$ is a rooted $b$-forest, (2) $S_i$ and $S_j$ are disjoint if $i \neq j$, and (3) the union of the vertex sets of $S_1, S_2, \ldots S_a$ is $[n]$.

Another set $C_n$ is introduced and will be put in one-to-one correspondence with both $A_n$ and $B_n$. First we say that a sequence $\mathbf{r} = (r_1, r_2, \ldots, r_{a+(n-1)b})$ of $a + (n-1)b$ non-negative integers is *balanced* if

$$\begin{cases} r_1 + r_2 + \cdots + r_{a+ib} & \geq & i + 1, \quad \text{for } i = 0, 1, \ldots, n-2, \\ r_1 + r_2 + \cdots + r_{a+(n-1)b} & = & n. \end{cases} \tag{1}$$

We also say that a permutation $\pi$ of $[n]$ is *compatible* with $\mathbf{r}$ if the terms in the inverse $\pi^{-1}$ of $\pi$ is increasing on every interval of the form $\{1 + \sum_{i=1}^{k} r_i, 2 + \sum_{i=1}^{k} r_i, \ldots, \sum_{i=1}^{k+1} r_i\}$ (if $r_{k+1} \neq 0$). The set $C_n$ is defined as the set of all couples $(\mathbf{r}, \pi)$ with $\mathbf{r} \in \mathbb{N}^{a+(n-1)b}$ balanced and $\pi$ a permutation of $[n]$ compatible with $\mathbf{r}$.

### 2.1 The mapping from $A_n$ to $C_n$

Throughout this paper, we fix $\mathbf{x} = (a, b, b, \ldots, b) \in \mathbb{N}^n$. Let $\alpha = (a_1, a_2, \ldots, a_n)$ be an $\mathbf{x}$-parking function of $A_n$. The couple $(\mathbf{r}_\alpha, \pi_\alpha)$ of $C_n$ associated with $\alpha$ is defined as follows. First, let $\mathbf{r}_\alpha = (r_1, r_2, \ldots, r_{a+(n-1)b})$ be the *specification* of $\alpha$, i.e.,

$$r_i = \text{Card}\{ a_j \in \alpha \mid a_j = i \}.$$

By the definition of $\mathbf{x}$-parking function, $\mathbf{r}_\alpha \in \mathbb{N}^{a+(n-1)b}$ is balanced.

The permutation $\pi_\alpha = (\pi_\alpha(1), \pi_\alpha(2), \ldots, \pi_\alpha(n))$ is defined by

$$\pi_\alpha(i) = \text{Card}\{j \in [n] \mid a_j < a_i, \text{ or } a_j = a_i \text{ and } j < i \}. \tag{2}$$

In other words, $\pi_\alpha(i)$ is the position of the term $a_i$ in the non-decreasing rearrangement of $\alpha$.

Another description of $\pi_\alpha$ is: the numbers $1, 2 \ldots, r_1$ appear in the successive positions left to right where $a_i = 1$, (if $r_1 \neq 0$); in general, the numbers

$$1 + \sum_{i=1}^{k} r_i, \; 2 + \sum_{i=1}^{k} r_i, \; \ldots, \; \sum_{i=1}^{k+1} r_i$$

appear in the successive positions left to right where $a_i = k+1$, for $k = 0, 1, 2, \ldots, a + (n-1)b - 1$, and $r_{k+1} \neq 0$. This is equivalent to say that

$$a_i = \begin{cases} 1, & \text{if } 1 \leq \pi_\alpha(i) \leq r_1, \\ k+1, & \text{if } 1 + \sum_{i=1}^{k} r_i \leq \pi_\alpha(i) \leq \sum_{i=1}^{k+1} r_i, \end{cases} \tag{3}$$

which implies $\pi_\alpha$ is compatible with $\mathbf{r}$.

**Example 1** *For $n = 3$, $\mathbf{x} = (2, 1, 1)$, the specification $\mathbf{r}_\alpha = (r_1, r_2, r_3, r_4)$ and permutations $\pi_\alpha$ on $\{1, 2, 3\}$ are as follows.*

| $\alpha$ | 111 | 112 | 121 | 211 | 113 | 131 | 311 | 114 | 141 | 411 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{r}_\alpha$ | 3000 | 2100 | 2100 | 2100 | 2010 | 2010 | 2010 | 2001 | 2001 | 2001 |
| $\pi_\alpha$ | 123 | 123 | 132 | 312 | 123 | 132 | 312 | 123 | 132 | 312 |

| $\alpha$ | 122 | 212 | 221 | 123 | 132 | 213 | 231 | 312 | 321 | 124 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{r}_\alpha$ | 1200 | 1200 | 1200 | 1110 | 1110 | 1110 | 1110 | 1110 | 1110 | 1101 |
| $\pi_\alpha$ | 123 | 213 | 231 | 123 | 132 | 213 | 231 | 312 | 321 | 123 |

| $\alpha$ | 142 | 214 | 241 | 412 | 421 | 133 | 313 | 331 | 134 | 143 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{r}_\alpha$ | 1101 | 1101 | 1101 | 1101 | 1101 | 1020 | 1020 | 1020 | 1011 | 1011 |
| $\pi_\alpha$ | 132 | 213 | 231 | 312 | 321 | 123 | 213 | 231 | 123 | 132 |

| $\alpha$ | 314 | 341 | 413 | 431 | 222 | 223 | 232 | 322 | 224 | 242 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{r}_\alpha$ | 1011 | 1011 | 1011 | 1011 | 0300 | 0210 | 0210 | 0210 | 0201 | 0201 |
| $\pi_\alpha$ | 213 | 231 | 312 | 321 | 123 | 123 | 132 | 312 | 123 | 132 |

| $\alpha$ | 422 | 233 | 323 | 332 | 234 | 243 | 324 | 342 | 423 | 432 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{r}_\alpha$ | 0201 | 0120 | 0120 | 0120 | 0111 | 0111 | 0111 | 0111 | 0111 | 0111 |
| $\pi_\alpha$ | 312 | 123 | 213 | 231 | 123 | 132 | 213 | 231 | 312 | 321 |

We have just constructed a map $\alpha \rightarrow (\mathbf{r}_\alpha, \pi_\alpha)$ from $A_n$ to $C_n$. Now assume $\alpha$ and $\beta$ are two distinct $\mathbf{x}$-parking functions of $A_n$. If $\mathbf{r}_\alpha \neq \mathbf{r}_\beta$, obviously $(\mathbf{r}_\alpha, \pi_\alpha) \neq (\mathbf{r}_\beta, \pi_\beta)$. If $\mathbf{r}_\alpha = \mathbf{r}_\beta$, then $\pi_\alpha \neq \pi_\beta$ by (3). Therefore the map $\alpha \rightarrow (\mathbf{r}_\alpha, \pi_\alpha)$ is *injective*.

To show that this map is *surjective* and at the same time define its inverse, let $(\mathbf{r}, \pi) \in C_n$. Define a sequence $\alpha = (a_1, a_2, \ldots, a_n)$ by letting

$$a_i = \begin{cases} 1, & \text{if } 1 \leq \pi(i) \leq r_1, \\ k+1, & \text{if } 1 + \sum_{i=1}^{k} r_i \leq \pi(i) \leq \sum_{i=1}^{k+1} r_i. \end{cases} \tag{4}$$

4

So $\mathbf{r}$ is the specification of $\alpha$, i.e., $\mathbf{r} = \mathbf{r}_\alpha$. As $\mathbf{r}$ is balanced, $\alpha$ defined above is an $\mathbf{x}$-parking function. Furthermore, $\pi$ and $\pi_\alpha$ are both compatible with $\mathbf{r}$. By the formulas (3) and (4), $\pi = \pi_\alpha$. This proves that the map $\alpha \to (\mathbf{r}_\alpha, \pi_\alpha)$ is surjective.

## 2.2 The mapping from $B_n$ to $C_n$

Let $\mathfrak{s} = (S_1, S_2, \ldots, S_a) \in B_n$. The couple in $C_n$ associated with $\mathfrak{s}$ is denoted by $(\mathbf{r}_\mathfrak{s}, \sigma_\mathfrak{s})$. First we describe the permutation $\sigma_\mathfrak{s}$. It is a natural generalization of the construction given in [1].

For any vertex $x$ in a rooted $b$-forest on $[n]$, there is a unique root $y$ which is connected with $x$. Define the *height* of $x$ to be the number of edges connecting $x$ and the root $y$. If $x$ is a root, then the height of $x$ is zero. If a vertex $z$ is the first vertex lying on the path from $x$ to the root $y$, we say that $z$ is the *predecessor* of $x$, $x$ is a *child* of $z$, and write as $z = pre(x)$ and $x \in child(z)$. Every non-rooted vertex $x$ uniquely determines an edge $zx$, which is denoted by $edge(x)$. Clearly $\mathfrak{s} = (S_1, S_2, \ldots, S_a)$ is fully determined by the sets of roots in $S_1, S_2, \ldots, S_a$, the function $pre(x)$ and the color of $edge(x)$ for each vertex $x$ which is not a root.

Fix a sequence of rooted $b$-forests $\mathfrak{s} \in B_n$, we define a linear order $<_\mathfrak{s}$ on $[n]$ by the following rules.

1. If two vertices $x$ and $y$ are both roots, then

$$
\begin{aligned}
x <_\mathfrak{s} y \quad \text{if either} \quad & x \in S_i,\ y \in S_j,\ \text{and}\ i < j, \\
\text{or} \quad & x, y \in S_i \ \text{and}\ x < y.
\end{aligned} \tag{5}
$$

2. If the height of $x$ is less than the height of $y$, then $x <_\mathfrak{s} y$.

3. If the height of $x$ equals the height of $y$, and $pre(x) <_\mathfrak{s} pre(y)$, then $x <_\mathfrak{s} y$.

4. If $pre(x) = pre(y)$, then

$$
\begin{aligned}
x <_\mathfrak{s} y \quad \text{if either} \quad & \text{color of}\ edge(x) < \text{color of}\ edge(y), \\
\text{or} \quad & \text{color of}\ edge(x) = \text{color of}\ edge(y)\ \text{and}\ x < y.
\end{aligned} \tag{6}
$$

The sequence formed by writing $\{1, 2, \ldots, n\}$ in the increasing order with respect to $<_\mathfrak{s}$ is denoted by $\sigma_\mathfrak{s}^{-1} = (\sigma_\mathfrak{s}^{-1}(1), \sigma_\mathfrak{s}^{-1}(2), \ldots, \sigma_\mathfrak{s}^{-1}(n))$. And the permutation $\sigma_\mathfrak{s}$ is the inverse of $\sigma_\mathfrak{s}^{-1}$.

**Example 2** *Take $n = 13$, $a = 2$ and $b = 2$. A sequence $(S_1, S_2)$ of rooted $b$-forests on $[n]$ is given in Figure 1, where the numbers in italic indicate the coloring of the edges.*
*We have*

$$
\begin{aligned}
\sigma_\mathfrak{s}^{-1} &= (7 \quad 8 \quad 4 \quad 5 \quad 1 \quad 12 \quad 10 \quad 9 \quad 11 \quad 13 \quad 2 \quad 6 \quad 3) \\
\sigma_\mathfrak{s} &= (5 \quad 11 \quad 13 \quad 3 \quad 4 \quad 12 \quad 1 \quad 2 \quad 8 \quad 7 \quad 9 \quad 6 \quad 10)
\end{aligned}
$$

Next we define the *forest specification* of $\mathfrak{s}$. Set $\mathbf{r}_\mathfrak{s} = (r_1, r_2, \ldots, r_{a+(n-1)b})$ as follows.

1. $r_i$ is the number of roots in $S_i$ for $i = 1, 2 \ldots, a$;

2. $r_{a+k}$ is the number of children of $\sigma_\mathfrak{s}^{-1}(1)$ with edge color $\overline{k-1}$, for $k = 1, 2, \ldots, b$;
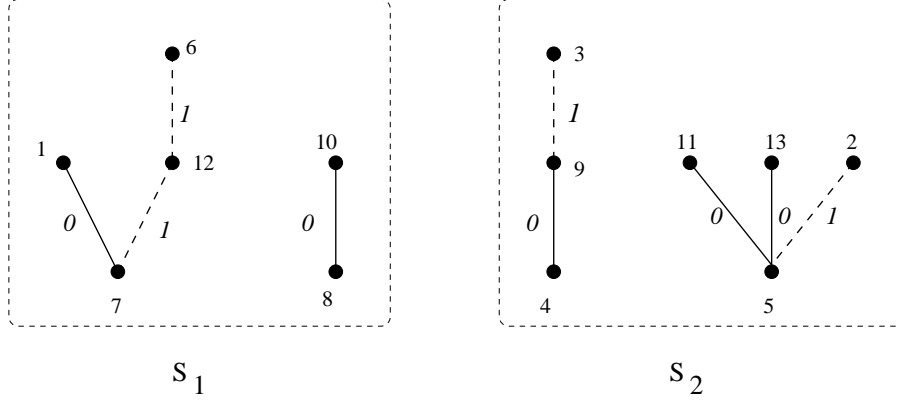
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Figure 1: A sequence of rooted 2-forests

3. In general, $r_{a+(i-1)b+k}$ is the number of children of $\sigma_{\mathfrak{s}}^{-1}(i)$ with edge color $\overline{k-1}$, for $k = 1, 2, \ldots, b$ and $i = 1, 2, \ldots, n-1$.

For instance, the forest specification $\mathbf{r}_{\mathfrak{s}}$ in Example 2 is

$$\mathbf{r}_{\mathfrak{s}} = (2, 2, \underbrace{1, 1}_{7}, \underbrace{1, 0}_{8}, \underbrace{1, 0}_{4}, \underbrace{2, 1}_{5}, \underbrace{0, 0}_{1}, \underbrace{0, 1}_{12}, \underbrace{0, 0}_{10}, \underbrace{0, 1}_{9}, \underbrace{0, 0}_{11}, \underbrace{0, 0}_{13}, \underbrace{0, 0}_{2}, \underbrace{0, 0}_{6}),$$

where the number under the braces are the corresponding $\sigma_{\mathfrak{s}}^{-1}(i)$. (It is obvious that $\sigma_{\mathfrak{s}}^{-1}(n)$ is a leaf in the rooted forest. So there is no need to record the number of children for $\sigma_{\mathfrak{s}}^{-1}(n)$.)

From the above construction, it is clear that $r_1 + r_2 + \cdots + r_a \geq 1$, $r_1 + r_2 + \cdots + r_{a+(n-1)b} = n$, and $\sigma_{\mathfrak{s}}$ is compatible with $\mathbf{r}_{\mathfrak{s}}$. Furthermore, knowing $\mathbf{r}_{\mathfrak{s}}$ and $\sigma_{\mathfrak{s}}$, we can recover the sequence of rooted $b$-forests by the following algorithm.

**Algorithm 1.**

1. $x$ is a root of $S_1$ if $1 \leq \sigma_{\mathfrak{s}}(x) \leq r_1$; $x$ is a root of $S_k$ if

$$1 + \sum_{j=1}^{k-1} r_i \leq \sigma_{\mathfrak{s}}(x) \leq \sum_{j=1}^{k} r_i, \qquad k = 2, 3, \ldots, a.$$

2. $x$ is a child of $\sigma_{\mathfrak{s}}^{-1}(i)$ if

$$1 + \sum_{j=1}^{a+(i-1)b} r_j \leq \sigma_{\mathfrak{s}}(x) \leq \sum_{j=1}^{a+ib} r_j.$$

3. After determined that $pre(x) = i$ by Step 2, the color of $edge(x)$ is $\overline{k-1}$ if

$$1 + \sum_{j=1}^{a+(i-1)b+k-1} r_j \leq \sigma_{\mathfrak{s}}(x) \leq \sum_{j=1}^{a+(i-1)b+k} r_j.$$

6

Next we claim that given a sequence of rooted $b$-forest, $\mathbf{r_s}$ is balanced. Note that from the above algorithm, for a non-rooted vertex $x$,

$$pre(x) = \sigma_{\mathfrak{s}}^{-1}(i) \qquad \text{iff} \qquad 1 + \sum_{j=1}^{a+(i-1)b} r_i \leq \sigma_{\mathfrak{s}}(x) \leq \sum_{j=1}^{a+ib} r_i.$$

The vertex $\sigma_{\mathfrak{s}}^{-1}(i)$ is less than any of its children in the linear order $<_{\mathfrak{s}}$, by the steps 1 and 2 of Algorithm 1,

$$\sigma_{\mathfrak{s}}(\sigma_{\mathfrak{s}}^{-1}(i)) \leq \sum_{j=1}^{a+(i-1)b} r_i,$$

that is,

$$i \leq \sum_{j=1}^{a+(i-1)b} r_j,$$

for $i = 1, 2, \ldots, n-1$. This proves that $\mathbf{r_s}$ is balanced.

In conclusion, we have constructed a mapping $\mathfrak{s} \to (\mathbf{r_s}, \sigma_{\mathfrak{s}})$ from $B_n$ to $C_n$. Let $\mathfrak{s}$ and $\mathfrak{t}$ be two distinct sequences of rooted $b$-forests in $B_n$. If $\mathbf{r_s} \neq \mathbf{r_t}$, then $(\mathbf{r_s}, \sigma_{\mathfrak{s}}) \neq (\mathbf{r_t}, \sigma_{\mathfrak{t}})$. On the other hand, if $\mathbf{r_s} = \mathbf{r_t}$, then $\sigma_{\mathfrak{s}} \neq \sigma_{\mathfrak{t}}$. Otherwise $\mathfrak{s} = \mathfrak{t}$ by the Algorithm 1. Hence the mapping from $B_n$ to $C_n$ is *injective*.

Conversely, given a couple $(\mathbf{r}, \sigma) \in C_n$, we can construct a sequence of rooted $b$-forests $\mathfrak{s} = (S_1, S_2, \ldots, S_a)$ by Algorithm 1 in which $\sigma_{\mathfrak{s}}$ is replaced by $\sigma$. From the preceding description, it is easy to check that $(\mathbf{r}, \sigma) = (\mathbf{r_s}, \sigma_{\mathfrak{s}})$. Therefore the mapping is *surjective*. In conclusion, the mapping $\mathfrak{s} \to (\mathbf{r_s}, \sigma_{\mathfrak{s}})$ is a bijection from $B_n$ to $C_n$.

## 2.3 The mapping from $A_n$ to $B_n$

Combining the results from the previous two subsections, we obtain an explicit mapping between the set $A_n$ of the $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, \ldots, b)$ and the set $B_n$ of sequences of rooted $b$-forests.

First, given a sequence $\mathfrak{s} = (S_1, S_2, \ldots, S_a)$ of rooted $b$-forests on $[n]$, and let $(\mathbf{r}, \pi) = (\mathbf{r_s}, \sigma_{\mathfrak{s}})$. Then $\alpha = (a_1, a_2, \ldots, a_n)$ defined by (4) is an $\mathbf{x}$-parking function, and the mapping $\mathfrak{s} \to \alpha$ gives the bijection from $B_n$ to $A_n$. Explicitly, we have

$$\begin{cases} a_i = j \leq a & \text{if in } \mathfrak{s}, \ i \text{ is a root of } S_j, \\ a_i = a + (\sigma_{\mathfrak{s}}(j) - 1)b + k + 1 & \text{if in } \mathfrak{s}, \ pre(i) = j \text{ and the color of } edge(i) \text{ is } \bar{k}, \end{cases} \tag{7}$$

where $0 \leq k < b$.

Conversely, given $\alpha = (a_1, a_2, \ldots, a_n)$, let $(\mathbf{r}, \sigma) = (\mathbf{r}_\alpha, \pi_\alpha)$, we get a sequence of rooted $b$-forests by the Algorithm 1. The explicit formula is

$$\begin{cases} i \text{ is a root of } S_j & \text{if} \quad a_i = j \leq a, \\ pre(i) = \pi_\alpha^{-1}(\lfloor \dfrac{j-a-1}{b} \rfloor + 1) & \text{if} \quad a_i = j > a, \end{cases} \tag{8}$$

and in the second case, the color of $edge(i)$ is $\overline{j - a - 1} \pmod{b}$.

REMARK.

1. The cardinality of the set $B_n$ is $a(a + nb)^{n-1}$. This result can be obtained by using a simple generalization of the Prüfer code on rooted forests ([18], Chapter 5.3). As a corollary, we again get that the number of $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, \ldots, b)$ is $a(a + nb)^{n-1}$.

2. In the case of $a = 1$, there is a simple bijection between $A_n$ and $B_n$, generalizing the first bijection of Foata and Riordan [1]. Note that in this case $B_n$ is just the set of all rooted $b$-forest on $[n]$. The bijection is defined as follows: for any parking function $(a_1, a_2, \ldots, a_n)$, let $C = (c_1, c_2, \ldots, c_{n-1})$ be its code where $c_i = a_{i+1} - a_i$, (mod $1 + nb$). Clearly $C \in \mathbb{Z}_{1+nb}^{n-1}$. It can be shown that every code $C \in \mathbb{Z}_{1+nb}^{n-1}$ uniquely determines an $\mathbf{x}$-parking function. On the other hand, vectors in $\mathbb{Z}_{a+nb}^{n-1}$ are the Prüfer codes for rooted $b$-forest on $[n]$ if one interprets the $(n-1)$-vectors by the usual definition of Prüfer codes and the following rules,

    (a) $x$ is a root if in the Prüfer code, $pre(x) = 0$;
    (b) $x$ is child of $y$ and the edge connecting $xy$ is of color $\overline{r-1}$ if in the Prüfer code, $pre(x) = (y-1)b + r$ and $1 \le r \le b$.

# 3 Inversions of sequence of rooted forests and x-parking functions

Kreweras studied a polynomial which enumerates labeled rooted forests by the number of inversions, as well as complements of the ordinary parking functions, which was named *suites majeurs* by Kreweras [10]. Such a polynomial also relates to the labeled connected graphs. For any positive integer $b$, there are known notions of $b$-parking functions and of labeled rooted $b$-forests, generalizing the case studied by Kreweras. In [22] the author showed that the enumerator $\bar{P}_n^{(b)}(q)$ for complements of $b$-parking functions by the sum of their terms is identical to the enumerator of $I_n^{(b)}(q)$ of rooted $b$-forests by the number of their inversions. In this section we generalize this result further to the inversion enumerator $I_n^{(a,b)}(q)$ of sequences of rooted $b$-forests and the sum enumerator $\bar{P}_n^{(a,b)}(q)$ of complements of $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, b, \ldots, b)$. Our method is an extension of that of [22]: We find the recurrence relations satisfied by $I_n^{(a,b)}(q)$ and $\bar{P}_n^{(a,b)}(q)$, and we introduce the concept of *multicolored $(a, b)$-graph* whose excess edges and roots are enumerated by a polynomial denoted by $C_n^{(a,b)}(q)$. We show that $C_n^{(a,b)}(q)$ satisfies the same recurrence relations as both $I_n^{(a,b)}(1 + q)$ and $\bar{P}_n^{(a,b)}(1 + q)$, proving that $I_n^{(a,b)}(q) = \bar{P}_n^{(a,b)}(q)$.

Given a rooted forest $F$ on $[n]$ for which every connected component is a rooted tree, an *inversion* is a pair $(j, k)$ for which $j > k$, and $j$ lies on the unique path connecting $k$ to $i$, where $i$ is the root of the tree to which $k$ belongs. Let $\text{inv}(F)$ denote the number of inversions of $F$. The *inversion enumerator* $I_n(q)$ for labeled rooted forests on $[n]$ is the polynomial defined by

$$I_n(q) = \sum_F q^{\text{inv}(F)},$$

where $F$ ranges over all labeled rooted forests on $[n]$. If $T$ is a labeled tree on $[n] \cup \{0\}$, then define $\text{inv}(T) := \text{inv}(F_T)$ where $F_T$ is the labeled rooted forest on $[n]$ obtained from $T$ by removing the vertex 0, and letting each neighbor of 0 be a root. It follows that $I_n(q) = \sum_T q^{\text{inv}(T)}$ where $T$ ranges over all labeled trees on $[n] \cup \{0\}$.

The notion of inversion enumerator can be generalized to the set $B_n$ — the sequences of rooted $b$-forests as follows: Let $\mathfrak{s} = (S_1, S_2, \ldots, S_a)$ be a sequence of rooted $b$-forests on $[n]$. Denote the

color of an edge $e$ by $\kappa(e)$. Define the $(a, b)$-inversion $\mathrm{inv}^{(a,b)}(s)$ by

$$\mathrm{inv}^{(a,b)}(\mathfrak{s}) = \mathrm{inv}(\mathfrak{s}) + \sum_{i=1}^{a}(i-1)|S_i| + \sum_{x \in [n]} \sum_{e \in K(x)} \kappa(e),$$

where $\mathrm{inv}(\mathfrak{s})$ is the number of inversions of $S_1 \cup S_2 \cdots \cup S_a$ as an ordinary rooted forest, $K(x)$ is the set of edges lying between the vertex $x$ and the root of the unique tree to which $x$ belongs. Define the $(a, b)$-*inversion enumerator* $I_n^{(a,b)}(q)$ by

$$I_n^{(a,b)}(q) = \sum_{\mathfrak{s} \in B_n} q^{\mathrm{inv}^{(a,b)}(\mathfrak{s})}.$$

For $a = 1$, $I_n^{(a,b)}(q) = I_n^{(1,b)}(q)$ is the $b$-inversion enumerator studied in [22], and $I_n^{(1,1)}(q) = I_n(q)$ is the ordinary inversion enumerator.

**Theorem 2** *The $(a, b)$-inversion enumerator $I_n^{(a,b)}(q)$ satisfies the recurrence relation*

$$I_0^{(a,b)}(q) = 1, \qquad I_1^{(a,b)}(q) = 1 + q + q^2 + \cdots + q^{a-1},$$

$$I_{n+1}^{(a,b)}(q) = \sum_{i=0}^{n} \binom{n}{i} \left(1 + q^{i+1} + q^{2(i+1)} + \cdots + q^{(a-1)(i+1)}\right)\left(1 + q + \cdots q^i\right) I_i^{(b,b)}(q) I_{n-i}^{(a,b)}(q). \tag{9}$$

**Proof.** For a sequence $\mathfrak{s} = (S_1, S_2, \ldots, S_a)$ of rooted $b$-forests on $[n+1]$, let $T$ be the rooted tree containing vertex 1. Assume $T$ contains $i$ vertices other than 1, and $T \in S_j$ for some $1 \le j \le a$. Then $\mathfrak{s} \setminus T = (S_1, \ldots, S_{j-1}, S_j \setminus T, S_{j+1}, \ldots, S_a)$ is a sequence of rooted $b$-forests on $n - i$ vertices. Let

$$\mathfrak{s}' = (\emptyset, \ldots, \emptyset, \underbrace{T}_{j\text{th position}}, \emptyset, \ldots, \emptyset),$$

and

$$K_i(q) = \sum_T q^{\mathrm{inv}^{(1,b)}(T)},$$

where $T$ ranges over all rooted $b$-trees on $[i+1]$. Then

$$I_{n+1}^{(a,b)}(q) = \sum_{i=0}^{n} \binom{n}{i} I_{n-i}^{(a,b)}(q) \left(\sum_{\mathfrak{s}'} q^{\mathrm{inv}^{(a,b)}(s')}\right) = \sum_{i=0}^{n} \binom{n}{i} I_{n-i}^{(a,b)}(q) \left(\sum_{j=1}^{a} q^{(j-1)(i+1)}\right) K_i(q). \tag{10}$$

The rooted $b$-tree $T$ on $[i+1]$ can be formed as follows: First assume the vertex 1 is the root. Other $i$ vertices form a sequence $\mathfrak{t}' = (T_1, T_2, \ldots, T_b)$ of rooted $b$-forests with length $b$. Then merge the vertex 1 and the forest $\mathfrak{t}'$ into the tree $T$ by connecting 1 to the roots of $T_j$ with edges of color $j - 1$. Note that in this case

$$\mathrm{inv}^{(b,b)}(\mathfrak{t}') = \mathrm{inv}^{(1,b)}(\mathfrak{t}') + \sum_{j=1}^{b}(j-1)|T_j| = \mathrm{inv}^{(1,b)}(T).$$

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If, instead of 1, $k$ is the root of $T$, ($1 \le k \le i+1$), then the number of inversion will increase by $k-1$. Therefore,

$$K_i(q) = (1 + q + \cdots + q^i)I_i^{(b,b)}(q). \tag{11}$$

Substitute (11) into (10), we get

$$I_{n+1}^{(a,b)}(q) = \sum_{i=0}^{n} \binom{n}{i} \left(1 + q^{i+1} + q^{2(i+1)} + \cdots + q^{(a-1)(i+1)}\right)(1 + q + q^2 + \cdots + q^i)I_i^{(b,b)}(q)I_{n-i}^{(a,b)}(q).$$

$\square$

Next we define the enumerator $\bar{P}_n^{(a,b)}(q)$ for the complements of $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, \dots, b)$. First, the enumerator $P_n^{(a,b)}(q)$ for the $\mathbf{x}$-parking functions is defined as

$$P_n^{(a,b)}(q) = \sum_{\alpha = (a_1, \dots, a_n)} q^{a_1 + a_2 + \cdots + a_n - n},$$

where $\alpha$ ranges over all $\mathbf{x}$-parking functions of length $n$.

Given an $\mathbf{x}$-parking function $\alpha = (a_1, a_2, \cdots, a_n)$, define its *complement* $\bar{\alpha} = (a + bn - a_1, a + bn - a_2, \cdots, a + bn - a_n)$. Clearly if $c_1 \le c_2 \le \cdots \le c_n$ is the monotonic rearrangement of the terms of $\bar{\alpha}$, then $bi \le c_i \le a + bn - 1$ for $1 \le i \le n$.

Define the *complement enumerator* $\bar{P}_n^{(a,b)}(q)$ of $\mathbf{x}$-parking functions to be the polynomial

$$
\begin{aligned}
\bar{P}_n^{(a,b)}(q) &= \sum_{\bar{\alpha}} q^{c_1 + c_2 + \cdots + c_n - b\binom{n+1}{2}} \\
&= \sum_{\alpha \in B_n} q^{b\binom{n}{2} + an - a_1 - a_2 - \cdots - a_n} = q^{b\binom{n}{2} + (a-1)n} P_n^{(a,b)}(1/q).
\end{aligned}
$$

For $a = 1$, $\bar{P}_n^{(a,b)}(q) = P_n^{(1,b)}(q)$ is the generating function of complements of $b$-parking function studied in [22].

**Theorem 3** *The complement enumerator $\bar{P}_n^{(a,b)}(q)$ of $\mathbf{x}$-parking functions satisfies the recurrence*

$$\bar{P}_0^{(a,b)}(q) = 1, \qquad \bar{P}_1^{(a,b)}(q) = 1 + q + q^2 + \cdots + q^{a-1},$$

$$\bar{P}_{n+1}^{(a,b)}(q) = \sum_{j=0}^{n} \binom{n}{j}(1 + q + q^2 + \cdots + q^{(n-j)b+a-1})(1 + q + \cdots + q^{b-1})^j \bar{P}_j^{(1,1)}(q^b) \bar{P}_{n-j}^{(a,b)}(q). \tag{12}$$

**Proof.** The proof is based on the same idea as the proof of Theorem 7 in [22]. Let $\bar{\alpha} = (a_1, a_2, \dots, a_{n+1})$ be a complement of an $\mathbf{x}$-parking function of length $n+1$. Let $\bar{\alpha}_i = (a_1, a_2, \dots, a_n, a_{n+1} - i)$ for $0 \le i \le a_{n+1}$, and let $\bar{\beta} = \bar{\alpha}_i$ if $\bar{\alpha}_i$ is a complement of an $\mathbf{x}$-parking function, but $\bar{\alpha}_{i+1}$ is not. We call $\bar{\beta}$ the *reduced complement of an $\mathbf{x}$-parking function of length $n+1$*. It is easy to see that if $\bar{\beta}$ is reduced, then the last term of $\bar{\beta}$ must be a multiple of $b$. Let $\mathcal{B}_j$ be the set of all reduced complements of $\mathbf{x}$-parking functions of length $n+1$ with the last term $b(j+1)$, for $j = 0, 1, \cdots, n$. If we define

$$\overline{P}_{n+1,j}^{(a,b)}(q) = \sum_{(a_1, a_2, \dots, a_{n+1}) \in \mathcal{B}_j} q^{a_1 + a_2 + \cdots + a_{n+1} - b\binom{n+2}{2}},$$

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then

$$\overline{P}_{n+1}^{(a,b)}(q) = \sum_{j=0}^{n} \left(1 + q + \cdots + q^{(n-j)b+a-1}\right) \overline{P}_{n+1,j}^{(a,b)}(q). \tag{13}$$

To compute $\overline{P}_{n+1,j}^{(a,b)}(q)$, for any $\bar{\beta} \in \mathcal{B}_j$, assume that $\bar{\beta} = (a_1, a_2, \ldots, a_n, b(j+1))$, and let $c_1 \leq c_2 \leq \cdots \leq c_{n+1}$ be the monotonic rearrangement of the terms of $\bar{\beta}$. Because $\bar{\beta}$ is reduced, it must satisfy the following conditions:

1. $c_{j+1} = b(j+1)$;

2. $bi \leq c_i \leq bj + b - 1$ for $1 \leq i \leq j$; and

3. $bi \leq c_i \leq b(n+1) + a - 1$ for $j + 2 \leq i \leq n+1$.

First consider $(c_1, c_2, \ldots, c_j)$. Every $c_i$ can be uniquely written as $bq_i + r_i$ with $0 \leq r_i < b$. Condition 2 listed above implies that $(c_1, c_2, \ldots, c_j) = b(q_1, q_2, \ldots, q_j) + (r_1, r_2, \ldots, r_j)$ where $q_1 \leq q_2 \leq \cdots \leq q_j$ is the monotonic rearrangement of an ordinary parking function, and $0 \leq r_i \leq b - 1$. Therefore

$$\sum_{i=1}^{j} (c_i - bi) = b \sum_{i=1}^{j} (q_i - i) + \sum_{i=1}^{j} r_i.$$

$$\sum_{(c_1, c_2, \ldots, c_j)} q^{\sum_{i=1}^{j}(c_i - bi)} = (1 + q + \cdots + q^{b-1})^j \sum_{(q_1, q_2, \ldots, q_j)} q^{b \sum_{i=1}^{j}(q_i - i)}.$$

The terms $c_1, \ldots, c_j$ will contribute a factor

$$(1 + q + \cdots + q^{b-1})^j \bar{P}_j^{(1,1)}(q^b)$$

to the enumerator $\bar{P}_{n+1,j}^{(a,b)}(q)$.

Next consider the terms $c_{j+2}, \ldots, c_{n+1}$ in $\bar{\alpha}$. By Condition 3, these terms can be expressed as the sum of the vector $(j+1)b \cdot \mathbf{1}$ and a complement of $\mathbf{x}'$-parking function, where $\mathbf{1} = (1, 1, \ldots, 1)$, $\mathbf{x}' = (a, b, \ldots, b)$, and both of them are of length $n - j$. These terms contribute a factor of $\bar{P}_{n-j}^{(a,b)}(q)$ to $\bar{P}_{n+1,j}^{(a,b)}(q)$.

Combining the above results, we have

$$P_{n+1,j}^{(a,b)}(q) = \binom{n}{j}(1 + q + \cdots + q^{b-1})^j \bar{P}_j^{(1,1)}(q^b) \bar{P}_{n-j}^{(a,b)}(q).$$

And the formula (12) follows immediately. $\square$

Our goal is to prove that the inversion enumerator of sequences of rooted $b$-forests is identical to the sum enumerator of complements of $\mathbf{x}$-parking functions. We do this by introducing certain graphs on $[n]$ which relate to both the sequences of rooted $b$-forests and the complements of $\mathbf{x}$-parking functions. To wit, define a *multicolored $(a,b)$-graph* on $[n]$ to be a graph $G$ on the vertex set $[n]$ such that

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1. The edges of $G$ are colored with colors $\bar{0}, \bar{1}, \ldots, \overline{b-1}$,

2. There are no loops or multiple edges in $G$. But $G$ may have edges with the same endpoints but different colors, and

3. every vertex $r$ is assigned with a subset $f(r)$ of $[a] = \{1, 2, \ldots, a\}$. We say that $r$ is a *root* of $G$ if $f(r) \neq \emptyset$.

4. For any subgraph $H$ of $G$, define $R(H) = \sum_{r \in H} |f(r)|$ to be the number of roots in $H$, counting multiplicity. Every connected component $G'$ of $G$ has at least one root, i.e., $R(G') > 0$.

Denote by $E(G)$ the number of edges of $G$, and by $R(G) = \sum_r |f(r)|$ the number of roots of $G$. Also denote by $V(G)$ the number of vertices of $G$, Let

$$C_n^{(a,b)}(q) = \sum_G q^{E(G)+R(G)-V(G)} = \sum_G q^{E(G)+R(G)-n},$$

where $G$ ranges over all multicolored $(a, b)$-graphs on $[n]$. Set $C_0^{(a,b)}(q) = 1$ for all $a, b \in \mathbb{N}$.

The following result for $a = 1$ is proved in [22].

**Theorem 4 (Yan)**

$$I_n^{(1,b)}(1+q) = \bar{P}_n^{(1,b)}(1+q) = C_n^{(1,b)}(q).$$

Note that there is a trivial bijection between multicolored $(1, 1)$-graphs on $[n]$ and the connected graphs on $[n] \cup \{0\}$: a vertex in a connected graph on $[n] \cup \{0\}$ is adjacent to the vertex $0$ if and only if it is a root in the multicolored $(1, 1)$-graph in $[n]$. Therefore we have the following lemma.

**Lemma 1**

$$C_n^{(1,1)}(q) = \sum_G q^{E(G)-n},$$

*where $G$ ranges over all connected graphs on $[n] \cup \{0\}$.*

**Theorem 5** (a) *We have*

$$I_n^{(a,b)}(1+q) = C_n^{(a,b)}(q) = \sum_G q^{E(G)+R(G)-n}$$

*where $G$ ranges over all multicolored $(a, b)$-graphs on $[n]$.*

(b) *We have*

$$\bar{P}_n^{(a,b)}(1+q)) = C_n^{(a,b)}(q) = \sum_G q^{E(G)+R(G)-n}$$

*where $G$ ranges over all multicolored $(a, b)$-graphs on $[n]$.*

(c) *It follows that*

$$
\begin{aligned}
\bar{P}_n^{(a,b)}(q) &= I_n^{(a,b)}(q), \\
q^{b\binom{n}{2}+(a-1)n} I_n^{(a,b)}(1/q) &= P_n^{(a,b)}(q) = \sum_{\alpha \in A_n} q^{a_1+a_2+\cdots+a_n-n}.
\end{aligned}
$$

**Proof.** (a) It is easy to check that

$$C_0^{(a,b)}(q) = I_0^{(a,b)}(1+q) = 1, \quad C_1^{(a,b)}(q) = \left((1+q)^a - 1\right)/q = I_1^{(a,b)}(1+q).$$

To show $C_n^{(a,b)}(q) = I_n^{(a,b)}(1+q)$, it suffices to show that $C_n^{(a,b)}(q)$ satisfies the same recurrence as $I_n^{(a,b)}(1+q)$. That is, (by Equation (9)),

$$C_{n+1}^{(a,b)}(q) = \sum_{i=0}^{n} \binom{n}{i} \frac{(q+1)^{a(i+1)} - 1}{q} C_i^{(b,b)}(q) C_{n-i}^{(a,b)}(q). \tag{14}$$

Once the recurrence (14) is proved, Theorem 4 implies that $C_n^{(b,b)}(q) = I_n^{(b,b)}(1+q)$ for all $n$. This in turn implies that $C_n^{(a,b)}(q) = I_n^{(a,b)}(1+q)$ by the recurrence (14).

Let $G$ be a multicolored $(a,b)$-graph on $[n+1]$, and let $G_0$ be the connected component containing the vertex 1. Assume $G_0$ has $i+1$ vertices. Then $G_1 = G \setminus G_0$ is a multicolored $(a,b)$-graph on $n - i$ vertices. Thus

$$
\begin{aligned}
C_{n+1}^{(a,b)}(q) &= \sum_{(G_0, G_1)} q^{E(G_0)+R(G_0)-V(G_0)} \cdot q^{E(G_1)+R(G_1)-V(G_1)} \\
&= \sum_{i=0}^{n} \binom{n}{i} C_{n-i}^{(a,b)}(q) \sum_{G_0 : |G_0| = i+1} q^{E(G_0)+R(G_0)-(i+1)}.
\end{aligned}
\tag{15}
$$

The edges of $G_0$ can be formed as follows. Take a multicolored $(b,b)$-graph $H$ on $G_0 \setminus \{1\}$. We merge the vertex 1 and the graph $H$ to get edges of $G_0$: First any edge in $H$ is unchanged in $G_0$; second, if $r$ is a root in $H$ whose assigned subset is $f_H(r) \subset \{1, 2, \dots, b\}$, then we connected $r$ with 1 by edges of colors $\overline{k-1}$, for all $k \in f_H(r)$, and view $r$ as an ordinary vertex.

Next we compute

$$\sum_{G_0} q^{R(G_0)} = \sum_{G_0} q^{\sum_{r \in G_0} |f(r)|}.$$

A vertex $r$ of $G_0$ may be assigned with any subset $f(r)$ of $[a] = \{1, 2, \dots, a\}$. By the definition of multicolored $(a,b)$-graph, the only case prohibited is $f(r) = \emptyset$ for all $r \in G_0$. Therefore

$$\sum_{G_0 : |G_0| = i+1} q^{E(G_0)+R(G_0)-(i+1)} = q^{-1} \sum_{G_0} q^{E(G_0)-i} \sum_{G_0} q^{R(G_0)} = q^{-1} C_i^{(b,b)}(q) \left((1+q)^a - 1\right).$$

Substituting this into Equation (15), we obtain the recurrence (14).

(b) To show $\bar{P}_n^{(a,b)}(1+q)) = C_n^{(a,b)}(q)$, it suffices to show that they satisfy the same recurrence relation. By Equation (12), we need to show

$$C_{n+1}^{(a,b)}(q) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{(1+q)^{(n-j)b+a} - 1}{q}\right) \left(\frac{(1+q)^b - 1}{q}\right)^j C_j^{(1,1)}((1+q)^b - 1) C_{n-j}^{(a,b)}(q). \tag{16}$$

Given a multicolored $(a,b)$-graph $G$ on $[n+1] = \{1, 2, \dots, n+1\}$, take away the vertex 1 and all the edges connected to 1. Assume that $F_1, F_2, \dots, F_t$ are the connected components of $G \setminus \{1\}$,

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where $F_1, F_2, \ldots, F_r$ ($r \leq t$) are those which do not have any root, i.e., $R(F_i) = 0$ for $1 \leq i \leq r$. Let $K$ be the induced subgraph containing $F_1 \cup F_2 \cup \cdots \cup F_r \cup \{1\}$, and let $L$ be $G \setminus K$. Furthermore assume the number of vertices of $K$ is $j + 1$. We have

$$
\begin{aligned}
C_{n+1}^{(a,b)}(q) &= \sum_G q^{E(G)+R(G)-(n+1)} \\
&= \sum_{K \cup L} q^{E(K)+R(K)-(j+1)} \cdot q^{E(L)+R(L)-(n-j)} \cdot q^{d(G)},
\end{aligned}
\tag{17}
$$

where $G$ ranges over all multicolored $(a, b)$-graphs on $[n + 1]$, and where $d(G)$ is the number of edges between vertex 1 and the subgraph $L$.

There are $\binom{n}{j}$ ways to choose $K$. Once the vertices of $K$ are fixed, since $F_1, F_2, \ldots, F_r$ do not contain any roots, they must connected to the vertex 1. So $K$ is a connected $(a, b)$-graph on $j + 1$ vertices with at most one non-empty root-set at vertex 1. By Lemma 1,

$$
q^j C_j^{(1,1)}(q) = \sum_P q^{E(P)}
\tag{18}
$$

where $P$ ranges over all connected $j + 1$ graphs. Apply Equation (18) to $K$, and note that in $K$, instead of a single edge between a pair of vertices, there may exist multiple edges with colors from the set $M$, where $\emptyset \neq M \subset \{\bar{0}, \bar{1}, \ldots, \overline{b-1}\}$. Hence

$$
\begin{aligned}
\sum_K q^{E(K)-(j+1)} &= q^{-(j+1)} \sum_G \left( \sum_{i=1}^k \binom{k}{i} q^i \right)^{E(K)} \\
&= q^{-(j+1)} C_j^{(1)} \left( (1+q)^b - 1 \right) \cdot \left( (1+q)^b - 1 \right)^j \\
&= q^{-1} C_j^{(1)} \left( (1+q)^b - 1 \right) \left( \frac{(1+q)^b - 1}{q} \right)^j.
\end{aligned}
\tag{19}
$$

It is obvious that $L$ is a multicolored $(a, b)$-graph on $n - j$ vertices. Hence

$$
\sum_L q^{E(L)+R(L)-(n-j)} = C_{n-j}^{(a,b)}(q).
$$

To count $d(G)$, the number of edges between vertex 1 and the subgraph $L$, we need to distinguish two cases.

1. $f(1) = \emptyset$. In this case, $1 \leq d(G) \leq b(n - j)$.

2. $f(1) \neq \emptyset$. In this case, $0 \leq d(G) \leq b(n - j)$. Also note that the set $f(1)$ assigned to the root 1 can be any non-empty subset of $[a]$.

Therefore we have

$$
\begin{aligned}
C_{n+1}^{(a,b)}(q) &= \sum_{K \cup L} q^{E(L)+R(L)-(n-j)} \cdot q^{E(K)-(j+1)} \cdot q^{R(K)+d(G)} \\
&= \sum_{j=0}^{n} \binom{n}{j} C_{n-j}^{(a,b)}(q) q^{-1} \left( \frac{(1+q)^b - 1}{q} \right)^j C_j^{(1,1)} \left( (1+q)^b - 1 \right) \\
&\qquad \cdot \left( \left( (1+q)^a - 1 \right) \cdot \sum_{i=0}^{b(n-j)} \binom{(n-j)b}{i} q^i + \sum_{i=1}^{b(n-j)} \binom{(n-j)b}{i} q^i \right) \\
&= \sum_{j=0}^{n} \binom{n}{j} \left( \frac{(1+q)^b - 1}{q} \right)^j \left( \frac{(1+q)^{(n-j)b+a} - 1}{q} \right) C_j^{(1,1)} \left( (1+q)^b - 1 \right) C_{n-j}^{(a,b)}(q).
\end{aligned}
$$

This finishes the proof.

(c) It is follows immediately from the above results. $\square$

The following corollary follows from Stanley [20], Theorem 3.3.

**Corollary 5.1** *The generating function identity holds:*

$$
\sum_{n \geq 0} I_n^{(a,b)}(q)(q-1)^n \frac{x^n}{n!} = \frac{\sum_{n \geq 0} q^{an + b\binom{n}{2}} \frac{x^n}{n!}}{\sum_{n \geq 0} q^{b\binom{n}{2}} \frac{x^n}{n!}}. \tag{20}
$$

**Proof.** The proof is a straightforward extension of Stanley's ([20], Theorem 3.3, (b)). Let

$$
T_n^{(b)}(q) = \sum_G q^{E(G)},
$$

where $G$ ranges over all connected graphs on $[n]$ with $b$-colors, with no loops and with no multiple edges of the same color. (We do not assign any root-set to vertices.) Without the condition that $G$ is connected, the corresponding generating function is $(1+q)^{b\binom{n}{2}}$. Hence by the exponential formula, (e.g., [18], Chapter 5), we have

$$
F^{(b)}(x) := \sum_{n \geq 1} T_n^{(b)}(q) \frac{x^n}{n!} = \log \left( \sum_{n \geq 1} (1+q)^{b\binom{n}{2}} \frac{x^n}{n!} \right).
$$

We can get a multicolored $(a,b)$-graph on $[n]$ by choosing a partition $\pi = \{B_1, \ldots, B_j\}$ of the set $[n]$, placing a graph enumerated by $T_n^{(b)}(q)$ on each block $B_i$, and assign a set $f(r) \subseteq [a]$ of roots to each vertex $r$ of $B_i$ such that $\bigcup_{r \in B_i} f(r) \neq \emptyset$. Hence

$$
q^n C_n^{(a,b)}(q) = \sum_{\pi = \{B_1, \ldots, B_j\}} T_{b_1}^{(b)}(q) \cdots T_{b_j}^{(b)}(q) [(1+q)^{ab_1} - 1] \cdots [(1+q)^{ab_j} - 1].
$$

15

where $\pi$ ranges over all partitions on $[n]$, and $b_i = \#B_i$. Again by the exponential formula we get

$$
\begin{aligned}
\sum_{n \geq 0} q^n C_n^{(a,b)}(q) \frac{x^n}{n!} &= \exp\left(F^{(b)}((1+q)^a x) - F^{(b)}(x)\right) \\
&= \exp\left(\log \sum_{n \geq 0}(1+q)^{b\binom{n}{2}} \frac{(1+q)^{an} x^n}{n!} - \log \sum_{n \geq 0}(1+q)^{b\binom{n}{2}} \frac{x^n}{n!}\right) \\
&= \frac{\sum_{n \geq 0}(1+q)^{an+b\binom{n}{2}} \frac{x^n}{n!}}{\sum_{n \geq 0}(1+q)^{b\binom{n}{2}} \frac{x^n}{n!}}.
\end{aligned}
$$

Now substitute $q - 1$ for $q$ and use Theorem 5 to get the desired formula. $\square$

# 4 Depth-first and breadth-first searches

In Section 3 we proved an identity between the inversion enumerator of sequences of rooted $b$-forests, and the sum enumerator of complements of $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, \ldots, b)$ by analyzing the recurrences satisfied by the corresponding generating functions. It is natural to ask for combinatorial correspondences between sequences of rooted forests, generalized parking functions, and labeled multicolored $(a, b)$-graphs. In this section we establish such correspondences using depth-first search and breadth-first search algorithms.

## 4.1 Depth-first search on multicolored graphs

Let $a = b = 1$. Then $I_n(q) = I_n^{(1,1)}(q)$ is the ordinary inversion enumerator of labeled rooted forests, and $C_n(q) = C_n^{(1,1)}(q)$ is the excess *edge and root* enumerator of multicolored $(1, 1)$-graphs. In each rooted structure described above, adjoining a vertex $0$ and replacing a root $i$ with an edge connecting $i$ and $0$, we obtain a labeled tree or a connected graph on $[n] \cup \{0\}$. As proved implicitly by Mallows and Riordan [11], and more explicitly by Kreweras [10],

$$
I_n(1+q) = C_n(q) = \sum_G q^{E(G)-n}, \tag{21}
$$

where $G$ ranges over all connected graphs on $[n] \cup \{0\}$, and $I_n(q) = \sum_T q^{\mathrm{inv}(T)}$ enumerates the inversions of labeled trees on $[n] \cup \{0\}$.

In [3] Gessel and Wang established a connection between labeled connected graphs and inversions of trees by a depth-first search (DFS) algorithm. This algorithm leads to an elegant combinatorial proof of the identity (21), which can be easily extended to an algorithm on the multicolored $(a, b)$-graphs.

To describe the DFS on multicolored graphs, we recall the construction on connected graphs by Gessel and Wang. Let $G$ be a connected graph on $[n] \cup \{0\}$. The DFS algorithm applies to $G$ and return a certain spanning tree $\mathcal{T}(G)$ by the following procedure: We start at vertex $0$, and at each step we go to the greatest adjacent unvisited vertex if there is one, otherwise, we backtrack. For example, from the graph in Figure. 2, we get the spanning tree in Figure. 3.
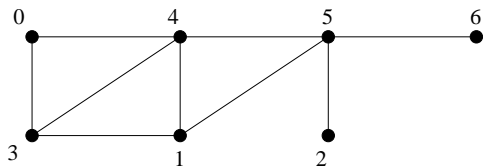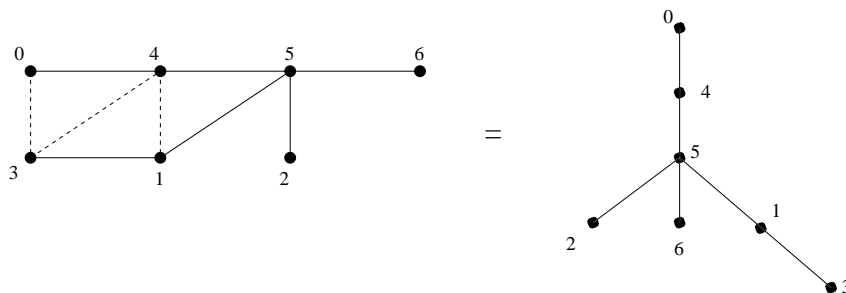
Figure 2: A connected graph on $[6] \cup \{0\}$



Figure 3: Spanning tree of Figure 2 found by DFS

Fix a labeled tree $T$ on $[n] \cup \{0\}$. Let $\mathcal{G}(T)$ be the set of connected graphs $G$ for which $\mathcal{T}(G) = T$. Define a set $\mathcal{E}(T)$ of edges not in $T$ whose elements are in one-to-one correspondence with the inversions of $T$: To every inversion $(j, k)$, $(j > k > 0)$, associate the edge between $k$ and the predecessor of $j$. For the above tree, the edge $\mathcal{E}(T)$ are indicated by dotted lines.
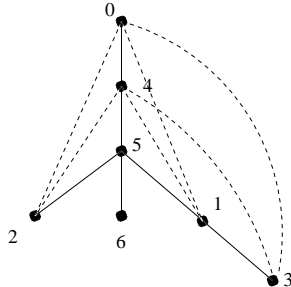


Figure 4: The set $\mathcal{E}(T)$ of edges associated to inversions

Gessel and Wang characterized the set of connected graphs in $\mathcal{G}(T)$.

**Theorem 6 (Gessel & Wang)** $\mathcal{G}(T)$ *consists of those graphs obtained from $T$ by adjoining some edges in $\mathcal{E}(T)$.*

An immediate consequence of Theorem 6 is

$$\sum_{G \in \mathcal{G}(T)} q^{E(G)-n} = (1+q)^{\mathrm{inv}(T)}.$$

Adding over all trees $T$ in $[n] \cup \{0\}$ yields

$$C_n(q) = \sum_G q^{E(G)-n} = \sum_T q^{\mathrm{inv}(T)} = I_n(1+q).$$

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Now we extend the DFS to the set of multicolored $(a,b)$-graphs. Given a multicolored $(a,b)$-graph $G$ on $[n]$, first replace the root-sets by colored edges as follows. Adjoin the vertex 0. For each vertex $r \in [n]$ with assigned root-set $f(r) \subseteq [a]$, replace $f(r)$ by edges connecting $r$ and 0 with colors $\{\overline{k-1}|k \in f(r)\}$. The resulted graph on $[n] \cup \{0\}$ has multicolored edges with no roots. Denote it by $G'$. Apply the DFS to the set of multicolored graphs $G'$. The algorithm starts at vertex 0. At each step, we go to the unvisited vertex for which the incident edge is of the greatest color. If there are more than one unvisited vertices with the same greatest edge color, we go to the largest one. If there is no such vertex, we backtrack. For example, for the multicolored graph $G'$ in Figure. 5 with $a = b = 2$, the DFS algorithm gives the multicolored spanning tree in Figure. 6.
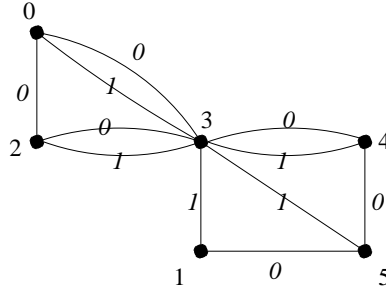


Figure 5: A multicolored graph $G'$ on $[5] \cup \{0\}$. Edge colors are in *italics*.
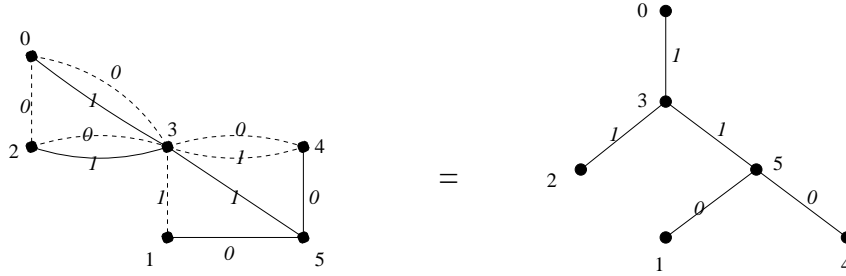


Figure 6: Spanning tree found by DFS for the multicolored graph in Figure 5

The output of the DFS algorithm is a spanning tree $T' = \mathcal{T}(G')$ on vertices $[n] \cup \{0\}$, where the edges of $T'$ are colored by $\bar{0}, \bar{1}, \ldots, \overline{b-1}$, and the edges connecting to 0 are colored by $\bar{0}, \bar{1}, \ldots, \overline{a-1}$. Call $T'$ a *multicolored tree* on $[n] \cup \{0\}$. There is a trivial bijection between such structures and the sequences $(S_1, S_2, \cdots, S_a)$ of rooted $b$-forests on $[n]$ by replacing edges incident to 0 with root-sets. Given a multicolored tree $T'$ on $[n] \cup \{0\}$, let $\mathcal{G}'(T')$ be the set of multicolored graphs $G'$ for which $\mathcal{T}(G') = T'$. As in the case of connected graphs, we define a set $\mathcal{E}'(T')$ of edges not in $T'$ whose elements are in one-to-one correspondence with the $(a,b)$-inversion of $T'$.

Recall for a labeled multicolored tree $T'$ on $[n] \cup \{0\}$, the $(a,b)$-inversion $\mathrm{inv}^{(a,b)}(T)$ is defined by

$$\mathrm{inv}^{(a,b)}(T') = \mathrm{inv}(T') + + \sum_{x \in [n]} \sum_{e \in K(x)} \kappa(e),$$

where $\mathrm{inv}(T')$ is the number of inversions of $T'$ as an ordinary labeled tree on $[n] \cup \{0\}$, $K(x)$ is

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the set of edges lying between $x$ and 0, and $\kappa(e)$ is the color of the edge $e$. The set $\mathcal{E}'(T')$ is formed by the following rules, (see Figure 7).

1. For every ordinary inversion $(j, k)$ where $j > k > 0$ and $j$ lies on the unique path from 0 to $k$, we associate an edge between $k$ and $i$, where $i$ is the predecessor of $j$. The color of this edge is $\kappa(ij)$.

2. For every pair $(p, m)$ where $p \geq 0$ lies on the unique path from 0 to $m$, let $n$ be the vertex on the path right after $p$. If the edge $pn$ is of color $k$, we associate $k$ edges connecting vertices $p$ and $m$ with colors $\bar{0}, \bar{1} \ldots, \overline{k-1}$.

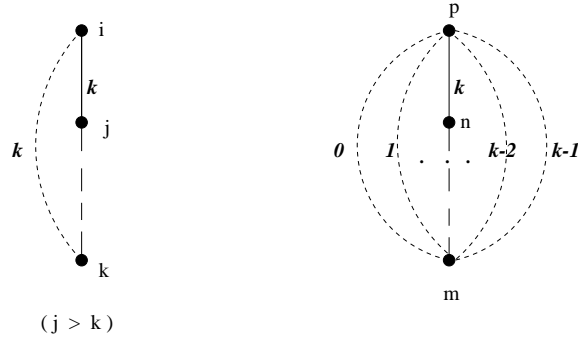(In the following figure, the dotted lines indicate the edges in $\mathcal{E}'(T')$. )



Figure 7: The edge set $\mathcal{E}'(T')$

For the multicolored tree in Figure 6, the edges of $\mathcal{E}'(T')$ are 02, 31, 01, 34 with color $\bar{1}$ by Rule 1; and 03, 02, 05, 01, 04, 32, 35, 31, 34 with color $\bar{0}$ by Rule 2.

Similar to the Theorem of Gessel and Wang's, we have

**Theorem 7** *For a multicolored tree $T'$ on $[n] \cup \{0\}$, the set $\mathcal{G}'(T')$ consists of those multicolored graphs $G'$ obtained from $T'$ by adjoining some edges in $\mathcal{E}'(T')$.*

**Proof.** Let $S'$ be a subset of $\mathcal{E}'(T)$ and let $G' = S' \cup T'$. If we perform the DFS on the multicolored graph $G'$, the spanning tree $\mathcal{T}(G')$ will be precisely $T'$ because

1. The first vertex to be visited is the same, namely, 0.

2. If $v_1, v_2, \ldots, v_k$, the first $k$ vertices visited by the DFS, and the edges connecting them used in the DFS, coincide with those of $T'$, then $v_{k+1}$, the $(k+1)^{\text{th}}$ vertex, and the color of $v_k v_{k+1}$, will still be the same as those of $T'$. The reason is, $v_k$ is connected by an edge in $\mathcal{E}'(T')$ to a vertex $j$ only if $\kappa(v_k j) < \kappa(v_k v_{k+1})$, or $\kappa(v_k j) = \kappa(v_k v_{k+1})$ but $j < v_{k+1}$. In DFS, we follow the edge with greatest color, and go to the largest vertex first. Thus we will follow exactly the same search order as in $T'$.

Conversely, if $v_1, v_2, \ldots, v_k$ are the first $k$ vertices visited in a multicolored graph $G'$ with $\mathcal{T}(G') = T'$, then in order to go to $v_{k+1}$ along the edge $v_k v_{k+1} \in T'$ in the next step, $v_k$ must not connect to any unvisited vertex $j$ such that $\kappa(v_k j) > \kappa(v_k v_{k+1})$, or $\kappa(v_k j) = \kappa(v_k v_{k+1})$ but $j > v_{k+1}$. This completes the proof. $\square$

**Corollary 7.1**

$$\sum_{G' \in \mathcal{G}'(T')} q^{E(G)-n} = (1+q)^{\mathrm{inv}^{(a,b)}(T')}.$$

*Adding over all multicolored trees $T'$ on $[n] \cup \{0\}$ yields*

$$C_n^{(a,b)}(q) = \sum_{\substack{G \ multicolored \\ (a,b)\text{-}graphs \ on \ [n]}} q^{E(G)+R(G)-n} = \sum_{T'} (1+q)^{\mathrm{inv}^{(a,b)}(T')} = I_n^{(a,b)}(1+q).$$

## 4.2 Breadth-first search on multicolored graphs

Another algorithm which gives a spanning tree in a connected graph is the breadth-first search (BFS) algorithm. It was used by J. Spencer [16] to develop an exact formula for the number of labeled connected graphs on $[n]$ with $n-1+k$ edges ($k$ fixed) in terms of appropriate expectations. Moving to asymptotics, Spencer showed that the expectations can be expressed in terms of a certain restricted Brownian motion. In this section, we will use the BFS to establish a combinatorial correspondence between labeled connected graphs and ordinary parking functions, and extend this correspondence to multicolored $(a,b)$-graphs and $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, \ldots, b)$.

First we state the BFS algorithm in the case $a = b = 1$, i.e., in connected graphs on $[n] \cup \{0\}$. It can be described as a queue $Q$ that starts at vertex 0. (We follow the description of [16]). At each stage we take the vertex $x$ at the head of the queue, remove $x$ from the queue, and add all unvisited neighbors of $x$ to the queue, in the numerical order. We will call that operation "processing $x$". For the connected graph in Figure 2, the BFS given the spanning tree
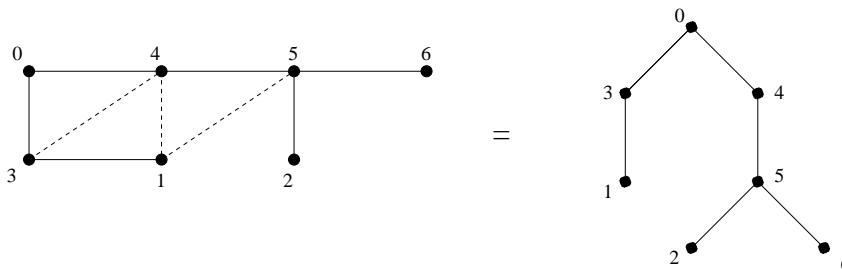


Figure 8: Spanning tree of Figure 2 found by BFS

The queue $Q$ at each stage $t$ is

| t | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Q | 0 | 3 | 4 | 1 | 5 | 2 | 6 | $\emptyset$ |
|   |   | 4 | 1 | 5 |   | 6 |   |   |
| $x_t$ |   | 2 | 1 | 1 | 0 | 2 | 0 | 0 |
| $q_t$ | 1 | 2 | 2 | 2 | 1 | 2 | 1 | 0 |

Let $x_t$ be the number of vertices found by the $t^{th}$ vertex processed (not vertex number $t$), which in our example, are $(x_1, x_2, \ldots, x_7) = (2, 1, 1, 0, 2, 0, 0)$. Note that $x_{n+1} = 0$ always. Let $q_t$ be the

size of the queue after the $t^{th}$ vertex is processed. Then $q_0 = 1$ and $q_t = q_{t-1} + x_t - 1$, which, in our example, are 1, 2, 2, 2, 1, 2, 1, 0. For a connected graph $G$, the necessary and sufficient conditions on the sequence $q_t$ are

$$q_{n+1} = 0, \qquad \text{and} \qquad q_i > 0 \text{ for } i \le n, \tag{22}$$

which are equivalent to

$$x_1 + \cdots x_i \ge i \text{ for } i < n, \qquad x_1 + \cdots + x_n = n.$$

Also note that the queue uniquely determines a permutation of $n$, namely, the order that the vertices are processed in the queue. In our example, it is (341526).

For a labeled tree $T$ with root 0, denote by $\sigma(T)$ the permutation described above, and let $x(T)$ be the vector $(x_1, x_2, \ldots, x_n)$. Clearly in the permutation $\sigma(T) = a_1 a_2 \cdots a_n$, the term $a_i$'s are increasing on every interval of the form $\{1 + \sum_{i=1}^{k} x_i, 2 + \sum_{i=1}^{k} x_i, \ldots, \sum_{i=1}^{k+1} x_i\}$. That is, the permutation $\sigma^{-1}(T)$ is *compatible* with the vector $x(T)$ (cf. Section 2, Page 3). In [1], Foata and Riordan gave an explicit bijection between the set of pairs $(x(T), \sigma^{-1}(T))$ and the set of ordinary parking functions. (Also see Section 2.1 with $a = b = 1$). The parking function $\alpha(T) = (b_1, \ldots, b_n)$ correspondents to $(x(T), \sigma^{-1}(T))$ is the sequence whose terms are $1^{x_1}, 2^{x_2}, \ldots, n^{x_n}$, and whose order is given by $\sigma^{-1}(T)$. Precisely,

$$\sigma^{-1}(T)(i) = \mathrm{Card}\{j \in [n] \mid b_j < b_i, \text{ or } b_j = b_i \text{ and } j < i\},$$

or equivalently,

$$b_j = \begin{cases} 1, & \text{if } 1 \le \sigma^{-1}(T)_j \le x_1, \\ i, & \text{if } \sum_{k=1}^{i-1} x_k < \sigma^{-1}(T)_j \le \sum_{k=1}^{i} x_k. \end{cases}$$

In the example, $\sigma^{-1}(T) = (351246)$, so the corresponding parking function is (251135) whose terms written in specification is $1^2 2^1 3^1 4^0 5^2 6^0$. The sum of the terms in the parking function $\alpha(T)$ is $\sum_i b_i = \sum_{i=1}^{n} i x_i$.

Let $\mathcal{G}_1(T)$ be the set of connected graphs for which the spanning tree found by the BFS is $T$. A crucial observation is made by Spencer [16]: An edge $(i, j)$ can be added to $T$ without changing the spanning tree under the BFS if and only if in the queue, when the first of the two vertices was processed, the other was currently in the queue. In our example, 34, 41, 15, 26 could be added to $T$. Let $\mathcal{E}_1(T)$ be the set of all such edges. It follows that

**Theorem 8 (Spencer)** $\mathcal{G}_1(T)$ *consists of these graphs obtained from $T$ by adjoining some edges in $\mathcal{E}_1(T)$.*

Thus

$$\sum_{G \in \mathcal{G}_1(T)} q^{E(G)-n} = (1+q)^{|\mathcal{E}_1(T)|}.$$

Now we compute $|\mathcal{E}_1(T)|$. From the queue $Q$, we have

$$|\mathcal{E}_1(T)| = \sum_{i=1}^{n} (q_i - 1) = \sum_{i=1}^{n} (x_1 + \cdots + x_i - i) = \sum_{i=1}^{n} (n+1-i) x_i - \binom{n+1}{2}.$$

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Since $\sum_{i=1}^{n} x_i = n$, the above number equals

$$(n+1)\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} ix_i - \binom{n+1}{2} = \binom{n+1}{2} - \sum_{i=1}^{n} ix_i.$$

Comparing with the parking function $\alpha(T)$, we have $|\mathcal{E}_1(T)| = \binom{n+1}{2} - \sum_{i=1}^{n} b_i$ if $\alpha(T) = (b_1, b_2, \ldots, b_n)$. Adding over all trees $T$ on $\{0\} \cup [n]$ yields

$$C_n(q) = \sum_{G \text{ connected}} q^{E(G)-n} = \sum_{(b_1,\ldots,b_n)\in\text{park(n)}} q^{\binom{n+1}{2}-\sum_{i=1}^{n} b_i} = \bar{P}_n^{(1,1)}(1+q),$$

where park$(n)$ is the set of all ordinary parking functions of length $n$.

To extend the BFS to multicolored $(a, b)$-graphs, we need some modifications on the algorithm. Given a multicolored $(a, b)$-graph $G_1$, as in the DFS, we replace the root-sets with colored edges by adjoining vertex 0, and for each vertex $r$ with assigned root-set $f(r) \subseteq [a]$, replacing the root-set be edges connecting 0 and $r$ with colors $\{\overline{k-1}|k \in f(r)\}$. The BFS is applied to the resulted multicolored graph $G_1'$ on $[n] \cup \{0\}$.

The BFS on $G_1'$ again is described as a queue $Q$ that starts at vertex 0. So $Q_0 = (0)$, and $q_0 = 1$, where $q_t$ is the size of the queue at time $t$. For $t = 1, \ldots, a - 1$, form $Q_t$ from $Q_{t-1}$ by adding all unvisited vertices which are connected to 0 with edges of color $\overline{t-1}$. The vertices are always added in numerical order. For $t = a$, remove vertex 0 from the queue, and add all unvisited vertices which are connected to 0 with edges of color $\overline{a-1}$. We say the vertex 0 is processed during time 0 to $a - 1$. It is the first vertex to be processed.

In general, for $t = a + ib + r$, $(1 \le r \le b - 1)$, let $x$ be the head of the queue at time $t - 1$. Form $Q_t$ from $Q_{t-1}$ by adding all unvisited vertices which are connected to $x$ with edges of color $\overline{r-1}$. For $t = a + (i + 1)b$, form $Q_t$ from $Q_{t-1}$ by removing $x$ from the queue and adding all unvisited vertices which are connected to $x$ with edges of color $\overline{b-1}$. We say that the vertex $x$ is processed during time $a + ib$ to $a + (i + 1)b - 1$. For the multicolored graph in Figure 5 with $a = b = 2$, the BFS gives the multicolored spanning tree $T'$
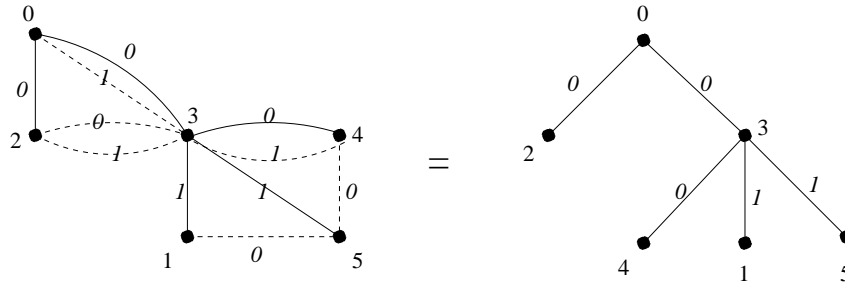


Figure 9: Spanning tree found by BFS for the multicolored graph in Figure 5

The queue for the BFS on the multicolored graph in Figure 5 is illustrated in the following table.

| t | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|
| Q | 0 | 0 | 2 | 2 | 3 | 3 | 4 | 4 | 1 | 1 | 5 | 5 | ∅ |
|   |   | 2 | 3 | 3 |   | 4 | 1 | 1 | 5 | 5 |   |   |   |
|   |   | 3 |   |   |   | 5 | 5 |   |   |   |   |   |   |
| $x_t$ |   | 2 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $q_t$ | 1 | 3 | 2 | 2 | 1 | 2 | 3 | 3 | 2 | 2 | 1 | 1 | 0 |

In the above table $x_t$ is the number of vertices that join the queue at time $t$. Note that we always have $x_{a+(n-1)b+r} = 0$ for $r = 1, \ldots, b$, since the last vertex in the queue has no unvisited neighbors in the multicolored graph $G_1'$. The relation between the sequence $x_t$ and $q_t$ are

$$q_0 = 1, \qquad q_i = q_{i-1} + x_i, \ (1 \le i \le a-1), \qquad q_a = q_{a-1} + x_a - 1. \tag{23}$$

And in general,

$$\begin{aligned} q_t &= q_{t-1} + x_t &&\text{for} \quad t = a + ib + r, \ 1 \le r \le b-1, \\ q_t &= q_{t-1} + x_t - 1, &&\text{for} \quad t = a + (i+1)b. \end{aligned}$$

A sequence $q_t$ is the sizes of $Q_t$ for a connected multicolored graph $G_1'$ if and only if

$$q_{a+nb} = 0, \qquad \text{and} \qquad q_i > 0 \text{ for } i < a + nb,$$

which are equivalent to

$$\begin{aligned} x_1 + x_2 + \cdots + x_{a+ib} &\ge i+1, \quad i = 0, 1, \ldots, n-2, \\ x_1 + x_2 + \cdots + x_{a+(n-1)b} &= n. \end{aligned}$$

(Compare with Formula (1) in Sec. 2.) Also note that the queue $Q$ uniquely determine a permutation of $n$, namely, the order that the vertices are processed (except 0). In the example, it is (23415). Denote by $\sigma(T')$ this permutation. Then the terms of $\sigma(T')$ are increasing on every interval of the form $\{1 + \sum_{i=1}^{k} x_i, 2 + \sum_{i=1}^{k} x_i, \ldots, \sum_{i=1}^{k+1} x_i\}$. That is, the permutation $\sigma^{-1}(T')$ is compatible with the vector $x(T') = (x_1, \ldots, x_{a+(n-1)b})$. In Section 2.1 we construct a bijection between the set of pairs $(x(T'), \sigma^{-1}(T'))$ and the set of $\mathbf{x}$-parking functions for $\mathbf{x} = (a, b, \ldots, b)$. The explicit formula is given in (4) where we substitute $\sigma^{-1}(T')$ for $\pi$. Denote by $\alpha(T') = (b_1, b_2 \ldots, b_n)$ the $\mathbf{x}$-parking function corresponding to $(x(T'), \sigma^{-1}(T'))$. In our example, $\sigma(T') = (23415)$, so $\sigma^{-1}(T') = (41235)$. Then $\alpha(T') = (6, 1, 1, 5, 6)$ whose terms written in specification is $(i^{x_i}) = 1^2 2^0 3^0 4^0 5^1 6^2 7^0 8^0 9^0 10^0 11^0 12^0$. The sum of the terms in $\alpha(T')$ is $\sum_i b_i = \sum_{i=1}^{a+(n-1)b} i x_i$.

Fixed a multicolored tree $T'$ on $[n] \cup \{0\}$. An edge $(m, n)$ with color $k$ can be added to $T'$ without changing the spanning tree under the breadth-first search if and only if in the queue, when the first of the two vertices, say $m$, is being processed, the other vertex, $n$, was currently in the queue, and the edge connecting $m$ and $n$ in $T'$ is of a color $\bar{k}' < \bar{k}$, (if there is such an edge). Another way to state this is, when $m$ is processed during time $t = a + ib$ to $t = a_{(}i+1)b - 1$, $n$ is in the queue at the time $t = a + ib + k$. Fix $m$, there are $q_{a+ib+k} - 1$ many such pairs $(m, n)$. In our example, the following edges can be added to $T'$ without changing the result of the BFS: 23 $(t = 2)$, 41, 45$(t = 6)$, 15 $(t = 8)$ with color $\bar{0}$; 02, 03 $(t = 1)$, 23$(t = 3)$, 34 $(t = 5)$, 41, 45$(t = 7)$, 15 $(t = 9)$ with color $\bar{1}$. Let $\mathcal{E}_1'(T')$ be the set of all such edges, and let $\mathcal{G}_1'(T')$ be the set of multicolored graphs for which the spanning tree found by the BFS is $T'$. The above argument shows

**Theorem 9** $\mathcal{G}'_1(T')$ *consists of those graphs obtained from* $T'$ *by adjoining some edges in* $\mathcal{E}'_1(T')$.

Finally we compute $|\mathcal{E}'(T')|$. From the structure of the queue $Q$, we have

$$
\begin{aligned}
|\mathcal{E}'_1(T')| &= \sum_{i=1}^{a+nb-1} (q_i - 1) = \left( \sum_{i=1}^{a+nb-1} q_i \right) - (a + nb - 1) \\
&= \sum_{i=1}^{a+nb-1} (a + nb - i)x_i - b\binom{n}{2} + (a-1) - (a + bn - 1) \\
&= (a + nb) \sum_{i=1}^{a+nb-1} x_i - \sum_{i=1}^{a+nb-1} ix_i - b\binom{n+1}{2} \\
&= an + b\binom{n}{2} - \sum_{i=1}^{a+nb-1} ix_i.
\end{aligned}
$$

Comparing with the **x**-parking function $\alpha(T')$, we have $|\mathcal{E}'_1(T')| = an + b\binom{n}{2} - \sum_i b_i$ if $\alpha(T') = (b_1, b_2, \ldots, b_n)$. Adding over all trees $T$ on $[n] \cup \{0\}$ yields

**Corollary 9.1**

$$
C_n^{(a,b)}(q) = \sum_{\substack{G'_1 \ multicolor \ on \\ [n] \cup \{0\}}} q^{E(G)-n} = \sum_{\substack{(b_1,\ldots,b_n) \\ \mathbf{x}\text{-}parking \ functions}} q^{an+b\binom{n}{2}-\sum_{i=1}^n b_i} = \bar{P}_n^{(a,b)}(1+q).
$$

Combining with the results of depth-first search in Section 4.1, one has

$$
I_n^{(a,b)}(1+q) = \sum_{\substack{G \ multicolored \\ (a,b)\text{-graph on } [n]}} q^{E(G)+R(G)-n} = \bar{P}_n^{(a,b)}(1+q).
$$

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