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Inverting operations in operads

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ABSTRACT

We construct a localization for operads with respect to one-ary operations based on the Dwyer-Kan hammock localization [2]. For an operad $\mathscr{O}$ and a sub-monoid of one-ary operations $\mathcal{W}$ we associate an operad $L\mathscr{O}$ and a canonical map $\mathscr{O} \to L\mathscr{O}$ which takes elements in $\mathcal{W}$ to homotopy invertible operations. Furthermore, we give a functor from the category of $\mathscr{O}$-algebras to the category of $L\mathscr{O}$-algebras satisfying an appropriate universal property.

1. Introduction

If $\mathscr{O}$ is an operad in simplicial sets with n-ary operations $\mathscr{O}(n)$ and X is an algebra over $\mathscr{O}$, each n-ary operation gives a map

$$X^n \to X.$$

Unless X is quite trivial, we do not expect these maps to be invertible for $n \neq 1$. In contrast, one might like to insist that (at least some of) the one-ary operations are invertible (up to homotopy). To facilitate the

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study of such algebras, we seek an operad that encodes such information. This leads us to search for a good definition of localization for an operad $\mathcal{O}$ with respect to a submonoid $\mathcal{W} \subset \mathcal{O}(1)$ of one-ary operations.

Localizations have been much studied in the literature, particularly in the context of model categories. An especially useful and well-studied construction of the localization of a category is the hammock localization of Dwyer and Kan [2]. We propose a variant of their construction where we consider not only hammocks of string type, that is a sequence of right and left pointing arrows, but also of tree type where left and right pointing arrows are assembled in a tree. This seems a necessary complication so that operad composition is well defined and associative after localization.

Indeed, the complication arises because hammock localization does not preserve the monoidal structure of a category. It is well-known that operads (with an action of the symmetric group) correspond to strict (symmetric) monoidal categories with object set $\mathbb{N}$ where Hom-sets between two objects a and b are monoidally generated from Hom-sets with source 1. Since the hammock localization of [2] does not preserve the monoidal structure, the outcome does not readily provide an operad.

The proposed tree hammock localization

$$L^{TH}_{\mathcal{W}}(\mathcal{O}) \quad \text{(or simply } L\mathcal{O}\text{)}$$

of $\mathcal{O}$ is functorial in the pair $(\mathcal{O}, \mathcal{W})$ and every operation in $\mathcal{W}$ is invertible up to homotopy in $L\mathcal{O}$; see Lemmas 5.1 and 5.2 and Proposition 6.1. Furthermore the derived tensor product defines a functor from $\mathcal{O}$-algebras to $L\mathcal{O}$-algebras satisfying a natural universal property with respect to $\mathcal{O}$-algebra maps to $L\mathcal{O}$-algebras; see Proposition 6.2.

Our study has been motivated by considerations in topological and conformal quantum field theory. From Atiyah and Segal's axiomatic point of view, this is the study of symmetric monoidal functors from suitably defined cobordism categories. A particularly well-studied theory is the field theory modeled by the 1+1 dimensional cobordism category where the objects are disjoint unions of circles and the morphisms are (oriented) surfaces with boundary. Often one is led to the question of how the theory behaves stably, and more or less equivalently, when the operation defined by the torus is invertible. More generally there has been much recent interest in invertible topological field theories in the context of the study of anomalies and topological phases. See, for example, [3] and the references therein.

Much of topological field theory is captured when restricting to the (maximal) symmetric monoidal sub-category of the cobordism category corresponding to an operad. Therefore, the study of the surface operad is essential to the study of 1+1 dimensional topological field theory and conformal field theory. It has many homotopy equivalent models; one, denoted by $\mathcal{M}$, is built as a subcategory of Segal's category of Riemann surfaces [6] and we will keep this operad in mind as an example. This was also the motivating example for our study [1] of operads with homological stability, compare [8]. Indeed our discussion there led us to consider localizations of the operad $\mathcal{M}$ in an attempt to answer an old question of Mike Hopkins.[2]

Content

In Section 2, we review the definition of an operad and then, in Section 3, we characterize them as a subcategory in the category of strict symmetric monoidal categories. Next, in Section 4, we recall the definition and some properties of the standard hammock localization for categories from [2]. Our main new construction is the tree hammock localization in Section 5 where we also study several important properties. In Section 6 we provide a functor from algebras over a given operad to algebras over an associated localized operad.

[2] Stringy Topology in Morelia, Morelia, Mexico, 2006.

2. Operads

Let $(\mathcal{D}, \otimes, U)$ be a closed symmetric monoidal category that is tensored over sets and has (finite) colimits. Our primary interest is the case where $\mathcal{D}$ is the category of simplicial sets $s\mathbf{S}\mathrm{et}$ or the category of based simplicial sets $s\mathbf{S}\mathrm{et}_*$.

Definition 2.1. An **operad** $\mathscr{O}$ in $\mathcal{D}$ is a collection of objects $\{\mathscr{O}(n)\}_{n\geq 0}$ in $\mathcal{D}$, a map $\epsilon\colon U \to \mathscr{O}(1)$, a right action of the symmetric group Σ_n on $\mathscr{O}(n)$ for each $n \geq 0$, and structure maps

$$\gamma\colon \mathscr{O}(k) \otimes \mathscr{O}(j_1) \otimes \ldots \otimes \mathscr{O}(j_k) \to \mathscr{O}(j)$$

for $k \geq 1$, $j_s \geq 0$ and $j = \Sigma_{s=1}^{k} j_s$ so the following diagrams commute for all i_t, j_s and k.

- γ is associative.

$$\begin{tikzcd}
\begin{array}{l} \mathscr{O}(k) \otimes \mathscr{O}(j_1) \otimes \ldots \otimes \mathscr{O}(j_k) \\ \quad \otimes \mathscr{O}(i_1) \otimes \ldots \otimes \mathscr{O}(i_{j_1}) \\ \quad \otimes \cdots \\ \quad \otimes \mathscr{O}(i_{j-j_k+1}) \ldots \otimes \mathscr{O}(i_j) \end{array} \arrow[r] \arrow[d] & \begin{array}{l} \mathscr{O}(j_1 + \ldots + j_k) \\ \otimes \mathscr{O}(i_1) \otimes \ldots \otimes \mathscr{O}(i_{j_1}) \\ \otimes \ldots \\ \otimes \mathscr{O}(i_{j-j_k+1}) \otimes \ldots \otimes \mathscr{O}(i_j) \end{array} \arrow[d] \\
\begin{array}{l} \mathscr{O}(k) \otimes \mathscr{O}(i_1 + \ldots + i_{j_1}) \\ \quad \otimes \cdots \\ \quad \otimes \mathscr{O}(i_{j-j_k+1} + \ldots + i_j) \end{array} \arrow[r] & \mathscr{O}(i_1 + \ldots + i_j)
\end{tikzcd}$$

- ϵ is a unit for γ.

$$\begin{tikzcd}
U \otimes \mathscr{O}(j) \arrow[d, "\epsilon\otimes\mathrm{id}"'] \arrow[dr, "\sim"] & \\
\mathscr{O}(1) \otimes \mathscr{O}(j) \arrow[r, "\gamma"] & \mathscr{O}(j)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{O}(k) \otimes U \otimes \ldots \otimes U \arrow[d, "\mathrm{id}\otimes\epsilon\otimes\ldots\otimes\epsilon"'] \arrow[dr, "\sim"] & \\
\mathscr{O}(k) \otimes \mathscr{O}(1) \otimes \ldots \otimes \mathscr{O}(1) \arrow[r, "\gamma"] & \mathscr{O}(k)
\end{tikzcd}$$

- γ is equivariant with respect to the symmetric group actions.

$$\begin{tikzcd}
\mathscr{O}(k) \otimes \mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_k) \arrow[r, "\sigma\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\sigma^*"] & \mathscr{O}(k) \otimes \mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_k) \arrow[dd, "\gamma"] \\
\mathscr{O}(k) \otimes \mathscr{O}(j_{\sigma^{-1}(1)}) \otimes \cdots \otimes \mathscr{O}(j_{\sigma^{-1}(k)}) \arrow[d, "\gamma"] & \\
\mathscr{O}(j_{\sigma^{-1}(1)} + \cdots + j_{\sigma^{-1}(k)}) \arrow[r, "\sigma_*(j_1,\ldots,j_k)"] & \mathscr{O}(j_1 + \cdots + j_k)
\end{tikzcd}$$

For $\sigma \in \Sigma_k$, $\sigma(j_1, \ldots, j_k) \in \Sigma_j$ permutes blocks of size j_s according to σ.

$$\begin{tikzcd}
\mathscr{O}(k) \otimes \mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_k) \arrow[r, "\mathrm{id}\otimes\tau_1\otimes\cdots\otimes\tau_k"] \arrow[d, "\gamma"] & \mathscr{O}(k) \otimes \mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_k) \arrow[d, "\gamma"] \\
\mathscr{O}(j_1 + \cdots + j_k) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_k"] & \mathscr{O}(j_1 + \cdots + j_k)
\end{tikzcd}$$

For $\tau_s \in \Sigma_{j_s}$ $\tau_1 \oplus \ldots \oplus \tau_k$ denotes the image of $(\tau_1, \ldots, \tau_k)$ under the natural inclusion of $\Sigma_{j_1} \times \ldots \times \Sigma_{j_k}$ into Σ_j

Example 2.2. Let M be a simplicial monoid. Then M defines an operad M_+ where

$$M_+(0) = \{*\},\ M_+(1) = M,\ \text{and } M_+(n) = \varnothing \text{ for } n \geq 2.$$

Operad composition is defined by monoid multiplication. Similarly, for any operad $\mathscr{O}$ the monoid of one-ary operations gives rise to a suboperad $\mathscr{O}(1)_+$.

Example 2.3. Let $\mathcal{M}_{g,n}$ denote the moduli spaces of Riemann surfaces of genus g with n parametrized and ordered boundary components. Segal [6] constructed a symmetric monoidal category where the objects are finite unions of circles and morphism spaces are disjoint unions of the spaces $\mathcal{M}_{g,n}$ with boundary circles divided into incoming and outgoing. Composition of morphisms is defined by gluing outgoing circles of one Riemann surface to incoming circles of another. A conformal field theory in the sense of [6] is then a symmetric monoidal functor from this surface category to an appropriate linear category.

By restricting the category and replacing the spaces by their total singular complexes, we define an operad $\mathcal{M}$ where

$$\mathcal{M}(n) := \coprod_{g \geq 0} \mathcal{S}ing_*(\mathcal{M}_{g,n+1}).$$

In order to study stable phenomena, one wants to invert the action of the torus $T \in \mathcal{S}ing_0(\mathcal{M}_{1,1+1}) \subset \mathcal{M}(1)$. See, for example, [3,7].

The structure and unit maps define a map $\circ_i \colon \mathscr{O}(k) \otimes \mathscr{O}(j) \to \mathscr{O}(k + j - 1)$ by requiring the following diagram to commute.

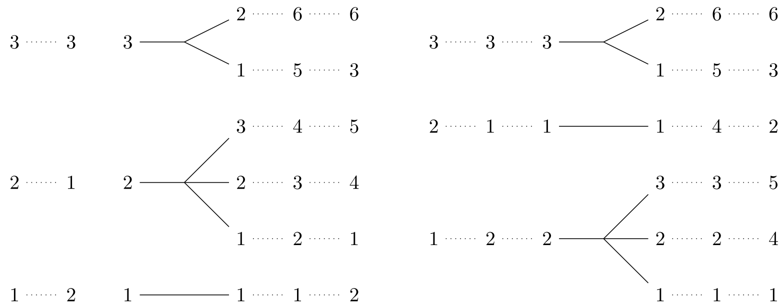
$$\begin{tikzcd}
\mathscr{O}(k) \otimes U \otimes \ldots \otimes U \otimes \mathscr{O}(j) \otimes U \otimes \ldots \otimes U \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\otimes\epsilon\otimes\ldots\otimes\epsilon\otimes\mathrm{id}\otimes\epsilon\otimes\ldots\otimes\epsilon"] & \mathscr{O}(k) \otimes \mathscr{O}(j) \arrow[d, dotted, "\circ_i"] \\
\mathscr{O}(k) \otimes \mathscr{O}(1) \otimes \ldots \otimes \mathscr{O}(1) \otimes \mathscr{O}(j) \otimes \mathscr{O}(1) \otimes \ldots \otimes \mathscr{O}(1) \arrow[r, "\gamma"] & \mathscr{O}(j + k - 1)
\end{tikzcd}$$

In the top left corner of the diagram $\mathscr{O}(j)$ is in the $(i + 1)$st spot. If $\otimes$ is the (categorical) product in $\mathcal{D}$, as it is for sets or unbased spaces, the maps $\circ_i$ determine the composition maps γ.

3. Operads as symmetric monoidal categories

It is well-known that operads give rise to symmetric monoidal categories. As the correspondence provides us with a useful comparison (see Section 5), we include a full description and proof of this correspondence.

For an operad $\mathscr{O}$ in $\mathcal{D}$, we define a strict symmetric monoidal category $\mathcal{C}_{\mathscr{O}}$ enriched in $\mathcal{D}$ whose objects are the natural numbers. Morphisms for $a > 0$ are given by



(a) $\sigma = (12)$, $\tau = (12)(345)$ (b) $\sigma(1,3,2)\tau = (24)(35)$

Fig. 1. The left symmetric group action.

$$\mathcal{C}_{\mathscr{O}}(a,b) := \coprod_{\Sigma k_i = b} (\mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_a)) \times_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_a}} \Sigma_b$$

where the coproduct is indexed by sequences of natural numbers $(k_1, \ldots, k_a)$ such that $\Sigma_i k_i = b$ and $k_i \geq 0$. Tensoring with Σ_b and using a coequalizer allows for all possible permutations of inputs, not only the ones that permute the inputs of each $\mathscr{O}(k_i)$ separately. In particular, there is a monoid homomorphism $U \times \Sigma_n \to \mathcal{C}_{\mathscr{O}}(n,n)$ and this defines a left Σ_a and a right Σ_b action on $\mathcal{C}_{\mathscr{O}}(a,b)$. See Fig. 1 for an example in a concrete category. We define $\mathcal{C}_{\mathscr{O}}(0,0)$ to be U and note that $\mathcal{C}_{\mathscr{O}}(0,b)$ is the initial object if $b > 0$ and $\mathcal{C}_{\mathscr{O}}(a,0)$ is $\mathscr{O}(0)^{\otimes a}$ for $a > 0$.

Composition. The map $\mathcal{C}_{\mathscr{O}}(a,b) \otimes \mathcal{C}_{\mathscr{O}}(b,c) \to \mathcal{C}_{\mathscr{O}}(a,c)$ is factored by the map

$$\begin{array}{c}
(\mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_a) \times_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_a}} \Sigma_b) \otimes (\mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_b) \times_{\Sigma_{j_1} \times \cdots \times \Sigma_{j_b}} \Sigma_c) \\
\big\downarrow \\
(\mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_a) \otimes \mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_b)) \times_{\Sigma_{k_1} \times \cdots \times \Sigma_{j_b}} \Sigma_b \times \Sigma_c
\end{array}$$

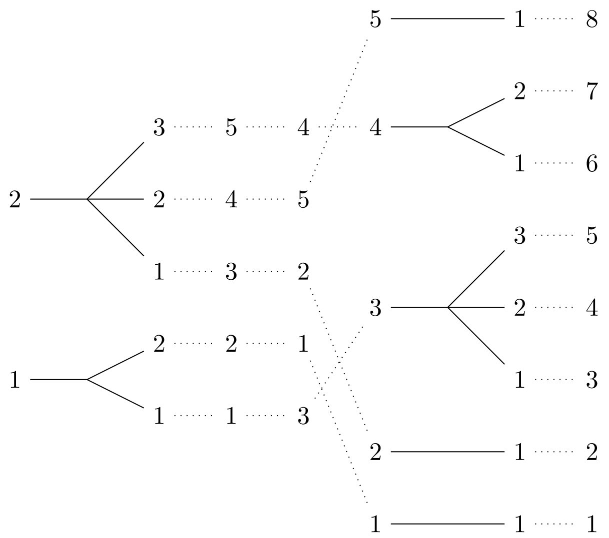
The composition takes $(\sigma, \tau) \in \Sigma_b \times \Sigma_c$ to $\sigma(k_1, \ldots, k_a)\tau$ (the block permutation described above) and is defined by $(\gamma \otimes \cdots \otimes \gamma) \circ \sigma^*$ on the tensor product (see Fig. 2).

$$\begin{array}{c}
\mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_a) \otimes \mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_b) \\
\big\downarrow{\scriptstyle \sigma^*} \\
\mathscr{O}(k_1) \otimes \mathscr{O}(j_{\sigma^{-1}(1)}) \otimes \cdots \otimes \mathscr{O}(j_{\sigma^{-1}(k_1)}) \otimes \cdots \otimes \mathscr{O}(k_a) \otimes \mathscr{O}(\ldots) \\
\big\downarrow{\scriptstyle \gamma \otimes \cdots \otimes \gamma} \\
\mathscr{O}(j_{\sigma^{-1}(1)} + \cdots + j_{\sigma^{-1}(k_1)}) \otimes \cdots \otimes \mathscr{O}(j_{\sigma^{-1}(b-k_a+1)} + \cdots + j_{\sigma^{-1}(b)})
\end{array}$$

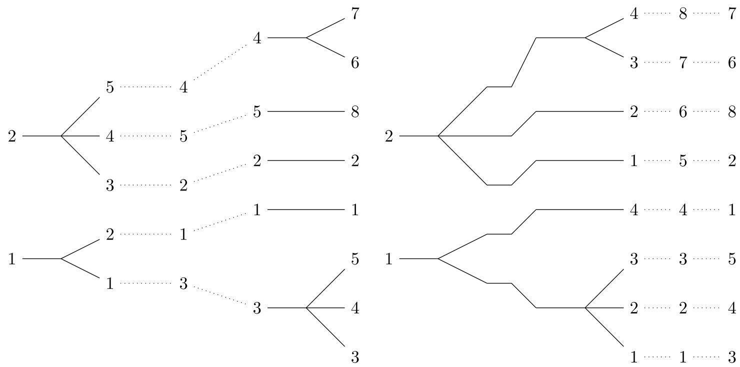
Monoidal structure. The addition of natural numbers defines the monoidal structure on the objects of $\mathcal{C}_{\mathscr{O}}$. On morphisms the monoidal product

$$\mathcal{C}_{\mathscr{O}}(a,b) \otimes \mathcal{C}_{\mathscr{O}}(c,d) \to \mathcal{C}_{\mathscr{O}}(a+c, b+d)$$

is the map



(a) Morphisms in $\mathcal{C}_{\mathscr{O}}(2,5)$ and $\mathcal{C}_{\mathscr{O}}(5,8)$.



(b) Acting by the symmetric group. (c) The final composite

Fig. 2. An example of composition in the category $\mathcal{C}_{\mathscr{O}}$.

$$
\begin{aligned}
&(\mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_a)) \times_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_a}} \Sigma_b \otimes (\mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_c)) \times_{\Sigma_{j_1} \times \cdots \times \Sigma_{j_c}} \Sigma_d \\
&\quad \to (\mathscr{O}(k_1) \otimes \cdots \otimes \mathscr{O}(k_a) \otimes \mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_c)) \times_{\Sigma_{k_1} \times \ldots \Sigma_{k_a} \times \Sigma_{j_1} \times \cdots \times \Sigma_{j_c}} \Sigma_{b+d}
\end{aligned}
$$

induced by the canonical inclusion $\Sigma_b \times \Sigma_d \to \Sigma_{b+d}$. Note that $\mathcal{C}_{\mathscr{O}}(a,b) \otimes \mathcal{C}_{\mathscr{O}}(c,d)$ has a left action by $\Sigma_a \times \Sigma_c$ and a right action by $\Sigma_b \times \Sigma_d$.

If $(a,b) \in \Sigma_{b+a}$ is the permutation that permutes the first a entries to the end,

$$
(U \otimes \ldots \otimes U) \times * \xrightarrow{\epsilon \otimes \ldots \otimes \epsilon \otimes (a,b)} \mathscr{O}(1) \otimes \ldots \otimes \mathscr{O}(1) \times_{\Sigma_1 \times \ldots \times \Sigma_1} \Sigma_{b+a} \subset \mathcal{C}_{\mathscr{O}}(a+b, b+a)
$$

defines an isomorphism from $a+b$ to $b+a$. Compatibility with block permutation makes this a natural transformation.

This construction extends to an equivalence of categories.

Proposition 3.1. *There is an equivalence of categories between the category of operads in $\mathcal{D}$ and the category of strict symmetric monoidal categories enriched in $\mathcal{D}$ such that*

1. *the monoid of objects is* $(\mathbb{N}, +, 0)$*;*
2. *there is a monoid homomorphism* $\Sigma_b \to \mathcal{C}(b,b)$ *for all objects* b*;*

3. *the map*

$$\coprod_{(k_1,\ldots,k_a),\Sigma k_i=b} (\mathcal{C}(1,k_1)\otimes\cdots\otimes\mathcal{C}(1,k_a))\times_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_a}}\Sigma_b\to\mathcal{C}(a,b)$$

defined by the monoidal structure and composition is an isomorphism for all $a\geq 1$ and b; the left action by $\Sigma_a\subset\mathcal{C}(a,a)$ corresponds to permuting the factors in the tensor product; and the identity map is the only map with source 0.

Proof. Note that $\mathcal{C}_{\mathscr{O}}$ satisfies the conditions above if $\mathscr{O}$ is an operad.

For the other implication, if $\mathcal{C}$ is a symmetric monoidal category satisfying the hypotheses above, define an operad $\mathscr{O}_{\mathcal{C}}$ by

$$\mathscr{O}_{\mathcal{C}}(n):=\mathcal{C}(1,n).$$

Then $\mathscr{O}_{\mathcal{C}}(n)$ has a right action by $\Sigma_n\subset\mathcal{C}(n,n)$. The operad structure maps

$$\mathscr{O}_{\mathcal{C}}(k)\otimes(\mathscr{O}_{\mathcal{C}}(j_1)\otimes\ldots\otimes\mathscr{O}_{\mathcal{C}}(j_k))\to\mathscr{O}_{\mathcal{C}}(j_1+\ldots+j_k)$$

are the composites

$$\mathcal{C}(1,k)\otimes(\mathcal{C}(1,j_1)\otimes\ldots\otimes\mathcal{C}(1,j_k))\subset\mathcal{C}(1,k)\otimes\mathcal{C}(k,j_1+\ldots+j_k)\to\mathcal{C}(1,j_1+\ldots+j_k).$$

By Assumption 3 this map is equivariant with respect to Σ_k and $\Sigma_{j_1}\times\ldots\times\Sigma_{j_k}$.

For an operad $\mathscr{O}$,

$$\mathscr{O}_{\mathcal{C}_{\mathscr{O}}}(b):=\mathcal{C}_{\mathscr{O}}(1,b)=\mathscr{O}(b)\times_{\Sigma_b}\Sigma_b=\mathscr{O}(b).$$

Starting with a symmetric monoidal category $\mathcal{C}$, $\mathscr{O}_{\mathcal{C}}(k):=\mathcal{C}(1,k)$ and

$$\begin{aligned}\mathcal{C}_{\mathscr{O}_{\mathcal{C}}}(a,b)&:=\coprod_{\Sigma k_i=b}(\mathscr{O}_{\mathcal{C}}(k_1)\otimes\cdots\otimes\mathscr{O}_{\mathcal{C}}(k_a))\times_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_a}}\Sigma_b\\&=\coprod_{\Sigma k_i=b}(\mathcal{C}(1,k_1)\otimes\cdots\otimes\mathcal{C}(1,k_a))\times_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_a}}\Sigma_b\end{aligned}$$

When $\mathcal{C}$ satisfies the third condition above, this agrees with $\mathcal{C}(a,b)$. $\square$

4. Hammock localization for categories enriched in simplicial sets

We recall the hammock localization of [2] and its extension to simplicially enriched categories [2, 2.5].

For a category $\mathcal{C}$ and a subcategory $\mathcal{W}$, a **reduced hammock** of height n in $\mathcal{C}$ is a commutative diagram in $\mathcal{C}$ (of arbitrary length $m\geq 0$) of the form

$$\begin{tikzcd}
 & k_{0,1} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] \arrow[d] & k_{0,m-1} \arrow[d] & \\
 & k_{1,1} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] \arrow[d] & k_{1,m-1} \arrow[d] & \\
a \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] \arrow[ddr, dash] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & b \arrow[uul, dash] \arrow[ul, dash] \arrow[dl, dash] \arrow[ddl, dash] \\
 & k_{n-1,1} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] \arrow[d] & k_{n-1,m-1} \arrow[d] & \\
 & k_{n,1} \arrow[r, dash] & \cdots \arrow[r, dash] & k_{n,m-1} &
\end{tikzcd}$$

satisfying the following conditions:

- horizontal arrows can point either direction,
- vertical and left-pointing arrows are in $\mathcal{W}$,
- in each column the horizontal arrows point in the same direction,
- arrows in adjacent columns point in different directions, and
- no column contains only identity maps.

The **hammock localization** of $\mathcal{C}$ at $\mathcal{W}$, $L^H_{\mathcal{W}}\mathcal{C}$, is a simplicially enriched category whose objects are the objects of $\mathcal{C}$. The n-simplices in $L^H_{\mathcal{W}}\mathcal{C}(a,b)$ are the reduced hammocks of height n. We define face and degeneracy maps by deleting or repeating a row, then reducing according to the conditions above. Composition in the category $L^H_{\mathcal{W}}\mathcal{C}$ is via concatenation of hammocks where the point of gluing is expanded to a vertical column of identity maps.

If O is a fixed set then $O - \mathbf{C}at$ is the category of small categories whose object set is O. Similarly, $sO - \mathbf{C}at$ is the categories enriched in $s\mathbf{S}et$ whose object set is O.

Theorem 4.1. *[2, 3.4] Localization is a functor*

$$\{(\mathcal{C},\mathcal{W})|\mathcal{W} \subset \mathcal{C} \in O - \mathbf{C}at\} \to sO - \mathbf{C}at$$

from the category of O-categories with an O-subcategory to the category of simplicial O-categories. For each pair $(\mathcal{C},\mathcal{W})$, there are canonical inclusion functors $\mathcal{C} \to L^H_{\mathcal{W}}\mathcal{C}$ and $\mathcal{W}^{op} \to L^H_{\mathcal{W}}\mathcal{C}$.

The map $\mathcal{C} \to L^H_{\mathcal{W}}\mathcal{C}$ is given by the inclusion of hammocks of length one and height zero with arrows pointing to the right. Similarly, the functor $\mathcal{W}^{op} \to L^H_{\mathcal{W}}\mathcal{C}$ is given by the inclusion of hammocks of length one and height zero with arrows pointing to the left.

We now generalize to simplicially enriched categories. Let $\mathcal{C}_*$ be a simplicially enriched O-category and $\mathcal{W}_*$ be an O-subcategory of $\mathcal{C}_*$. For each simplicial degree k, let $\mathcal{C}_k$ be the O-category whose morphisms are the k simplices of $\mathcal{C}_*(a,b)$. The face and degeneracy maps in the hom sets of $\mathcal{C}$ define functors $\mathcal{C}_k \to \mathcal{C}_{k-1}$ and $\mathcal{C}_k \to \mathcal{C}_{k+1}$. We also have corresponding subcategories $\mathcal{W}_k$, which form a simplicial subcategory.

The hammock localization of $\mathcal{C}_k$ with respect to $\mathcal{W}_k$ is a simplicially enriched category $L^H_{\mathcal{W}_k}\mathcal{C}_k$. The **hammock localization** of $\mathcal{C}_*$ at $\mathcal{W}_*$ is a category $L^H_{\mathcal{W}_*}\mathcal{C}_*$, enriched in bisimplicial sets, where $L^H_{\mathcal{W}_*}\mathcal{C}_*(a,b)_{k,\ell}$ is the height ℓ hammocks of $L^H_{\mathcal{W}_k}\mathcal{C}_k(a,b)$. The required simplicial structure maps are defined using the functoriality of the hammock localization.

Proposition 4.2. *Let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be simplicially enriched O-categories with O-subcategories $\mathcal{W}_* \subset \mathcal{C}_*$ and $\mathcal{W}'_* \subset \mathcal{C}'_*$. A functor $F\colon \mathcal{C}_* \to \mathcal{C}'_*$ of simplicially enriched categories such that $F(\mathcal{W}_*) \subset \mathcal{W}'_*$ induces a functor of bisimplicially enriched categories*

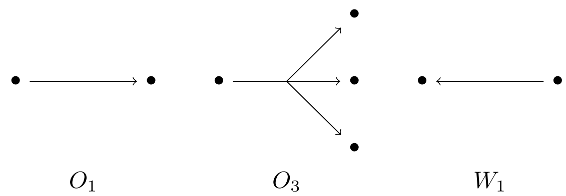


Fig. 3. Atomic building blocks. The root node is on the left for each tree.

$$L^H_{\mathcal{W}_*}\mathcal{C}_* \to L^H_{\mathcal{W}'_*}\mathcal{C}'_*.$$

Proof. By the corresponding result for unenriched categories [2, 3.4], there are maps $L^H_{\mathcal{W}_k}\mathcal{C}_k \to L^H_{\mathcal{W}'_k}\mathcal{C}'_k$. These assemble to a functor $L^H_{\mathcal{W}_*}\mathcal{C}_* \to L^H_{\mathcal{W}'_*}\mathcal{C}'_*$ of categories enriched in bisimplicial sets. Applying the diagonal functor induces a simplicial functor as in the statement. □

Proposition 4.3. *Let $\phi\colon b \to c$ be in $\mathcal{W}_*$. Then for every object a in $\mathcal{C}$, ϕ induces weak homotopy equivalences*

$$\phi_*\colon L^H_{\mathcal{W}_*}\mathcal{C}_*(a,b) \to L^H_{\mathcal{W}_*}\mathcal{C}_*(a,c) \text{ and } \phi^*\colon L^H_{\mathcal{W}_*}\mathcal{C}_*(c,a) \to L^H_{\mathcal{W}_*}\mathcal{C}_*(b,a)$$

Proof. By [2, 3.3], the maps $(\phi_k)_*\colon L^H_{\mathcal{W}_k}\mathcal{C}_k(a,b) \to L^H_{\mathcal{W}_*}\mathcal{C}_k(a,c)$ and $(\phi_k)^*\colon L^H_{\mathcal{W}_*}\mathcal{C}_k(c,a) \to L^H_{\mathcal{W}_*}\mathcal{C}_k(b,a)$ are weak homotopy equivalences. Then [4, 4.1.7] implies they are weak homotopy equivalences. □

5. Localization for operads

Ideally, we would like to consider an operad as one of the symmetric monoidal categories identified in Section 3, and apply the hammock localization as described in Section 4. However, in order to translate back to operads, we would need to show that the localized category is of the form prescribed by Proposition 3.1. Unfortunately, there is an obstacle. In order to define the monoidal product on hammocks one has to struggle with the fact that different hammocks have different lengths, and any naive attempt to extend hammocks via identities seems to lead to a monoidal product that no longer commutes with composition and hence will not be functorial.

Here we instead give a construction working directly with the operad. The standard hammock localization applied to the monoid of 1-ary operations $\mathscr{O}(1)$ is a subcategory of the localization of $\mathscr{O}(1)$ constructed in this section. Indeed, our procedure will construct a localization of the associated strict symmetric monoidal category as described in Proposition 3.1. We expect that this localization will yield a category that is homotopic to the standard hammock localization.

The major difference between this section and the previous is that we now allow hammocks where the rows have been replaced by trees with one root and n labeled leaves. For $n \geq 0$, let $\mathbb{T}(n)$ be the set of directed planar trees with one root and n leaves labeled 1 through n and possibly some other leaves without label. Each tree $\tau \in \mathbb{T}(n)$ is constructed from atomic, directed pieces. These atomic pieces are

$$O_n\colon 1 \to n \text{ for } n \geq 0, \text{ and } W_1\colon 1 \leftarrow 1.$$

See Fig. 3. Each tree has a root node (the source for O_n and the target for W_1) and leaf nodes; each tree is constructed by gluing together atomic pieces identifying the root node of one tree with a labeled leaf node of another tree; the resulting node will be called an internal node.[3]

[3] The nodes are not the vertices in the underlying graph. For example $O_n\colon 1 \to n$ has $n+1$ nodes corresponding to its boundary but no internal one.

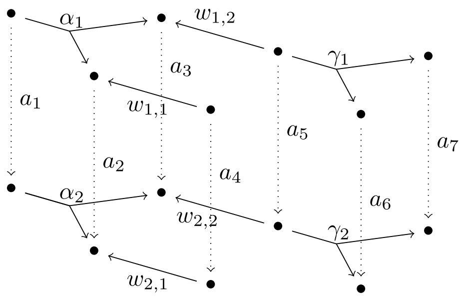


Fig. 4. A reduced tree hammock of height one. We require $w_{1,1}$, $w_{1,2}$, $w_{2,1}$, $w_{2,2}$ and a_i, $i = 1, \ldots 7$, to be operations in $\mathcal{W}$ and $\gamma(\alpha_1; a_2, a_3) = \gamma(a_1; \alpha_2)$, $\gamma(w_{1,1}; a_2) = \gamma(a_4; w_{2,1})$, $\gamma(w_{1,2}; a_3) = \gamma(a_5; w_{2,2})$, and $\gamma(\gamma_1; a_6, a_7) = \gamma(a_5; \gamma_2)$.

Let $\mathscr{O}$ be an operad and let $\mathcal{W}$ be a sub-monoid of operations in $\mathscr{O}(1)$. A **reduced tree hammock** of height k and type $\tau \in \mathbb{T}(n)$ is a three dimensional diagram consisting of k copies of τ arranged in parallel (horizontal) planes with additional (vertical and downward pointing) directed edges connecting corresponding roots, leaves, and internal nodes in consecutive copies of τ. Each atomic piece in τ and each vertical edge is labeled by an element in $\mathscr{O}$ so that the diagram commutes. We further require that

1. atomic pieces $O_n : 1 \to n$ are labeled by elements in $\mathscr{O}(n)$;
2. atomic pieces $W_1 : 1 \leftarrow 1$ and vertical arrows are labeled by elements in $\mathcal{W}$;
3. arrows in adjacent columns of atomic pieces point in different directions[4]; and
4. no column of atomic pieces corresponding to O_1 or W_1 contains arrows all labeled by the identity element. (Note that the vertical arrows can all be identity maps.)

Note that if τ is made up of atomic pieces $O_1 : 1 \to 1$ and $W_1 : 1 \leftarrow 1$, this is a reduced hammock in the sense of section 4 for the monoid $\mathscr{O}(1)$ viewed as a category.

We have removed the initial and final maps of the hammocks. See Fig. 4 for an example of this type of hammock.

For each nonnegative integer n, let $L\mathscr{O}(n)$ be the simplicial set whose k simplices are the reduced tree hammocks of height k and type $\tau \in \mathbb{T}(n)$. Simplicial face maps are given by deleting a plane and composing adjacent vertical maps. Degeneracy maps are given by repeating a plane. The simplicial sets $L\mathscr{O}(n)$ assemble to an operad where the ith composition $\circ_i$ is defined by grafting the underlying trees, that is, identifying the leaf node labeled i with the root node of the ith tree and relabeling leaf nodes as appropriate. If the tree hammock resulting from any of these operations is not in reduced form it can easily be made so by composing operations for columns of neighboring arrows pointing in the same direction, and by deleting columns of identities.

As for the hammocks defined in the previous section, the construction above extends to simplicially enriched operads.[5] For a simplicially enriched operad $\mathscr{O}$ and a sub-monoid $\mathcal{W} \subset \mathscr{O}(1)$, let $L^{TH}_{\mathcal{W}}\mathscr{O}(n)$ (or $L\mathscr{O}(n)$) be the bisimplicial set of all reduced hammocks of tree type τ for some $\tau \in \mathbb{T}(n)$. The symmetric group acts on $L^{TH}_{\mathcal{W}}\mathscr{O}(n)$ by relabeling the labeled leaf nodes. Grafting of tree hammocks (and reduction if necessary) defines an associative and equivariant composition, and we thus define the **tree hammock localization** of a simplicially enriched operad $\mathscr{O}$ with respect to the sub-monoid $\mathcal{W} \subset \mathscr{O}(1)$ to be the operad

$$L^{TH}_{\mathcal{W}}\mathscr{O}$$

enriched in bisimplicial sets.

[4] In particular, for an atomic piece $O_n : 1 \to n$ all $n+1$ adjacent pieces will have to be of the form $W_1 : 1 \leftarrow 1$.
[5] If $\mathcal{D}$ is the category of simplicial sets, we will say that such an operad in $\mathcal{D}$ is a **simplicial operad** or is **simplicially enriched**.

We have the following analogues of Proposition 4.2 and 4.3.

Lemma 5.1 *(Functoriality). Let $\mathscr{O}$ and $\mathscr{O}'$ be two simplicial operads with sub-monoids $\mathcal{W}$ and $\mathcal{W}'$ of the respective 1-ary operations. Let $\phi\colon \mathscr{O} \to \mathscr{O}'$ be a map of operads with $\phi(\mathcal{W}) \subset \mathcal{W}'$. Then ϕ induces a map on localizations*

$$\phi\colon L^{TH}_{\mathcal{W}}\mathscr{O} \longrightarrow L^{TH}_{\mathcal{W}'}\mathscr{O}'.$$

In particular we may consider the map of pairs $(\mathscr{O}, \{1\}) \subset (\mathscr{O}, \mathcal{W})$. This defines a natural map

$$\mathscr{O} \longrightarrow L^{TH}_{\mathcal{W}}\mathscr{O}$$

where $\mathscr{O}$ is identified with the bisimplicial set $L^{TH}_{\{1\}}\mathscr{O}$ constant in one simplicial direction.

Lemma 5.2 *(Weak invertibility). Let $w \in \mathcal{W}$ be a one-ary operation. Then w induces weak homotopy equivalences*

$$w \circ \_\colon L^{TH}_{\mathcal{W}}\mathscr{O}(n) \to L^{TH}_{\mathcal{W}}\mathscr{O}(n) \;\; \textit{and} \;\; \_ \circ_i w\colon L^{TH}_{\mathcal{W}}\mathscr{O}(n) \to L^{TH}_{\mathcal{W}}\mathscr{O}(n)$$

Proof. Composition with w corresponds to grafting with a tree of type $O_1\colon 1 \to 1$ and label w. The inverse is given by grafting with a tree of type $W_1\colon 1 \leftarrow 1$ with label w. The composition of these two operations is homotopic to the identity as can be deduced from the following hammocks of height 1

$$\begin{tikzcd}
\bullet \arrow[d, "w"'] & \bullet \arrow[l, "1"'] \arrow[r, "1"] \arrow[d, "1"] & \bullet \arrow[d, "w"] \\
\bullet & \bullet \arrow[l, "w"] \arrow[r, "w"'] & \bullet
\end{tikzcd}$$

and

$$\begin{tikzcd}
\bullet \arrow[r, "w"] \arrow[d, "w"'] & \bullet \arrow[d, "1"] & \bullet \arrow[l, "w"'] \arrow[d, "w"] \\
\bullet \arrow[r, "1"'] & \bullet & \bullet \arrow[l, "1"]
\end{tikzcd}$$ □

We will now compare the tree hammock localization for operads with the hammock localization for categories.

By Example 2.2, the monoid of one-ary operations in an operad $\mathscr{O}$ forms a sub-operad M_+ with

$$M_+(1) = \mathscr{O}(1),\; M_+(0) = \{*\},\; \text{and } M_+(n) = \varnothing \text{ for } n > 1\;.$$

Vice versa, any monoid M gives rise to such an operad M_+ with one-ary operations $M_+(1) = M$. As every monoid is also a category (with one object), we thus have two potentially different ways of localizing.

Lemma 5.3. *Let M be a monoid and W a submonoid. Then the hammock localization of M with respect to W agrees with the tree hammock localization of M_+ with respect to W. More precisely, there is an isomorphism of simplicial monoids*

$$L^H_W(M) \cong L^{TH}_W(M_+)(1). \qquad \square$$

Thus, for example, the torus T from Example 2.3 generates a (free) monoid $W \simeq \mathbb{N} \subset \mathcal{M}(1)$. Its localization is homotopy equivalent to $\mathbb{Z}$.

More generally, let $\mathcal{O}$ be an operad and $\mathcal{W}$ be a submonoid of $\mathcal{O}(1)$ which we extend to the suboperad $\mathcal{W}_+$. By Proposition 3.1, $\mathcal{O}$ and $\mathcal{W}_+$ uniquely define a strict symmetric monoidal category $\mathcal{C}_{\mathcal{O}}$ with a strict monoidal subcategory $\mathcal{C}_{\mathcal{W}_+}$.

Proposition 5.4. *Hammock reduction defines a full functor of (bi)simplicially enriched* $\mathbb{N}$*-categories*

$$R\colon L^H_{\mathcal{C}_{\mathcal{W}_+}}\mathcal{C}_{\mathcal{O}} \longrightarrow \mathcal{C}_{L^{TH}_{\mathcal{W}}\mathcal{O}}.$$

Note that the target of the reduction map R is a strict symmetric monoidal category. Thus tree hammock localization provides a localization in strict symmetric monoidal categories for categories of the form $\mathcal{C}_{\mathcal{O}}$.

Proof. By definition, R is the identity on objects.

By the description of the morphisms in $\mathcal{C}_{\mathcal{O}}$ in section 3, the statement of the proposition will follow from the special case where the source is the object 1.

Let $n \geq 0$. It is not hard to see that the reduced hammocks defining $L^H_{\mathcal{C}_{\mathcal{W}_+}}\mathcal{C}_{\mathcal{O}}(1,n)$ are precisely the tree hammocks where the geodesic paths from the root node to any of the n labeled leaf nodes have exactly the same length, that is the same number of atomic pieces: The union of all the atomic pieces that are precisely i steps away from the root make up the ith column of the standard hammock described in section 4. Note that there might be more or less than n pieces i steps away from the root; the former case may arise when there are contributions from paths to unlabeled leaf nodes. Even though we start with a reduced hammock, it may not be reduced as a tree hammock. This is because hammock reduction will only remove the ith column in the hammock if the individual columns of all ith pieces in the tree hammock are identities. On the other hand the tree hammock reduction removes identity columns defined by any atomic piece in the tree. Thus every element in $L^H_{\mathcal{C}_{\mathcal{W}_+}}\mathcal{C}_{\mathcal{O}}(1,n)$ defines an element in $L^{TH}_{\mathcal{W}}\mathcal{O}(n)$ but possibly only after tree hammock reduction:

$$R\colon L^H_{\mathcal{C}_{\mathcal{W}_+}}\mathcal{C}_{\mathcal{O}}(1,n) \longrightarrow L^{TH}_{\mathcal{W}}\mathcal{O}(n).$$

It is easy to see that every reduced tree hammock is the reduction of a tree hammock with equal length geodesics from root nodes to labeled leaf nodes. Indeed, any tree hammock can be extended to such a tree hammock by adding identity columns to the labeled nodes where necessary. Hence R is surjective.

As reduction preserves the simplicial structure, R defines a (bi)simplicial map. Finally, reduction also commutes with composition (gluing of hammocks and grafting of trees). Hence R is functorial. □

In the special case of Lemma 5.3, R induces an isomorphism on morphism spaces. In general, we believe that R induces a homotopy equivalence on morphism spaces. This would imply that properties of the usual hammock localization (such as homotopy invariance and calculus of fractions) can be transferred to the tree hammock localization of operads. We hope to return to this question elsewhere.

6. Localization for algebras

We now turn to the study of algebras over $L\mathcal{O} = L^{TH}_{\mathcal{W}}(\mathcal{O})$. Correspondingly we restrict our attention to operads in $s\mathbf{Set}$.

Recall, a **$\mathcal{O}$-algebra** is a based simplicial set $(X, *)$ with structure maps

$$\theta\colon \mathcal{O}(j) \times X^j \to X$$

for all $j \geq 0$ such that

1. For all $c \in \mathscr{O}(k)$, $d_s \in \mathscr{O}(j_s)$, and $x_t \in X$
$$\theta(\gamma(c; d_1, \dots, d_k); x_1, \dots, x_j) = \theta(c; y_1, \dots, y_k)$$
where $y_s = \theta(d_s; x_{j_1+\dots+j_{s-1}+1}, \dots, x_{j_1+\dots+j_s})$.
2. For all $x \in X$
$$\theta(1; x) = x \text{ and } \theta(*) = *.$$
3. For all $c \in \mathscr{O}(k)$, $x_s \in X$, $\sigma \in \Sigma_k$
$$\theta(c\sigma; x_1, \dots, x_k) = \theta(c; x_{\sigma^{-1}1}, \dots, x_{\sigma^{-1}k}).$$

Proposition 6.1. *For any $L\mathscr{O}$-algebra Y, the action of any $w \in \mathcal{W}$ on Y is a homotopy equivalence.*

Proof. As w has a homotopy inverse w^{-1} in $L\mathscr{O}$, the associativity in condition 1 above implies that we have a homotopy equivalence
$$\theta(w^{-1}, -) \circ \theta(w, -) \simeq \theta(1, -) \colon Y \to Y. \quad \square$$

Proposition 6.2. *There is a functor*
$$L(= L^{TH}_{\mathcal{W}}) \colon \mathscr{O}\text{-algebras} \to L\mathscr{O}\text{-algebras}$$
with the following three properties.

1. *For each $\mathscr{O}$-algebra X, there is a zig-zag of natural $\mathscr{O}$-algebra maps between X and LX.*
2. *If X is an $L\mathscr{O}$-algebra, there is a map $X \to LX$ which has a left inverse up to homotopy.*
3. *For any $\mathscr{O}$-algebra X, $L\mathscr{O}$-algebra Y, and map of $\mathscr{O}$-algebras $X \to Y$, there is a canonical map $LX \to Y$ so that*
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ LX \arrow[ur] & \end{tikzcd}$$
commutes up to homotopy.

This result follows from a very general construction for monads associated to operads. We now recall the relevant definitions needed for the construction.

Definition 6.3. If $\mathscr{O}$ is an operad and $(X, *)$ is a based simplicial set, the **free $\mathscr{O}$-algebra on X** is
$$\mathbb{O}(X) := \coprod_{n \geq 0} \left(\mathscr{O}(n) \times_{\Sigma_n} X^n\right) / \sim$$
where $\sim$ is a base point relation generated by
$$(\sigma_i c; x_1, \dots, x_{n-1}) \sim (c; s_i(x_1, \dots, x_{n-1}))$$

for all $c \in \mathscr{O}(n)$, $x_i \in X$, and $0 \leq i < n$ where $\sigma_i c = \gamma(c, e_i)$ with

$$e_i = (1^i, *, 1^{n-i-1}) \in \mathscr{O}(1)^i \times \mathscr{O}(0) \times \mathscr{O}(1)^{n-i-1},$$

and $s_i(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_{n-1})$.

This construction defines a monad in $s\mathbf{S}\mathrm{et}_*$ and associates a natural transformation of monads $\mathbb{O} \to \mathbb{P}$ to a map of operads $\mathscr{O} \to \mathscr{P}$. Hence, it defines a functor from operads in $s\mathbf{S}\mathrm{et}$ to monads in $s\mathbf{S}\mathrm{et}_*$.

For a monad $\mathbb{O}$ in the category $s\mathbf{S}\mathrm{et}_*$, an **$\mathbb{O}$-algebra** is a pair (X, ξ) consisting of a simplicial set X and a simplicial map $\xi \colon \mathbb{O}(X) \to X$ that is unital and associative. A **map of $\mathbb{O}$-algebras** is a map of simplicial sets $f \colon X \to Y$ where the following diagram commutes.

$$\begin{tikzcd}
\mathbb{O}(X) \arrow[r, "\mathbb{O}(f)"] \arrow[d, "\xi_X"'] & \mathbb{O}(Y) \arrow[d, "\xi_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The functor from operads to monads described above defines an isomorphism between the category of $\mathscr{O}$-algebras and the category of $\mathbb{O}$-algebras.

A **$\mathbb{O}$-functor** [5, 2.2, 9.4] in a category $\mathcal{D}$ is a functor $F \colon s\mathbf{S}\mathrm{et}_* \to \mathcal{D}$ and a unital and associative natural transformation $\lambda \colon F\mathbb{O} \to F$. The **bar construction** [5, 9.6] for a monad $\mathbb{O}$, $\mathbb{O}$-algebra X and $\mathbb{O}$-functor F, denoted $B_\bullet(F, \mathbb{O}, X)$, is the simplicial object in $\mathcal{D}$ where

$$B_q(F, \mathbb{O}, X) := F\mathbb{O}^q X.$$

If $\mathcal{D}$ is the category of simplicial sets, $B_\bullet(F, \mathbb{O}, X)$ is a bisimplicial set. We define

$$B(F, \mathbb{O}, X) := |B_\bullet(F, \mathbb{O}, X)|$$

where the geometric realization is the diagonal simplicial set.

Then Proposition 6.2 is a special case of the following familiar and more general result. We include a proof for completeness.

Proposition 6.4. *A map of operads $\mathscr{O} \to \mathscr{P}$ defines a functor*

$$\mathscr{P} \otimes_{\mathscr{O}} (-) \colon \mathscr{O}\text{-algebras} \to \mathscr{P}\text{-algebras}$$

satisfying the following properties.

1. *For each $\mathscr{O}$-algebra X, there is a zig-zag of natural $\mathscr{O}$-algebra maps between X and $\mathscr{P} \otimes_{\mathscr{O}} X$.*
2. *If X is an $\mathscr{P}$-algebra there is a map $X \to \mathscr{P} \otimes_{\mathscr{O}} X$ which has a left inverse up to homotopy.*
3. *For any $\mathscr{O}$-algebra X, $\mathscr{P}$-algebra Y and map of $\mathscr{O}$-algebras $X \to Y$, there is a canonical map $\mathscr{P} \otimes_{\mathscr{O}} X \to Y$ so that*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \\
\mathscr{P} \otimes_{\mathscr{O}} X \arrow[ur] &
\end{tikzcd}$$

commutes up to homotopy.

Proof. The map of operads, $\mathscr{O} \to \mathscr{P}$, the associated natural transformation of monads, $\mathbb{O} \to \mathbb{P}$, and the composite

$$\mathbb{P}\mathbb{O} \to \mathbb{P}\mathbb{P} \to \mathbb{P},$$

give $\mathbb{P}$ the structure of an $\mathbb{O}$ functor. Thus, for an $\mathscr{O}$-algebra X, we may define

$$\mathscr{P} \otimes_{\mathscr{O}} X := B(\mathbb{P}, \mathbb{O}, X)$$

and observe that $\mathscr{P} \otimes_{\mathscr{O}} X$ inherits the structure of a $\mathscr{P}$-algebra from the copy of $\mathbb{P}$ in the bar construction. Then there is a zig-zag of $\mathscr{O}$-algebra maps

$$X \xleftarrow{\sim} B(\mathbb{O}, \mathbb{O}, X) \to B(\mathbb{P}, \mathbb{O}, X) = \mathscr{P} \otimes_{\mathscr{O}} X$$

where the left hand map is the natural homotopy equivalence between X and its free resolution, and the right hand map is induced by the map $\mathscr{O} \to \mathscr{P}$. This construction is natural in X, and hence it defines a functor

$$\mathscr{P} \otimes_{\mathscr{O}} - : \mathscr{O} - algebras \to \mathscr{P} - algebras.$$

If X is a $\mathscr{P}$-algebra, consider the composite

$$\mathscr{P} \otimes_{\mathscr{O}} X := B(\mathbb{P}, \mathbb{O}, X) \to B(\mathbb{P}, \mathbb{P}, X) \to X.$$

Then

$$B(\mathbb{O}, \mathbb{O}, X) \to B(\mathbb{P}, \mathbb{O}, X) \to B(\mathbb{P}, \mathbb{P}, X) \to X$$

is the natural homotopy equivalence $B(\mathbb{O}, \mathbb{O}, X) \xrightarrow{\sim} X$.

For a map $X \to Y$ of $\mathscr{O}$-algebras, naturality gives a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathscr{P} \otimes_{\mathscr{O}} X \arrow[r] & \mathscr{P} \otimes_{\mathscr{O}} Y \end{tikzcd}$$

As Y is a $\mathscr{P}$-algebra, the right vertical map has a left inverse. Then the composite

$$\mathscr{P} \otimes_{\mathscr{O}} X \to \mathscr{P} \otimes_{\mathscr{O}} Y \to Y$$

makes the following diagram commute up to homotopy

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ \mathscr{P} \otimes_{\mathscr{O}} X \arrow[ur] & \end{tikzcd}$$

□

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