

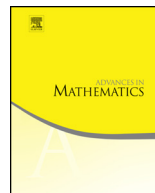


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# Infinite loop spaces from operads with homological stability



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## ABSTRACT

Motivated by the operad built from moduli spaces of Riemann surfaces, we consider a general class of operads in the category of spaces that satisfy certain homological stability conditions. We prove that such operads are infinite loop space operads in the sense that the group completions of their algebras are infinite loop spaces.

The recent, strong homological stability results of Galatius and Randal-Williams for moduli spaces of even dimensional manifolds can be used to construct examples of operads with homological stability. As a consequence diffeomorphism groups and mapping class groups are shown to give rise to infinite loop spaces. Furthermore, the map to  $K$ -theory defined by the action of the diffeomorphisms on the middle

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dimensional homology is shown to be a map of infinite loop spaces.

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## 1. Introduction

The theory of operads has its origin in the systematic study of loop spaces [1,3,16]. Operads give a tool for the construction and detection of loop spaces and, crucially, facilitate the study of maps between them. An important example is the little  $n$ -cube operad  $\mathcal{C}_n$  which detects  $n$ -fold loop spaces in the sense that the group completion of a  $\mathcal{C}_n$ -algebra is an  $n$ -fold loop space, and in particular  $\mathcal{C}_\infty$ -algebras group complete to infinite loop spaces. These are examples of  $E_n$ - and  $E_\infty$ -operads.

Let  $\mathcal{O} = \{\mathcal{O}(j)\}_{j \geq 0}$  be an operad where each  $\mathcal{O}(j)$  is a topological space equipped with a compatible action of the symmetric group  $\Sigma_j$ . In our setting, the space  $\mathcal{O}(0)$  has a base point  $*$  but may be much larger. Because the study of infinite loop spaces is equivalent to the study of connective spectra (in a sense that can be made precise)  $E_\infty$ -operads are of particular interest. An operad  $\mathcal{O}$  is an  $E_\infty$ -operad if each  $\mathcal{O}(j)$  is contractible and has a free  $\Sigma_j$ -action. In contrast, an operad  $\mathcal{O}$  is an  $E_1$ -operad (also called  $A_\infty$ -operad) if each  $\mathcal{O}(j)$  is homotopy equivalent to a free orbit of  $\Sigma_j$ .

In general there are obstructions to extending a given  $n$ -fold loop space structure to an  $n + 1$ -fold structure. Motivated by examples of operads that arise in the study of conformal and topological field theories, we prove here a very general theorem showing that the obstructions vanish in the presence of homological stability.

**Definition 1.1.** An operad  $\mathcal{O}$  is an **operad with homological stability (OHS)** if the following conditions are satisfied:

- (1) the operad is graded, in the sense that each  $\mathcal{O}(j)$  is the disjoint union of connected spaces  $\mathcal{O}_g(j)$  for  $g \in \mathbb{N}$  and the grading is compatible with the operad structure;
- (2) the maps  $D: \mathcal{O}_g(j) \rightarrow \mathcal{O}_g(0)$  induced by the operation of  $*$  in  $\mathcal{O}(0)$  are homology isomorphisms in degrees  $q \leq \phi(g)$  where  $\phi(g)$  tends to infinity with  $g$ ;
- (3) there is an  $A_\infty$ -operad  $\mathcal{A}$  concentrated in degree zero, and a map of graded operads  $\mu: \mathcal{A} \rightarrow \mathcal{O}$  such that the image of  $\mathcal{A}(2)$  in  $\mathcal{O}(2)$  is path-connected.

We refer to §4 for a more general definition. The  $\mathbb{N}$ -grading of the operad is for convenience and is evident in our main examples, but we can also consider operads graded by other abelian monoids. The third condition ensures that any  $\mathcal{O}$ -algebra is a monoid up to homotopy with a group completion and the condition on  $\mu(\mathcal{A}(2))$  is a very mild commutativity condition that allows an application of the group completion theorem; see §3. The second condition is the crucial one which enables us to prove the following very general theorem. Importantly, this condition can be weakened, see §4.

We need only ask that the map  $D$  induces a homology isomorphism on the homotopy colimits as  $g$  goes to infinity.

**Theorem 1.2.** *Every operad  $\mathcal{O}$  with homological stability (OHS) is an infinite loop space operad: Group completion defines a functor  $\mathcal{G}$  from the category of  $\mathcal{O}$ -algebras to the category of infinite loop spaces.*

A special case of this theorem was proved in [31] for a discrete model of the surface operad. The above theorem allows us to simplify the model and extend the theorem to operads associated to non-orientable surfaces and surfaces with other tangential structures. We discuss this in the Appendix. However, our main new examples are operads of higher dimensional manifolds. We now give a brief description; the details are given in §7. Let  $W_{g,j+1}$  be the connected sum of  $g$  copies of  $S^k \times S^k$  with  $j + 1$  open disks removed. We will think of it as a cobordism from  $j$  spheres of dimension  $2k - 1$  to one such sphere. As each  $W_{g,j+1}$  is  $k - 1$ -connected, the map to  $BO(2k)$  that classifies the tangent bundle factors through the  $k$ -connected cover  $\theta: B \rightarrow BO(2k)$ . We construct a graded operad  $\mathcal{W}^{2k}$  with

$$\mathcal{W}_g^{2k}(j) \simeq \mathcal{M}_k^\theta(W_{g,j+1}, \ell_{W_{g,j+1}})$$

the connected component of the moduli space of  $W_{g,j+1}$  with  $\theta$ -structure containing a given  $\theta$ -structure  $\ell_{W_{g,j+1}}$ , as defined and studied by Galatius and Randal-Williams. Analogously to the surface operad, the structure maps of  $\mathcal{W}^{2k}$  are defined by gluing the outgoing boundary sphere of one cobordism to an incoming sphere of another. Using the recent, strong homological stability results of [9] we prove the following.

**Theorem 1.3.** *For  $2k \geq 2$ , the operad  $\mathcal{W}^{2k}$  is an operad with homological stability (OHS).*

As an immediate consequence we construct infinite loop spaces from the classifying spaces of diffeomorphism groups of manifolds and their mapping class groups. Furthermore, we prove that the map induced by the action of  $\text{Diff}(W_{g,1}; \partial)$  on the middle dimensional homology  $H_k(W_{g,1}) \simeq \mathbb{Z}^{2g}$  induces a map of infinite loop spaces (or more precisely a zig-zag of such) from  $\Omega^\infty \mathbf{MT}(\theta)$  to  $K$ -theory.

The operad  $\mathcal{W}^{2k}$  can be considered as a subcategory of the  $2k$ -dimensional cobordism category as studied in [10] (after a slight modification). Thus the above theorem can be interpreted to say that for  $2k \geq 2$ , topological quantum field theories (TQFT) in topological spaces, i.e., symmetric monoidal functors from the  $2k$ -cobordism category to the category of spaces, take the  $(2k - 1)$ -sphere to an  $A_\infty$ -monoid whose group completion is an infinite loop space. One may thus reasonably restrict attention to studying TQFTs with values in (connected) spectra.

Finally, we note some restrictions imposed by our definition of an OHS. Above we have only mentioned new examples of operads for even dimensional manifolds. Even though homological stability has been extended to classes of odd dimensional manifolds by Perlmutter [22], at the moment there is no homological stability result available that

allows the number of boundary components to be reduced (attaching a disk). However this is the homological stability that is needed for condition 2 of our definition. The same remark holds for diffeomorphism groups for handlebodies which also have been shown to have homological stability by Perlmutter [23], and the stable homology been identified by Botvinnik and Perlmutter [4]. Similarly, Nariman [21] proves homological stability results for diffeomorphism groups of manifolds made discrete. Interestingly, for surfaces he proves the weaker stable homology stability needed [20]. However, for the discrete diffeomorphisms groups, condition 3 also poses a challenge and needs additional thought.

*Outline*

In §2 we recall familiar constructions from the theory of operads and monads. We define a group completion functor for homotopy associative monoids and establish several important basic properties in §3. In §4 we define operads with homological stability and in §5 we show that the group completions of free algebras over such operads are infinite loop spaces. We prove our main result in §6 and show that algebras over operads with homological stability group complete to infinite loop spaces. Our most important new example, the higher dimensional analog of the surface operad [31], is constructed and studied in §7. We describe various surface examples in the appendix.

*Acknowledgments*

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**2. Operads, monads, and the bar construction**

We recall some basic definitions and constructions from [16].

Let  $\mathcal{U}$  denote the category of compactly generated weakly Hausdorff spaces and continuous maps, and let  $\mathcal{T}$  denote the category of well pointed compactly generated Hausdorff spaces and based continuous maps.

An **operad** is a collection of spaces

$$\mathcal{O} = \prod_{n \geq 0} \mathcal{O}(n)$$

in  $\mathcal{U}$  with a nondegenerate base point  $*$   $\in$   $\mathcal{O}(0)$ , a distinguished element  $1 \in \mathcal{O}(1)$ , a right action of the symmetric group  $\Sigma_n$  on  $\mathcal{O}(n)$  for each  $n \geq 0$ , and structure maps

$$\gamma: \mathcal{O}(k) \times \mathcal{O}(j_1) \times \dots \times \mathcal{O}(j_k) \rightarrow \mathcal{O}(j_1 + \dots + j_k)$$

for  $k \geq 1$  and  $j_s \geq 0$ . The structure maps are required to be associative, unital, and equivariant in the following sense:

- (1) For all  $c \in \mathcal{O}(k)$ ,  $d_s \in \mathcal{O}(j_s)$ , and  $e_t \in \mathcal{O}(i_t)$

$$\gamma(\gamma(c; d_1, \dots, d_k); e_1, \dots, e_j) = \gamma(c; f_1, \dots, f_k)$$

where  $f_s = \gamma(d_s; e_{j_1+\dots+j_{s-1}+1}, \dots, e_{j_1+\dots+j_s})$  and  $f_s = d_s$  if  $j_s = 0$ .

- (2) For all  $d \in \mathcal{O}(j)$  and  $c \in \mathcal{O}(k)$

$$\gamma(1; d) = d \text{ and } \gamma(c; 1^k) = c.$$

- (3) For all  $c \in \mathcal{O}(k)$ ,  $d_s \in \mathcal{O}(j_s)$ ,  $\sigma \in \Sigma_k$  and  $\tau_s \in \Sigma_{j_s}$

$$\gamma(c\sigma; d_1, \dots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \dots, d_{\sigma^{-1}(k)})\sigma(j_1, \dots, j_k)$$

and

$$\gamma(c; d_1\tau_1, \dots, d_k\tau_k) = \gamma(c; d_1, \dots, d_k)(\tau_1 \oplus \dots \oplus \tau_k)$$

where  $\sigma(j_1, \dots, j_k) \in \Sigma_j$  permutes blocks of size  $j_s$  according to  $\sigma$ , and  $\tau_1 \oplus \dots \oplus \tau_k$  denotes the image of  $(\tau_1, \dots, \tau_k)$  under the natural inclusion of  $\Sigma_{j_1} \times \dots \times \Sigma_{j_k}$  into  $\Sigma_j$ .

While [16] and most of the literature assumes the space  $\mathcal{O}(0)$  of 0-ary operations is a single point, it is essential for us to allow more general spaces of 0-ary operations.

**Example 2.1.** There are two discrete operads that play particularly important roles.

- The  $n$ th space of the operad Com is a single point and all structure maps are the unique map to a point. This has trivial actions of the symmetric groups.
- The  $n$ th space of the operad Ass is  $\Sigma_n$  and the structure maps are given by block sum of permutations.

**Example 2.2** (Little  $n$ -cubes operad, [16, §4]). Let  $I^n$  denote the unit  $n$ -cube and let  $J^n$  denote its interior. An **(open) little  $n$ -cube** is a linear embedding  $\alpha_1 \times \dots \times \alpha_n : J^n \rightarrow J^n$  where  $\alpha_i : J^1 \rightarrow J^1$  is defined by  $\alpha_i(t) = a_it + b_i$  for  $0 < a_i \leq 1$ .

The  $j$ th space of the little  $n$ -cubes operad,  $\mathcal{C}_n(j)$ , is the set of  $j$ -tuples  $\langle c_1, \dots, c_j \rangle$  of little  $n$ -cubes where the images of the  $c_r$  are pairwise disjoint. The topology on  $\mathcal{C}_n(j)$  is induced from the topology on the space of continuous maps

$$\coprod_j J^n \rightarrow J^n.$$

The space  $\mathcal{C}_n(0)$  is a single point regarded as the unique “embedding” of the empty set in  $J^n$ . The structure maps are defined by composition of maps

$$\gamma(c; d_1, \dots, d_k) := c \circ (d_1 \amalg \dots \amalg d_k) : \left( \prod_{j_1} J^n \right) \amalg \dots \amalg \left( \prod_{j_k} J^n \right) \rightarrow J^n$$

for  $c \in \mathcal{C}_n(k)$  and  $d_s \in \mathcal{C}_n(j_s)$ . The identity object is  $1 \in \mathcal{C}_n(1)$  and the action of  $\sigma \in \Sigma_j$  is given by  $\langle c_1, \dots, c_j \rangle \sigma = \langle c_{\sigma(1)}, \dots, c_{\sigma(j)} \rangle$ .

**Example 2.3.** Let  $\mathcal{M}_{g,n}$  denote the moduli spaces of Riemann surfaces of genus  $g$  with  $n$  parametrized and ordered boundary components. Segal [28] constructed a symmetric monoidal category where the objects are finite unions of circles and morphism spaces are disjoint unions of the spaces  $\mathcal{M}_{g,n}$  with boundary circles divided into incoming and outgoing. Composition of morphisms is defined by gluing outgoing circles of one Riemann surface to incoming circles of another. A conformal field theory in the sense of [28] is then a symmetric monoidal functor from this surface category to an appropriate linear category.

By restricting the category we can define an operad  $\mathcal{M}$  where

$$\mathcal{M}(n) := \coprod_{g \geq 0} \mathcal{M}_{g,n+1}.$$

Note that  $\mathcal{M}(0)$  is non-trivial in this example. The operad  $\mathcal{M}$  contains a natural suboperad  $\mathcal{P}$  of Riemann surfaces of genus zero. The operad  $\mathcal{P}$  is levelwise homotopic to the framed little two disk operad [11, p. 282].

We describe closely related operads in [Appendix A](#).

A **map of operads**  $\mathcal{O} \rightarrow \mathcal{P}$  is a collection of  $\Sigma_n$ -equivariant maps  $\mathcal{O}(n) \rightarrow \mathcal{P}(n)$  which commute with the structure maps and preserve 1 and  $*$ .

**Example 2.4.** The map  $\phi_n : \mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$  defined by

$$\phi_n(j) (\langle c_1, \dots, c_j \rangle) := \langle c_1 \times 1, \dots, c_j \times 1 \rangle : J^{n+1} \rightarrow J^{n+1}$$

is a map of operads. Each  $\phi_n(j)$  is an inclusion, and we let  $\mathcal{C}_\infty(j)$  denote the colimit of  $\phi_n(j)$ . These spaces assemble to an operad we denote by  $\mathcal{C}_\infty$ .

There are maps of operads  $\mathcal{C}_1 \rightarrow \text{Ass}$  and  $\mathcal{C}_\infty \rightarrow \text{Com}$ . The first is a levelwise  $\Sigma_n$ -equivariant homotopy equivalence and the second is a levelwise homotopy equivalence [16, 3.5]. Motivated by these examples, an operad  $\mathcal{O}$  is an  $A_\infty$ -**operad** if there is a levelwise  $\Sigma_n$ -equivariant homotopy equivalence  $\mathcal{O} \rightarrow \text{Ass}$  and  $\mathcal{O}$  is an  $E_\infty$ -**operad** if the symmetric groups act freely levelwise and there is a levelwise homotopy equivalence  $\mathcal{O} \rightarrow \text{Com}$ . This implies that an  $E_\infty$ -operad is levelwise contractible.

There are many examples where a direct map of operads is difficult to construct and we can only relate operads through a zig-zag of maps. In these cases we use the product of operads.

**Definition 2.5.** The **product** of operads  $\mathcal{O}$  and  $\mathcal{P}$ , denoted  $\mathcal{O} \times \mathcal{P}$ , is defined by  $(\mathcal{O} \times \mathcal{P})(j) := \mathcal{O}(j) \times \mathcal{P}(j)$  with the following structure:

- (1) for  $c \times c' \in \mathcal{O}(k) \times \mathcal{P}(k)$  and  $d_s \times d'_s \in \mathcal{O}(j_s) \times \mathcal{P}(j_s)$

$$(\gamma \times \gamma')(c \times c'; d_1 \times d'_1, \dots, d_k \times d'_k) := \gamma(c; d_1, \dots, d_k) \times \gamma'(c'; d'_1, \dots, d'_k);$$

- (2)  $1 := 1 \times 1 \in \mathcal{O}(1) \times \mathcal{P}(1);$
- (3)  $* := * \times * \in \mathcal{O}(0) \times \mathcal{P}(0);$  and
- (4)  $(c \times c')\sigma := c\sigma \times c'\sigma$  for  $c \times c' \in \mathcal{O}(j) \times \mathcal{P}(j)$  and  $\sigma \in \Sigma_j.$

For  $A_\infty$ -operads we refine this definition to operads over Ass. If  $\mathcal{A}$  and  $\mathcal{A}'$  are  $A_\infty$ -operads,  $(\mathcal{A} \times_{\text{Ass}} \mathcal{A}')(n)$  is the pullback of the maps  $\mathcal{A}(n) \rightarrow \text{Ass}(n)$  and  $\mathcal{A}'(n) \rightarrow \text{Ass}(n)$ . The structure maps are defined so that  $\mathcal{A} \times_{\text{Ass}} \mathcal{A}'$  is a sub-operad of  $\mathcal{A} \times \mathcal{A}'$ . Then  $\mathcal{A} \times_{\text{Ass}} \mathcal{A}'$  is an  $A_\infty$ -operad and there is a canonical map to  $\mathcal{A} \times \mathcal{A}'$ .

An  $\mathcal{O}$ -algebra is a based space  $(X, *)$  with structure maps

$$\theta: \mathcal{O}(j) \times X^j \rightarrow X$$

for all  $j \geq 0$  such that

- (1) For all  $c \in \mathcal{O}(k)$ ,  $d_s \in \mathcal{O}(j_s)$ , and  $x_t \in X$

$$\theta(\gamma(c; d_1, \dots, d_k); x_1, \dots, x_j) = \theta(c; y_1, \dots, y_k)$$

where  $y_s = \theta(d_s; x_{j_1+\dots+j_{s-1}+1}, \dots, x_{j_1+\dots+j_s}).$

- (2) For all  $x \in X$

$$\theta(1; x) = x \text{ and } \theta(*) = *.$$

- (3) For all  $c \in \mathcal{O}(k)$ ,  $x_s \in X$ ,  $\sigma \in \Sigma_k$

$$\theta(c\sigma; x_1, \dots, x_k) = \theta(c; x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)}).$$

A **map of  $\mathcal{O}$ -algebras** is a map  $f: X \rightarrow Y$  in  $\mathfrak{T}$  that commutes with the structure maps. We will denote the category of  $\mathcal{O}$ -algebras by  $\mathcal{O}[\mathfrak{T}]$ .

An operad map  $\mathcal{O} \rightarrow \mathcal{P}$  defines a  $\mathcal{O}$ -algebra structure on any  $\mathcal{P}$ -algebra. Indeed, it defines a functor from  $\mathcal{P}[\mathfrak{T}]$  to  $\mathcal{O}[\mathfrak{T}]$ .

**Example 2.6.** The point  $*$  is trivially an  $\mathcal{O}$ -algebra, the space  $\mathcal{O}(0)$  is an  $\mathcal{O}$ -algebra with structure maps given by the operad structure maps, and every  $\mathcal{O}$ -algebra  $X$  receives a unique  $\mathcal{O}$ -algebra map  $\mathcal{O}(0) \rightarrow X$ . Additionally, the map  $X \rightarrow *$  is an  $\mathcal{O}$ -algebra map for all  $\mathcal{O}$ -algebras  $X$ , but the map  $* \rightarrow X$  is an  $\mathcal{O}$ -algebra map if and only if the map  $\mathcal{O}(0) \rightarrow X$  factors through  $*$ .

For each space  $X$  in  $\mathfrak{T}$  we can define a free  $\mathcal{O}$ -algebra on  $X$ . This associates a monad to each operad allowing us to use familiar constructions from the theory of monads in the context of operads [16, §2].

If  $\mathcal{O}$  is an operad and  $(X, *)$  is a based space, the **free  $\mathcal{O}$ -algebra on  $X$**  is

$$\mathbb{O}(X) := \coprod_{n \geq 0} (\mathcal{O}(n) \times_{\Sigma_n} X^n) / \sim$$

where  $\sim$  is a base point relation generated by

$$(\sigma_i c; x_1, \dots, x_{n-1}) \sim (c; s_i(x_1, \dots, x_{n-1}))$$

for all  $c \in \mathcal{O}(n)$ ,  $x_i \in X$ , and  $0 \leq i < n$  where  $\sigma_i c = \gamma(c, e_i)$  with

$$e_i = (1^i, *, 1^{n-i-1}) \in \mathcal{O}(1)^i \times \mathcal{O}(0) \times \mathcal{O}(1)^{n-i-1},$$

and  $s_i(x_1, \dots, x_{n-1}) = (x_1, \dots, x_i, *, x_{i+1}, \dots, x_{n-1})$ .

The class of  $(1, *) \in \mathcal{O}(1) \times X$  is the base point of  $\mathbb{O}(X)$ . Note that it coincides with the class of  $* \in \mathcal{O}(0)$ .

**Remark 2.7.** We will occasionally abuse notation and identify points in  $\mathbb{O}(X)$  with their preimage in  $\mathcal{O}(n) \times X^n$ . In particular, we will identify elements of  $\mathcal{O}(0)$  with their images in  $\mathbb{O}(X)$ . This gives an identification of  $\mathcal{O}(0)$  and  $\mathbb{O}(*)$ .

The free algebra construction is functorial in  $X$  and in  $\mathcal{O}$ . Any map of pointed spaces  $X \rightarrow Y$  induces a map of  $\mathcal{O}$ -algebras  $\mathbb{O}(X) \rightarrow \mathbb{O}(Y)$ . Any map of operads  $\mathcal{O} \rightarrow \mathcal{P}$  gives rise to a map of  $\mathcal{O}$ -algebras  $\mathbb{O}(X) \rightarrow \mathbb{P}(X)$ . The construction defines a monad in  $\mathfrak{T}$  and associates a natural transformation of monads  $\mathbb{O} \rightarrow \mathbb{P}$  to a map of operads  $\mathcal{O} \rightarrow \mathcal{P}$ . Hence, it defines a functor from operads in  $\mathfrak{U}$  to monads in  $\mathfrak{T}$ .

For a monad  $\mathbb{T}$  in a category  $\mathfrak{D}$ , a  **$\mathbb{T}$ -algebra** is a pair  $(X, \xi)$  consisting of an object  $X$  in  $\mathfrak{D}$  and a map  $\xi: \mathbb{T}(X) \rightarrow X$  in  $\mathfrak{D}$  that is unital and associative. A **map of  $\mathbb{T}$ -algebras** is a map of spaces  $f: X \rightarrow Y$  where the following diagram commutes.

$$\begin{array}{ccc} \mathbb{T}(X) & \xrightarrow{\mathbb{T}(f)} & \mathbb{T}(Y) \\ \xi_X \downarrow & & \downarrow \xi_Y \\ X & \xrightarrow{f} & Y \end{array}$$



The functor from operads to monads described above defines an isomorphism between the categories of  $\mathcal{O}$ -algebras and  $\mathbb{O}$ -algebras. The proofs of these facts in [16, 2.8] carry over verbatim to the more general case where  $\mathcal{O}(0)$  is not necessarily a single point.

A  **$\mathbb{T}$ -functor** [16, 2.2, 9.4] in a category  $\mathfrak{C}$  is a functor  $F: \mathfrak{D} \rightarrow \mathfrak{C}$  and a unital and associative natural transformation  $\lambda: F\mathbb{T} \rightarrow F$ . The **bar construction** [16, 9.6] for a monad  $\mathbb{T}$ ,  $\mathbb{T}$ -algebra  $X$  and  $\mathbb{T}$ -functor  $F$ , denoted  $B_{\bullet}(F, \mathbb{T}, X)$ , is the simplicial object in  $\mathfrak{C}$  where  $B_q(F, \mathbb{T}, X) := F\mathbb{T}^q X$ . If  $\mathfrak{C}$  is the category  $\mathfrak{T}$  of based spaces, we define

$$B(F, \mathbb{T}, X) := |B_{\bullet}(F, \mathbb{T}, X)|.$$

**Lemma 2.8.** [16, 9.7–9.10] *For  $\mathbb{T}$ ,  $F$  and  $X$  as above, the bar construction has the following properties:*

- (1) *For any functor  $G: \mathfrak{T} \rightarrow \mathfrak{T}$ ,  $GF$  is a  $\mathbb{T}$ -functor in  $\mathfrak{T}$  and there is a homeomorphism*

$$B(GF, \mathbb{T}, X) \cong |GB_{\bullet}(F, \mathbb{T}, X)|.$$

- (2) *The structure map  $\xi: \mathbb{T}(X) \rightarrow X$  induces a map of  $\mathbb{T}$ -algebras  $B(\mathbb{T}, \mathbb{T}, X) \xrightarrow{\sim} X$  that is a strong deformation retract of spaces with inverse induced by the unit  $\eta_{\mathbb{T}}: X \rightarrow \mathbb{T}(X)$ .*
- (3)  *$F(\xi)$  induces a strong deformation retract of spaces  $B(F, \mathbb{T}, \mathbb{T}(X)) \xrightarrow{\sim} F(X)$ .*
- (4) *If  $\delta: \mathbb{T} \rightarrow \mathbb{T}'$  is a natural transformation of monads, then  $\mathbb{T}'$  is a  $\mathbb{T}$ -functor and  $B_{\bullet}(\mathbb{T}', \mathbb{T}, X)$  is a simplicial  $\mathbb{T}'$ -algebra. The composite*

$$\tau: X \rightarrow B(\mathbb{T}, \mathbb{T}, X) \xrightarrow{B(\delta, 1, 1)} B(\mathbb{T}', \mathbb{T}, X)$$

*where the first map is induced by the unit  $\eta_{\mathbb{T}}: X \rightarrow \mathbb{T}(X)$ , coincides with the map induced by the unit  $\eta_{\mathbb{T}'}: X \rightarrow \mathbb{T}'(X)$ .*

We finish this section with the observation that the bar construction for a monad is closely related to the classifying space construction. Given a strict monoid  $M$ , we define an operad  $\mathcal{M}$  where

$$\mathcal{M}(0) = *, \mathcal{M}(1) = M, \text{ and } \mathcal{M}(n) = \{ \} \text{ for } n \geq 2.$$

The structure maps of the operad are given by the monoidal product and 1 is the monoidal unit. (Note that most of the structure maps for this operad are maps from the empty set to itself.)  $\mathcal{M}$ -algebras are spaces with an  $M$ -action. The monadic bar construction for the trivial  $\mathcal{M}$ -algebra  $S^0$  coincides with the realization of the nerve of the category with a single object and morphisms given by  $M$  with an additional disjoint base point:

$$B(\text{Id}, \mathbb{M}, S^0) = BM_+ \quad \text{where} \quad BM = |N_{\bullet}M|.$$

### 3. Monoid rectification and group completion

For every  $A_\infty$ -operad  $\mathcal{A}$  we define a functor from  $\mathcal{A}$ -algebras to groups up to homotopy that serves as the ‘group-completion’ alluded to in our main theorem. Our first step is to functorially replace the monoid up to homotopy defined by the  $A_\infty$ -structure by a homotopy equivalent monoid with a strictly associative multiplication. The second step is to define a group completion for strictly associative monoids.

#### 3.1. Monoid rectification

Let  $\mathcal{A}$  be an  $A_\infty$ -operad and let  $\delta: \mathbb{A} \rightarrow \mathbb{A}ss$  be the map of monads associated to the augmentation of operads  $\mathcal{A} \rightarrow \mathbb{A}ss$ . For an  $\mathcal{A}$ -algebra  $X$  define

$$\mathbf{M}_{\mathcal{A}}(X) := B(\mathbb{A}ss, \mathbb{A}, X).$$

**Proposition 3.1.** *For any  $A_\infty$ -operad  $\mathcal{A}$ , the construction  $\mathbf{M}_{\mathcal{A}}$  defines a functor from the category of  $\mathcal{A}$ -algebras to the category of strictly associative monoids. For every  $\mathcal{A}$ -algebra  $X$ , there is a strong deformation retract*

$$\rho: X \rightarrow \mathbf{M}_{\mathcal{A}}(X)$$

that is natural in  $X$ . Furthermore,  $X$  and  $\mathbf{M}_{\mathcal{A}}(X)$  are related by a zig-zag of  $\mathcal{A}$ -algebras. Hence  $\rho$  induces an isomorphism of monoids on connected components and an isomorphism of homology Pontryagin rings.

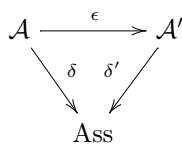
**Proof.** The simplicial space  $B_\bullet(\mathbb{A}ss, \mathbb{A}, X)$  is a simplicial  $\mathbb{A}ss$ -algebra (i.e. a simplicial topological monoid), and geometric realization preserves this structure [16, 9.10, 11.7]. Thus  $\mathbf{M}_{\mathcal{A}}$  defines a functor from  $\mathcal{A}$ -algebras to strict topological monoids.

Lemma 2.8 and properties of the bar construction give maps of  $\mathcal{A}$ -algebras

$$X \xleftarrow{\xi} B(\mathbb{A}, \mathbb{A}, X) \xrightarrow{B(\delta, 1, 1)} B(\mathbb{A}ss, \mathbb{A}, X) \tag{3.2}$$

where the first map is a strong deformation retract of spaces with right inverse induced by the unit  $\eta_{\mathbb{A}}: X \rightarrow \mathbb{A}(X)$ , and the second is a homotopy equivalence [17, A.2, A.4]. Let  $\rho$  be the map  $B(\delta, 1, 1) \circ \eta_{\mathbb{A}}$ . This is the desired homotopy equivalence of spaces which is natural in  $X$ .  $\square$

**Lemma 3.3.** *Suppose  $\mathcal{A}$  and  $\mathcal{A}'$  are  $A_\infty$ -operads and  $\epsilon: \mathcal{A} \rightarrow \mathcal{A}'$  is a map of operads so that the diagram*



commutes. If  $X$  is an  $\mathcal{A}'$ -algebra (regarded as an  $\mathcal{A}$ -algebra via  $\epsilon$ ), there is a map of monoids  $\mathbf{M}_{\mathcal{A}}(X) \rightarrow \mathbf{M}_{\mathcal{A}'}(X)$  that is a homotopy equivalence.

In particular, rectification is independent of the choice of  $A_{\infty}$ -operad. When clear from context, we will drop the subscript from the notation for the rectification functor.

**Proof.** There is a commutative diagram

$$\begin{array}{ccccc}
 & & B(\mathbb{A}, \mathbb{A}, X) & \xrightarrow{B(\delta, 1, 1)} & B(\mathbb{A}_{\text{ss}}, \mathbb{A}, X) \\
 & \swarrow \xi & \downarrow B(\epsilon, \epsilon, 1) & & \downarrow B(1, \epsilon, 1) \\
 X & & B(\mathbb{A}', \mathbb{A}', X) & \xrightarrow{B(\delta', 1, 1)} & B(\mathbb{A}_{\text{ss}}, \mathbb{A}', X) \\
 & \searrow \xi' & & & 
 \end{array}$$

where all arrows are homotopy equivalences and  $B(1, \epsilon, 1)$  is a map of Ass-algebras.  $\square$

The functoriality of the rectification functor is well-understood and we leave the proof of the following result as an exercise.

**Lemma 3.4.** *Let  $\mathcal{A}$  and  $\mathcal{A}'$  be  $A_{\infty}$ -operads.*

(1) *If  $X$  and  $X'$  are  $\mathcal{A}$ - and  $\mathcal{A}'$ -algebras, then  $X \times X'$  is a  $\mathcal{A} \times_{\text{Ass}} \mathcal{A}'$ -algebra and*

$$\mathbf{M}_{\mathcal{A} \times_{\text{Ass}} \mathcal{A}'}(X \times X') \simeq \mathbf{M}_{\mathcal{A}}(X) \times \mathbf{M}_{\mathcal{A}'}(X').$$

(2) *If  $X_{\bullet}$  is a simplicial  $\mathcal{A}$ -algebra then its realization is an  $\mathcal{A}$ -algebra and*

$$\mathbf{M}_{\mathcal{A}}|X_{\bullet}| \simeq |\mathbf{M}_{\mathcal{A}}X_{\bullet}|.$$

### 3.2. Constructing a group completion

For an algebraic monoid, group completion is defined by formally inverting elements; for a topological monoid, a natural analog is to use  $\Omega B$ , the based loops of the bar construction. Motivated by this observation, we define the **group completion** of a strictly associative topological monoid  $M$  to be

$$\mathcal{G}M := \Omega BM$$

where  $BM = |N_{\bullet}M|$ , the geometric realization of the nerve of  $M$  considered as a category with one object. The identification of  $M$  with the 1 simplices in the nerve  $N_{\bullet}M$  extends to a canonical map  $[0, 1] \times M \rightarrow BM$  which factors through  $S^1 \wedge M$ . Taking its adjoint defines a natural map

$$g: M \longrightarrow \Omega BM = \mathcal{G}M.$$

**Theorem 3.5.** *The assignment  $M \mapsto \mathcal{G}M$  defines a functor  $\mathcal{G}$  from well-pointed strictly associative topological monoids to  $\Omega$ -spaces and  $g: \text{Id} \rightarrow \mathcal{G}$  is a natural transformation.*

*Additionally, group completion has the following properties:*

- (1) *For monoids  $M$  and  $M'$ ,  $\mathcal{G}(M \times M') \simeq \mathcal{G}(M) \times \mathcal{G}(M')$ .*
- (2) *If  $M$  is a grouplike monoid, then  $g: M \rightarrow \mathcal{G}(M)$  is a weak equivalence.*
- (3) *If a map of monoids  $M \rightarrow M'$  is a homotopy equivalence, then  $\mathcal{G}M \rightarrow \mathcal{G}M'$  is a homotopy equivalence.*
- (4) *If  $M_\bullet$  is a simplicial monoid, then  $|\mathcal{G}M_\bullet| \simeq \mathcal{G}|M_\bullet|$ .*

We assume that all our monoids are well-pointed in the sense of [16, A.8]. This will ensure that the bar construction is a ‘good’ simplicial space in the sense of [28, A.4]. Thus the fat realization agrees with the usual realization and levelwise homotopy equivalences induce homotopy equivalences on total spaces.

**Proof of Theorem 3.5.** These are standard facts. Assertion (1) is an immediate consequence of the fact that the nerve  $N_\bullet(-)$ , geometric realization  $|-|$ , and taking loops  $\Omega(-)$  are all functors that take products to products. For (2) see, for example, McDuff–Segal [19, 1]. For (3) note that the induced map of nerves  $N_\bullet M \rightarrow N_\bullet M'$  is a levelwise homotopy equivalence, and hence induces a homotopy equivalence on realizations. Finally for (4), note that for any simplicial monoid  $M_\bullet$ ,

$$|BM_\bullet| \cong B|M_\bullet|$$

as the order in which the realization of bisimplicial objects is taken can be interchanged. Furthermore, as each  $BM_n$  is connected, taking based loops  $\Omega$  commutes with the realization functor, compare [16, 12.3], and hence  $|\Omega BM_\bullet| \cong \Omega|BM_\bullet|$  and so

$$|\Omega BM_\bullet| \cong \Omega B|M_\bullet|. \quad \square$$

We extend the above discussion to  $A_\infty$ -algebras by first applying the rectification functor. Thus, given an  $A_\infty$ -operad  $\mathcal{A}$  and an  $\mathcal{A}$ -algebra  $X$  define

$$\mathcal{G}(X) := \Omega B(\mathbf{M}_\mathcal{A}(X)) = \Omega B(B(\text{Ass}, \mathbb{A}, X)),$$

and let  $g: X \rightarrow \mathcal{G}(X)$  be the composite

$$X \xrightarrow{\rho} B(\text{Ass}, \mathbb{A}, X) = \mathbf{M}_\mathcal{A}(X) \longrightarrow \Omega B\mathbf{M}_\mathcal{A}(X) = \mathcal{G}(X).$$

With these definitions, group completion defines a functor from the category of  $\mathcal{A}$ -algebras to the category of loop spaces, and Theorem 3.5 remains valid. Indeed, (1) follows from Lemma 3.4(1), (2) and (3) hold since weak equivalences are preserved by the monoid rectification functor  $\mathbf{M}_\mathcal{A}$ , and (4) is a consequence of the observation that  $\mathbf{M}_\mathcal{A}$  commutes up to homotopy with realization for simplicial  $\mathcal{A}$ -algebras in Lemma 3.4(2).

### 3.3. Group completion theorem

We recall the group completion theorem from [7]; also see [19,25].

**Theorem 3.6.** *Let  $M$  be a well-pointed topological monoid. If  $\pi_0(M)$  is in the center of  $H_*(M; k)$  then*

$$g_* : H_*(M; k)[(\pi_0 M)^{-1}] \rightarrow H_*(\mathcal{G}M; k)$$

is an isomorphisms of  $k$ -algebras for all commutative coefficient rings  $k$ .

**Remark 3.7.** This result also holds under the hypothesis that  $H_*(M; k)[\pi_0(M)^{-1}]$  can be constructed by right fractions [7,25].

If  $\pi_0(M)$  is finitely generated by  $s_1, \dots, s_n$ , the localization can be achieved by inverting the product  $s := s_1 \dots s_n$ :

$$H_*(M; k)[(\pi_0 M)^{-1}] = H_*(M; k)[s^{-1}].$$

The right term can be identified with the homology of the telescope

$$\text{hocolim}_{\tilde{s}} M = \text{Tel}(M \xrightarrow{\tilde{s}} M \xrightarrow{\tilde{s}} M \xrightarrow{\tilde{s}} \dots)$$

where  $\tilde{s} \in M$  is in the connected component corresponding to  $s$ . Thus, if we let

$$M_\infty := (\text{hocolim}_{\tilde{s}} M)_{\text{Id}}$$

denote the connected component of the identity of  $\text{hocolim}_{\tilde{s}} M$ , the homology of a connected component of the group completion is given by

$$H_*((\mathcal{G}M)_{\text{Id}}; k) = H_*(M_\infty; k). \tag{3.8}$$

If the monoid is homotopy commutative, there is a refinement of the above theorem which allows for abelian coefficient systems. We thus have the following corollary, compare [25, 1.2].

**Corollary 3.9.** *Let  $M$  be a homotopy commutative monoid with  $\pi_0(M)$  finitely generated and  $M_\infty$  defined as above. Then the commutator subgroup of the fundamental group  $\pi_1(M_\infty)$  is perfect and there is a weak homotopy equivalence*

$$(\mathcal{G}M)_{\text{Id}} \simeq M_\infty^+,$$

where  $Z^+$  denotes Quillen’s plus construction.

Because the Pontryagin homology ring of an  $A_\infty$ -algebra is isomorphic to that of its rectification, the discussion above immediately also extends to any such monoid up to homotopy.

**Remark 3.10.** Any  $\mathcal{C}_2$ -algebra (or more generally any  $E_2$ -algebra) is homotopy commutative and hence the conditions of the group completion theorem, [Theorem 3.6](#), and its corollary are immediately satisfied.

#### 4. Operads with homological stability (OHS)

In this section we define the operads of primary interest. In particular, they are graded and receive a map from an  $A_\infty$ -operad.

**Definition 4.1.** Let  $I$  be a commutative, finitely generated monoid with identity element 0. An  $I$ -grading on an operad  $\mathcal{O}$  is a decomposition

$$\mathcal{O}(n) = \coprod_{g \in I} \mathcal{O}_g(n)$$

for each  $n$  so that:

- (1) the basepoint  $*$  lies in  $\mathcal{O}_0(0)$ ,
- (2) the  $\Sigma_n$  action on  $\mathcal{O}(n)$  restricts to an action on each  $\mathcal{O}_g(n)$ , and
- (3) the structure maps restrict to maps

$$\gamma: \mathcal{O}_g(k) \times \mathcal{O}_{g_1}(j_1) \times \dots \times \mathcal{O}_{g_k}(j_k) \rightarrow \mathcal{O}_{g+g_1+\dots+g_k}(j_1 + \dots + j_k).$$

In particular,  $1 \in \mathcal{O}_0(1)$  and the maps

$$\sigma_i := \gamma(-; 1^i, *, 1^{n-i-1}): \mathcal{O}(n) \rightarrow \mathcal{O}(n-1)$$

restrict to maps  $\sigma_i: \mathcal{O}_g(n) \rightarrow \mathcal{O}_g(n-1)$ .

We will mainly be concerned with  $\mathbb{N}$ -graded operads. Note that any ordinary operad is an  $I$ -graded operad concentrated in degree 0. Furthermore, if  $J$  is a submonoid of  $I$  then the graded pieces of  $\mathcal{O}$  corresponding to  $J$  form a suboperad of  $\mathcal{O}$ .

Let  $\mathcal{O}$  be an  $I$ -graded operad and assume there is a map of graded operads  $\mu: \mathcal{A} \rightarrow \mathcal{O}$  for some  $A_\infty$ -operad  $\mathcal{A}$ . We refer to  $\mu$  as a **multiplication map**. Then every  $\mathcal{O}$ -algebra is an  $A_\infty$ -algebra and we can apply the group completion functor defined in the previous section. Note that the image of  $\mathcal{A}$  is contained in the degree 0 part of  $\mathcal{O}$ , and for most of the remainder of this paper we will assume that  $\mathcal{O}$  satisfies the following **weak homotopy commutativity condition**:

$$\mu(\mathcal{A}(2)) \subset \mathcal{O}_0(2) \text{ is contained in a path component.} \tag{4.2}$$

**Remark 4.3.** Condition (4.2) implies there is a path between any two points in the image of  $\mu$ . Thus the induced multiplication on any  $\mathcal{O}$ -algebra is homotopy commutative and the group completion theorem, Theorem 3.6, and its corollary apply.

Let  $D: \mathcal{O}(n) \rightarrow \mathcal{O}(0)$  be defined by  $D(-) = \gamma(-; *, \dots, *)$ . For each  $n \in \mathbb{N}$ ,  $g \in I$  and  $\tilde{s} \in \mathcal{O}_s(1)$ , the diagram

$$\begin{CD} \mathcal{O}_g(n) @>\tilde{s}>> \mathcal{O}_{g+s}(n) \\ @V D VV @VV D V \\ \mathcal{O}_g(0) @>\tilde{s}>> \mathcal{O}_{g+s}(0) \end{CD}$$

commutes by the associativity of  $\gamma$ . Here we abuse notation and use  $\tilde{s}$  to denote the map  $\gamma(\tilde{s}; -)$ .

Now let  $s$  be the product of a set of generators for  $I$  and choose  $\tilde{s} \in \mathcal{O}_s(1)$ . We will refer to both the element  $\tilde{s}$  and the map  $\gamma(\tilde{s}, -)$  as the **propagator**. Define

$$\mathcal{O}_\infty(n) := \operatorname{hocolim}_{\tilde{s}} \mathcal{O}_g(n).$$

The commutative square above defines a map

$$D_\infty: \mathcal{O}_\infty(n) \longrightarrow \mathcal{O}_\infty(0).$$

Note that any other choice of  $\tilde{s}$  in the same path component of  $\mathcal{O}_s(1)$  will produce a homotopic map, but a choice from a different component may not.

**Remark 4.4.** In later sections it will be convenient to view the map  $\tilde{s}$  as the multiplication with an element  $\tilde{s}_0 \in \mathcal{O}_s(0)$ . For this purpose we construct  $\tilde{s}$  as follows: let  $\mu_0$  be an element in the path-connected image  $\mu(\mathcal{A}(2)) \subset \mathcal{O}_0(2)$  and pick  $\tilde{s}_0 \in \mathcal{O}_s(0)$ . Then define

$$\tilde{s} := \gamma(\mu_0; \tilde{s}_0, 1) \in \mathcal{O}_s(1).$$

For any  $\mathcal{O}$ -algebra  $X$ , the map  $\theta(\tilde{s}; -): X \rightarrow X$  is homotopic to the multiplication by the image of  $\tilde{s}_0$  under the map  $\theta: \mathcal{O}(0) \rightarrow X$ .

**Definition 4.5.** Let  $\mathcal{O}$  be an  $I$ -graded operad together with a map  $\mu: \mathcal{A} \rightarrow \mathcal{O}$  that satisfies condition (4.2). Then  $\mathcal{O}$  is an **operad with homological stability (OHS)** if the maps

$$D_\infty: \mathcal{O}_\infty(n) \longrightarrow \mathcal{O}_\infty(0)$$

induce homology isomorphisms.

For operads  $\mathcal{O}$  and  $\mathcal{P}$  with homological stability, a **map of operads with homological stability** is a pair of graded operad maps  $f: \mathcal{O} \rightarrow \mathcal{P}$  and  $a: \mathcal{A} \rightarrow \mathcal{A}'$  such that the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{a} & \mathcal{A}' \\
 \mu \downarrow & & \downarrow \mu' \\
 \mathcal{O} & \xrightarrow{f} & \mathcal{P}.
 \end{array}$$

For  $I = \mathbb{N}$ ,  $D_\infty$  is a homology isomorphism if the maps

$$D: \mathcal{O}_g(n) \rightarrow \mathcal{O}_g(0)$$

are homology isomorphisms in degrees  $* < \phi(g)$  where  $\phi$  goes to infinity as  $g$  goes to infinity. For operads concentrated in degree 0,  $\mathcal{O}_\infty(n) \simeq \mathcal{O}(n)$  and the condition is that  $D: \mathcal{O}(n) \rightarrow \mathcal{O}(0)$  is a homology isomorphism.

**Example 4.6.**  $\mathcal{C}_\infty$  is an OHS with multiplication  $\mu: \mathcal{C}_1 \rightarrow \mathcal{C}_\infty$  defined by the inclusion. Note that  $\mu$  factors through  $\mathcal{C}_2$  and so satisfies condition (4.2). (In fact, it is enough to note that  $\mathcal{C}_\infty(2)$  is contractible.) As all levels  $\mathcal{C}_\infty(n)$  are contractible, the homological stability condition is trivially satisfied.

**Example 4.7.** The Riemann surface operad  $\mathcal{M}$  receives a map from the genus zero operad  $\mathcal{P}$  and hence it satisfies condition (4.2).

By Harer’s homological stability theorem, the homology of the mapping class groups  $H_*(\Gamma_{g,n+1})$  are independent of  $g$  and  $n$  as long as  $*$  is small relative to  $g$  [13]. Since  $\mathcal{M}_g(n) \simeq B\Gamma_{g,n+1}$ , the maps  $D: \mathcal{M}_g(n) \rightarrow \mathcal{M}_g(0)$  are homology isomorphisms in degrees  $*$  growing with  $g$ .

In particular,  $\mathcal{M}$  is an operad with homological stability.

Taking the operadic product  $\mathcal{O} \times \mathcal{P}$  of operads  $\mathcal{O}$  and  $\mathcal{P}$  is functorial in both factors and the projection maps allow us to consider  $\mathcal{O}$ -algebras and  $\mathcal{P}$ -algebras as  $(\mathcal{O} \times \mathcal{P})$ -algebras. We can upgrade the product of operads to a product in the category of OHSs.

**Lemma 4.8.** *Let  $\mathcal{O}$  and  $\mathcal{P}$  be two OHSs. Then the product operad  $\mathcal{O} \times \mathcal{P}$  is an OHS with compatible maps of OHSs*

$$\mathcal{O} \xleftarrow{\pi_1} \mathcal{O} \times \mathcal{P} \xrightarrow{\pi_2} \mathcal{P}$$

where  $\pi_1$  and  $\pi_2$  are the projections to the first and second factors, respectively.

**Proof.** If  $\mathcal{O}$  is  $I$ -graded and  $\mathcal{P}$  is  $I'$ -graded, the product  $\mathcal{O} \times \mathcal{P}$  is  $I \times I'$ -graded with graded pieces

$$(\mathcal{O} \times \mathcal{P})_{(g,g')}(n) = \mathcal{O}_g(n) \times \mathcal{P}_{g'}(n).$$



The symmetric group  $\Sigma_n$  acts diagonally.

Let  $\mu: \mathcal{A} \rightarrow \mathcal{O}$  and  $\mu': \mathcal{A}' \rightarrow \mathcal{P}$  be the multiplications for  $\mathcal{O}$  and  $\mathcal{P}$ . Then  $\mathcal{A} \times_{\text{Ass}} \mathcal{A}'$  is an  $A_\infty$ -operad and the composite

$$\tilde{\mu}: \mathcal{A} \times_{\text{Ass}} \mathcal{A}' \longrightarrow \mathcal{A} \times \mathcal{A}' \xrightarrow{\mu \times \mu'} \mathcal{O} \times \mathcal{P}$$

defines a multiplication map for  $\mathcal{O} \times \mathcal{P}$ . Since  $\mathcal{O}$  and  $\mathcal{P}$  satisfy condition (4.2),  $\mathcal{O} \times \mathcal{P}$  also satisfies this condition.

Suppose  $I$  is generated by  $s_1, \dots, s_n$  and  $I'$  is generated by  $s'_1, \dots, s'_m$ . Let  $\tilde{s} = s_1 \times \dots \times s_n$  and  $\tilde{s}' = s'_1 \times \dots \times s'_m$ , then  $(\tilde{s}, \tilde{s}')$  is a generator for  $I \times I'$ . Thus  $\text{hocolim}_{(\tilde{s}, \tilde{s}')}(\mathcal{O} \times \mathcal{P})(n) \simeq \text{hocolim}_{\tilde{s}} \mathcal{O}(n) \times \text{hocolim}_{\tilde{s}'} \mathcal{P}(n)$  and we have

$$D_\infty \times D'_\infty: \mathcal{O}_\infty(n) \times \mathcal{P}_\infty(n) \longrightarrow \mathcal{O}_\infty(0) \times \mathcal{P}_\infty(0)$$

is a homology isomorphism. Then  $\mathcal{O} \times \mathcal{P}$  is an OHS.  $\square$

In the next sections we will focus on operads  $\mathcal{O}$  with homological stability that admit a map of OHSs<sup>1</sup>

$$\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty.$$

Any element  $\tilde{c} \in \mathcal{C}_\infty(1)$  is a propagator for  $\mathcal{C}_\infty$ , and we have that

$$\text{hocolim}_{\tilde{c}} \mathcal{C}_\infty(n) \simeq \mathcal{C}_\infty(n). \tag{4.9}$$

In particular if  $\tilde{s}$  is the propagator for the OHS  $\mathcal{O}$ , then  $\pi(\tilde{s})$  is a propagator for  $\mathcal{C}_\infty$ . To simplify notation and in light of (4.9) we write  $\mathcal{C}_\infty(n)$  for  $\text{hocolim}_{\pi(\tilde{s})} \mathcal{C}_\infty(n)$  and we have maps  $\pi_\infty: \mathcal{O}_\infty(n) \rightarrow \mathcal{C}_\infty(n)$ .

The following result allows us to replace  $\mathcal{O}$ -algebras with  $\tilde{\mathcal{O}}$ -algebras, where  $\tilde{\mathcal{O}}$  is an operad with an OHS map to  $\mathcal{C}_\infty$ .

**Corollary 4.10.** *Let  $\mathcal{O}$  be an OHS. Then the product operad  $\tilde{\mathcal{O}} := \mathcal{O} \times \mathcal{C}_\infty$  is an OHS with compatible maps of operads*

$$\mathcal{O} \xleftarrow{\pi_1} \tilde{\mathcal{O}} \xrightarrow{\pi} \mathcal{C}_\infty$$

where  $\pi_1$  and  $\pi$  are the projections to the first and second factors respectively.

Indeed,  $\mathcal{O} \mapsto \tilde{\mathcal{O}}$  defines a functor from the category of OHSs to the category of OHSs over  $\mathcal{C}_\infty$ .

---

<sup>1</sup>  $\mathcal{C}_\infty$  is a convenient  $E_\infty$ -operad, but any other  $E_\infty$ -operad could be used.

**Proof.** This follows immediately from [Lemma 4.8](#) since  $\mathcal{C}_\infty$  is an OHS.  $\square$

Thus any  $\mathcal{O}$ -algebra is an  $\tilde{\mathcal{O}}$ -algebra, and the statements and properties we prove for  $\tilde{\mathcal{O}}$ -algebras will equally apply to  $\mathcal{O}$ -algebras. Likewise, any  $\mathcal{C}_\infty$ -algebra is a  $\tilde{\mathcal{C}}$ -algebra, and for each  $n \geq 0$ , the symmetric group  $\Sigma_n$  acts freely on  $\tilde{\mathcal{C}}(n)$ .

### 5. The group completion of free algebras over an OHS

The goal of this section is to describe the group completion of a free  $\mathcal{O}$ -algebra for an OHS with a compatible map  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$ . (Recall from the previous section that there is no loss of generality in assuming we have the map  $\pi$ .) We will show that, up to homotopy, the group completion of the free  $\mathcal{O}$ -algebra on a based space  $X$  is a product of the group completion of  $\mathcal{O}(0)$  and the free infinite loop space on  $X$ . This will be used in the following section where we replace  $X$  by its free resolution as an  $\mathcal{O}$ -algebra. First however, we will prove [Lemma 5.2](#) which uses the homological stability condition in the definition of an OHS in an essential way.

For each  $n$  and  $\Sigma_n$ -space  $Y$ , we have  $\Sigma_n$ -equivariant maps

$$D_\infty \times *: \mathcal{O}_\infty(n) \times Y \rightarrow \mathcal{O}_\infty(0) \times *$$

inducing maps  $\bar{D}_\infty: \mathcal{O}_\infty(n) \times_{\Sigma_n} Y \rightarrow \mathcal{O}_\infty(0)$ . Taking the product with the compatible map of operads  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$  gives a map

$$\bar{D}_\infty \times (\pi_\infty \times 1_Y): \mathcal{O}_\infty(n) \times_{\Sigma_n} Y \rightarrow \mathcal{O}_\infty(0) \times (\mathcal{C}_\infty(n) \times_{\Sigma_n} Y). \tag{5.1}$$

**Lemma 5.2.** *Suppose  $\mathcal{O}$  is an OHS and there is a compatible map  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$ . Then the induced map*

$$H_*(\mathcal{O}_\infty(n) \times_{\Sigma_n} Y) \rightarrow H_*(\mathcal{O}_\infty(0) \times (\mathcal{C}_\infty(n) \times_{\Sigma_n} Y))$$

*is an isomorphism for all  $\Sigma_n$ -spaces  $Y$ .*

**Proof.** The space  $\mathcal{C}_\infty(n)$  is contractible, and so the maps

$$\mathcal{O}_\infty(n) \xrightarrow{1 \times *Y} \mathcal{O}_\infty(n) \times Y \xrightarrow{\pi_\infty \times 1_Y} \mathcal{C}_\infty(n) \times Y \tag{5.3}$$

induce a long exact sequence on homotopy groups. Since  $\pi_\infty$  is  $\Sigma_n$ -equivariant and the actions of  $\Sigma_n$  on  $\mathcal{O}_\infty(n)$  and  $\mathcal{C}_\infty(n)$  are free, the long exact sequence in homotopy groups for (5.3) implies the maps

$$\mathcal{O}_\infty(n) \xrightarrow{1 \times *Y} \mathcal{O}_\infty(n) \times_{\Sigma_n} Y \xrightarrow{\pi_\infty \times 1_Y} \mathcal{C}_\infty(n) \times_{\Sigma_n} Y$$

induce a long exact sequence on homotopy groups.

Consider the following diagram where the second and third rows are fibrations and the first row induces a long exact sequence on homotopy groups.

$$\begin{array}{ccccc}
 \mathcal{O}_\infty(n) & \xrightarrow{1 \times *Y} & \mathcal{O}_\infty(n) \times_{\Sigma_n} Y & \xrightarrow{\pi_\infty \times 1Y} & \mathcal{C}_\infty(n) \times_{\Sigma_n} Y \\
 \parallel & & \downarrow (1 \times \pi) \times 1 & & \parallel \\
 \mathcal{O}_\infty(n) & \xrightarrow{1 \times *Y} & (\mathcal{O}_\infty(n) \times \mathcal{C}_\infty(n)) \times_{\Sigma_n} Y & \xrightarrow{\text{proj}_2 \times 1Y} & \mathcal{C}_\infty(n) \times_{\Sigma_n} Y \\
 D_\infty \downarrow & & \downarrow \bar{D}_\infty \times 1 \times 1Y & & \parallel \\
 \mathcal{O}_\infty(0) & \xrightarrow{1 \times *} & \mathcal{O}_\infty(0) \times (\mathcal{C}_\infty(n) \times_{\Sigma_n} Y) & \xrightarrow{\text{proj}_2} & \mathcal{C}_\infty(n) \times_{\Sigma_n} Y
 \end{array}$$

The top left square commutes up to homotopy and the remaining small squares all commute strictly.

As the  $\Sigma_n$  action on  $\mathcal{O}_\infty(n)$  is free, the map

$$\mathcal{O}_\infty(n) \times_{\Sigma_n} Y \rightarrow (\mathcal{O}_\infty(n) \times \mathcal{C}_\infty(n)) \times_{\Sigma_n} Y$$

is a weak homotopy equivalence. The map between the bottom two rows defines a map of fibrations and hence a map of Serre spectral sequences. On the  $E_2$ -page, the map

$$H_p(\mathcal{C}_\infty(n) \times_{\Sigma_n} Y; H_q(\mathcal{O}_\infty(n))) \rightarrow H_p(\mathcal{C}_\infty(n) \times_{\Sigma_n} Y; H_q(\mathcal{O}_\infty(0)))$$

is an isomorphism since  $D_\infty$  induces an isomorphism in homology by the assumption that  $\mathcal{O}$  is an OHS. This induces an isomorphism on the  $E_\infty$ -page and hence the desired isomorphism between the homology of the total spaces [35, 3.26].  $\square$

As we assume we have a compatible map of operads  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$ , any  $\mathcal{C}_\infty$ -algebra is an  $\mathcal{O}$ -algebra. For any space  $X$ , there is a map of  $\mathcal{O}$ -algebras

$$\tau \times \pi: \mathbb{O}(X) \rightarrow \mathbb{O}(*) \times \mathbb{C}_\infty(X),$$

where the map  $\tau$  is induced by the unique map  $X \rightarrow *$  and the diagonal action of  $\mathcal{O}$  provides the  $\mathcal{O}$ -algebra structure on the product  $\mathbb{O}(*) \times \mathbb{C}_\infty(X)$ . Being maps of  $\mathcal{O}$ -algebras,  $\tau$  and  $\pi$  are maps of  $\mathcal{A}$ -algebras, so they induce a map on group completions

$$\mathcal{G}(\tau) \times \mathcal{G}(\pi): \mathcal{G}\mathbb{O}(X) \rightarrow \mathcal{G}(\mathbb{O}(*)) \times \mathcal{G}(\mathbb{C}_\infty(X)).$$

We can now state the main result of this section.

**Theorem 5.4.** *Suppose  $\mathcal{O}$  is an OHS with a compatible operad map  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$ . For any based space  $X$ ,*

$$\mathcal{G}(\tau) \times \mathcal{G}(\pi): \mathcal{G}(\mathbb{O}(X)) \rightarrow \mathcal{G}(\mathbb{O}(*)) \times \mathcal{G}(\mathbb{C}_\infty(X))$$

*is a weak homotopy equivalence.*

**Proof.** For any based space  $X$ , the grading on  $\mathcal{O}$  defines a decomposition of  $\mathbb{O}(X)$  as  $\coprod_{g \in I} \mathbb{O}_g(X)$  where

$$\mathbb{O}_g(X) := \left( \prod_n \mathcal{O}_g(n) \times_{\Sigma_n} X^n \right) / \sim$$

and the equivalence relation is as in the definition of the free  $\mathcal{O}$ -algebra.

Similarly, we have a natural decomposition of  $\mathbb{O}(\ast) \times \mathbb{C}_\infty(X)$  with graded pieces

$$\mathbb{O}_g(\ast) \times \mathbb{C}_\infty(X).$$

Left multiplication by the propagator defines maps

$$\tilde{s}: \mathbb{O}_g(X) \rightarrow \mathbb{O}_{g+s}(X) \quad \text{and} \quad \tilde{s} \times 1: \mathbb{O}_g(\ast) \times \mathbb{C}_\infty(X) \rightarrow \mathbb{O}_{g+s}(\ast) \times \mathbb{C}_\infty(X)$$

Let

$$\mathbb{O}_\infty(X) \quad \text{and} \quad \mathbb{O}_\infty(\ast) \times \mathbb{C}_\infty(X)$$

denote the homotopy colimits over these maps. Then, under the homeomorphism  $\mathbb{O}_g(\ast) \cong \mathcal{O}_g(0)$ , Remark 4.4 shows  $\mathbb{O}_\infty(\ast) \cong \mathcal{O}_\infty(0)$ .

It will be enough to show that the map

$$\tau_\infty \times \pi_\infty: \mathbb{O}_\infty(X) \longrightarrow \mathbb{O}_\infty(\ast) \times \mathbb{C}_\infty(X) \tag{5.5}$$

is a homology isomorphism. Since, if we let  $s$  denote the component of  $\tilde{s}$ , it will follow that

$$\tau \times \pi: H_*(\mathbb{O}(X))[s^{-1}] \longrightarrow H_*(\mathbb{O}(\ast) \times \mathbb{C}_\infty(X))[s^{-1}]$$

is an isomorphism. By the group completion theorem,

$$\mathcal{G}(\tau) \times \mathcal{G}(\pi): \mathcal{G}(\mathbb{O}(X)) \rightarrow \mathcal{G}(\mathbb{O}(\ast)) \times \mathcal{G}(\mathbb{C}_\infty(X))$$

induces an isomorphism in homology. As both source and target are simple spaces, the statement of the theorem will then follow by the Whitehead theorem.

We have filtrations

$$F_n = \left( \prod_{k \leq n} \mathcal{O}_\infty(k) \times_{\Sigma_k} X^k \right) / \sim$$

of  $\mathbb{O}_\infty(X)$  and

$$\tilde{F}_n = \left( \prod_{k \leq n} \mathcal{O}_\infty(0) \times \mathcal{C}_\infty(k) \times_{\Sigma_k} X^k \right) / \sim$$

of  $\mathbb{O}_\infty(*) \times \mathbb{C}_\infty(X)$ . The associated subquotients are

$$F_n/F_{n-1} = \mathcal{O}_\infty(n) \times_{\Sigma_n} X^{\wedge n} \text{ and } \tilde{F}_n/\tilde{F}_{n-1} = \mathcal{O}_\infty(0) \times \mathcal{C}_\infty(n) \times_{\Sigma_n} X^{\wedge n}$$

where  $\times$  is the **half smash product** defined by  $Y \times Z := (Y \times Z)/(Y \times *)$ . The map in (5.5) is compatible with these filtrations, and so there are induced maps

$$F_n/F_{n-1} \rightarrow \tilde{F}_n/\tilde{F}_{n-1}. \tag{5.6}$$

There is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_\infty(n) & \longrightarrow & \mathcal{O}_\infty(n) \times Y \\ \downarrow \bar{D} \times \pi & & \downarrow (\bar{D} \times \pi) \times 1_Y \\ \mathcal{O}_\infty(0) \times \mathcal{C}_\infty(n) & \longrightarrow & \mathcal{O}_\infty(0) \times (\mathcal{C}_\infty(n) \times Y) \end{array}$$

where the maps to  $Y$  are inclusion of the base point. Taking the quotient by the  $\Sigma_n$  action (and observing that  $\Sigma_n$  acts trivially on  $\mathcal{O}_\infty(0)$ ), we have a map of cofiber sequences

$$\begin{array}{ccccc} \mathcal{O}_\infty(n)/\Sigma_n & \longrightarrow & \mathcal{O}_\infty(n) \times_{\Sigma_n} Y & \longrightarrow & \mathcal{O}_\infty(n) \times_{\Sigma_n} Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_\infty(0) \times (\mathcal{C}_\infty(n) \times_{\Sigma_n} *) & \longrightarrow & \mathcal{O}_\infty(0) \times (\mathcal{C}_\infty(n) \times_{\Sigma_n} Y) & \longrightarrow & \mathcal{O}_\infty(0) \times (\mathcal{C}_\infty(n) \times_{\Sigma_n} Y). \end{array}$$

The first two vertical maps are as in (5.1) and, if we take  $Y = X^{\wedge n}$ , the last map is (5.6). The long exact sequence in homology, Lemma 5.2, and the five lemma imply (5.6) is an isomorphism in homology. The filtrations above define a pair of spectral sequences and (5.6) defines a map between them. These spectral sequences converge to the homology of  $\mathbb{O}_\infty(X)$  and  $\mathbb{O}_\infty(*) \times \mathbb{C}_\infty(X)$  and (5.5) induces (5.6). Since (5.6) is an isomorphism in homology, [18, 3.4] implies (5.5) is an isomorphism in homology.  $\square$

### 6. OHS as infinite loop space operad

In this section we use Theorem 5.4 to show that, for any OHS  $\mathcal{O}$ , the group completion of an  $\mathcal{O}$ -algebra  $X$  is an infinite loop space. The idea is to replace  $X$  by its free resolution as an  $\mathcal{O}$ -algebra using the bar construction. In Lemma 6.2, we show that, after group completing each level, the (homotopy) quotient of  $\mathcal{G}X$  by  $\mathcal{G}\mathcal{O}(*)$  is an infinite loop space. To see that  $\mathcal{G}X$  is also an infinite loop space, in Proposition 6.4 we consider  $\mathcal{O}(*) \times X$  with the diagonal  $\mathcal{O}$ -algebra structure and deduce that the (homotopy) quotient

of  $\mathcal{G}(\mathcal{O}(\ast) \times X) \simeq \mathcal{G}(\mathcal{O}(\ast)) \times \mathcal{G}(X)$  by  $\mathcal{G}(\mathcal{O}(\ast))$  is  $\mathcal{G}(X)$  and hence an infinite loop space. Our main theorem, [Theorem 6.5](#), summarizes these results.

We start by proving the naturality of our construction.

**Proposition 6.1.** *Suppose that  $\mathcal{O}$  is an OHS with compatible operad map  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$ . The assignment  $X \mapsto |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, X)|$  defines a functor from  $\mathcal{O}$ -algebras to  $\Omega^\infty$ -spaces.*

**Proof.** There is a map of monads  $\alpha: \mathbb{C}_\infty \rightarrow \Omega^\infty \Sigma^\infty$  [[16, 5.2](#)] and, for every based space  $Z$ , the map  $\alpha: \mathbb{C}_\infty Z \rightarrow \Omega^\infty \Sigma^\infty Z$  is a group completion [[17, 2.2](#)].

For a map  $f: X \rightarrow Y$  of  $\mathcal{O}$ -algebras, we have a commutative diagram where all horizontal maps are induced by  $f$ .

$$\begin{array}{ccc}
 |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, X)| & \longrightarrow & |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, Y)| \\
 \downarrow & & \downarrow \\
 |\mathcal{G}B_\bullet(\Omega^\infty \Sigma^\infty, \mathbb{O}, X)| & \longrightarrow & |\mathcal{G}B_\bullet(\Omega^\infty \Sigma^\infty, \mathbb{O}, Y)| \\
 \uparrow & & \uparrow \\
 |\mathcal{G}\Omega^\infty B_\bullet(\Sigma^\infty, \mathbb{O}, X)| & \longrightarrow & |\mathcal{G}\Omega^\infty B_\bullet(\Sigma^\infty, \mathbb{O}, Y)| \\
 \uparrow & & \uparrow \\
 |\Omega^\infty B_\bullet(\Sigma^\infty, \mathbb{O}, X)| & \longrightarrow & |\Omega^\infty B_\bullet(\Sigma^\infty, \mathbb{O}, Y)| \\
 \downarrow & & \downarrow \\
 \Omega^\infty |B_\bullet(\Sigma^\infty, \mathbb{O}, X)| & \longrightarrow & \Omega^\infty |B_\bullet(\Sigma^\infty, \mathbb{O}, Y)|
 \end{array}$$

The first vertical maps are induced by  $\alpha: \mathbb{C}_\infty \rightarrow \Omega^\infty \Sigma^\infty$  and are levelwise weak equivalences since they are levelwise homology isomorphisms between simple spaces. The second maps are equivalences by [Lemma 2.8](#). Since infinite loop spaces are grouplike, the third maps are equivalences by [Theorem 3.5](#). The fourth maps are equivalences by [[16, 12.3](#)].

Note that the last horizontal map in the diagram above is a map of  $\Omega^\infty$ -spaces.  $\square$

The inclusion of the base point  $\ast \rightarrow X$  induces a map of  $\mathbb{O}$ -algebras  $\mathbb{O}(\ast) \rightarrow \mathbb{O}(X)$  and a simplicial map  $\mathcal{G}\mathbb{O}(\ast) \rightarrow \mathcal{G}B_\bullet(\mathbb{O}, \mathbb{O}, X)$ , where  $\mathcal{G}\mathbb{O}(\ast)$  is a constant simplicial space.

The operad maps  $\mathcal{A} \rightarrow \mathcal{O} \xrightarrow{\pi} \mathcal{C}_\infty$  induce maps of monads  $\mathbb{A} \rightarrow \mathbb{O} \xrightarrow{\pi} \mathbb{C}_\infty$  and a map of simplicial  $\mathbb{A}$ -algebras  $B_\bullet(\mathbb{O}, \mathbb{O}, X) \rightarrow B_\bullet(\mathbb{C}_\infty, \mathbb{O}, X)$ . Applying the functor  $\mathcal{G}$  levelwise, we have a simplicial map

$$\mathcal{G}(B(\pi, 1, 1)): \mathcal{G}B_\bullet(\mathbb{O}, \mathbb{O}, X) \rightarrow \mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, X).$$

**Lemma 6.2.** *Let  $\mathcal{O}$  be an OHS with a compatible operad map  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$ . For any  $\mathcal{O}$ -algebra  $X$  there is a homotopy fibration sequence*

$$\mathcal{G}\mathbb{O}(\ast) \rightarrow |\mathcal{G}B_{\bullet}(\mathbb{O}, \mathbb{O}, X)| \xrightarrow{\mathcal{G}B(\pi,1,1)} |\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, X)|.$$

**Proof.** For each fixed  $n$  we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{G}\mathbb{O}(\ast) & \longrightarrow & \mathcal{G}\mathbb{O}^{n+1}X & \longrightarrow & \mathcal{G}\mathbb{C}_{\infty}\mathbb{O}^nX \\ \parallel & & \simeq \downarrow & & \parallel \\ \mathcal{G}\mathbb{O}(\ast) & \xrightarrow{\theta} & \mathcal{G}\mathbb{O}(\ast) \times \mathcal{G}\mathbb{C}_{\infty}\mathbb{O}^nX & \xrightarrow{\text{proj}} & \mathcal{G}\mathbb{C}_{\infty}\mathbb{O}^nX \end{array}$$

where the map  $\theta$  is the identity on the first factor and the composite

$$\mathcal{G}\mathbb{O}(\ast) \rightarrow \mathcal{G}\mathbb{O}(\mathbb{O}^nX) \xrightarrow{\pi} \mathcal{G}\mathbb{C}_{\infty}\mathbb{O}^nX$$

on the second factor. The center vertical arrow is the homotopy equivalence of [Theorem 5.4](#). The bottom row is a fiber sequence up to homotopy, and so is the top row.

In order to deduce that the top row is a homotopy fibration sequence after realization, we need to show that the conditions of [\[5, B.4\]](#) are satisfied, namely, that the simplicial spaces  $\mathcal{G}B_{\bullet}(\mathbb{O}, \mathbb{O}, X)$  and  $\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, X)$  satisfy the  $\pi_{\ast}$ -Kan condition and that the map between them is a fibration on  $\pi_0$ . This follows since both the total and base simplicial spaces are levelwise loop spaces, and the structure maps for both simplicial spaces as well as the map between them are maps of loop spaces. To be more explicit, let  $Z_{\bullet}$  be a simplicial space with  $Z_m$  simple for  $m \geq 0$ , and let  $\pi_t(Z_m)_{free}$  denote the set of unpointed homotopy classes of maps from the  $t$ -sphere to  $Z_m$ . Then  $Z_{\bullet}$  satisfies the  $\pi_{\ast}$ -Kan condition if and only if the simplicial map  $\pi_t(Z_{\bullet})_{free} \rightarrow \pi_0(Z_{\bullet})$  is a fibration for each  $t \geq 1$  [\[5, B.3.1\]](#). Thus both  $\mathcal{G}B_{\bullet}(\mathbb{O}, \mathbb{O}, X)$  and  $\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, X)$  satisfy the  $\pi_{\ast}$ -Kan condition, and the induced map on connected components

$$\pi_0\mathcal{G}B_{\bullet}(\mathbb{O}, \mathbb{O}, X) \rightarrow \pi_0\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, X)$$

is a map of simplicial groups and hence a fibration.  $\square$

If  $X$  is an  $\mathcal{O}$ -algebra, there is a map of  $\mathbb{O}$ -algebras  $\mathbb{O}(\ast) \rightarrow \mathbb{O}(X) \xrightarrow{\xi} X$  and, applying group completion, this defines a map  $\mathcal{G}\mathbb{O}(\ast) \rightarrow \mathcal{G}X$ . An operad map  $\mathcal{A} \rightarrow \mathcal{O}$  defines a monoid structure on  $X$  so the composite

$$\mathcal{G}\mathbb{O}(\ast) \times \mathcal{G}X \rightarrow \mathcal{G}X \times \mathcal{G}X \rightarrow \mathcal{G}X, \tag{6.3}$$

where the second map is loop sum, is an action  $\mathcal{G}\mathbb{O}(\ast)$  on  $\mathcal{G}X$  up to homotopy.

**Proposition 6.4.** *Let  $\mathcal{O}$  be an operad with homological stability and suppose there is a compatible map  $\pi: \mathcal{O} \rightarrow \mathcal{C}_{\infty}$ . For any  $\mathcal{O}$ -algebra  $X$ , there is a weak homotopy equivalence between  $\mathcal{G}X$  and  $|\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, \mathbb{O}(\ast) \times X)|$  where  $\mathbb{O}(\ast) \times X$  has the product  $\mathcal{O}$ -algebra structure.*

**Proof.** Lemma 6.2, applied to the product  $\mathcal{O}$ -algebra  $\mathbb{O}(*)\times X$ , implies the top horizontal row in the following diagram is a fibration sequence:

$$\begin{array}{ccccc}
 \mathcal{G}\mathbb{O}(*) & \xrightarrow{\iota} & |\mathcal{G}B_{\bullet}(\mathbb{O}, \mathbb{O}, \mathbb{O}(*)\times X)| & \longrightarrow & |\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, \mathbb{O}(*)\times X)| \\
 \parallel & & \downarrow \mathcal{G}(\text{eval}) & & \\
 & & \mathcal{G}(\mathbb{O}(*)\times X) & & \\
 & & \downarrow \simeq & & \\
 \mathcal{G}\mathbb{O}(*) & \xrightarrow{\Delta} & \mathcal{G}\mathbb{O}(*)\times \mathcal{G}X & & 
 \end{array}$$

Let  $x_0$  be the base point of  $X$ . The map  $\iota$  is obtained by mapping  $*$  to the basepoint  $((1, *), x_0) \in \mathbb{O}(*)\times X$  and extending freely to a map  $\mathbb{O}(*)\rightarrow \mathbb{O}((\mathbb{O}(*)\times X))$ . The map  $\text{eval}$  is induced by the diagonal action of the monad  $\mathbb{O}$  on  $\mathbb{O}(*)\times X$ . For the diagram above to commute, the image of  $a \in \mathcal{G}(\mathbb{O}(*))$  under  $\Delta$  must be  $(a, a\bar{x}_0) \in \mathcal{G}\mathbb{O}(*)\times \mathcal{G}X$  where  $\bar{x}_0$  denotes the image of  $x_0$  in  $\mathcal{G}X$  and  $a\bar{x}_0$  is the image of  $(a, \bar{x}_0)$  under (6.3). The vertical maps are homotopy equivalences, and so there is a fibration sequence

$$\mathcal{G}\mathbb{O}(*) \xrightarrow{\Delta} \mathcal{G}\mathbb{O}(*)\times \mathcal{G}X \longrightarrow |\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, \mathbb{O}(*)\times X)|.$$

Now consider the **shearing** map  $\text{sh}: \mathcal{G}\mathbb{O}(*)\times \mathcal{G}X \rightarrow \mathcal{G}\mathbb{O}(*)\times \mathcal{G}X$  defined by  $\text{sh}((a, x)) = (a, ax)$ . As  $\mathcal{G}\mathbb{O}(*)$  is a group up to homotopy,  $\text{sh}$  is a homotopy equivalence with homotopy inverse given by  $\text{sh}^{-1}((a, x)) = (a, a^{-1}x)$ . There is a commutative diagram

$$\begin{array}{ccccc}
 \mathcal{G}\mathbb{O}(*) & \xrightarrow{\text{incl}_1} & \mathcal{G}\mathbb{O}(*)\times \mathcal{G}X & \longrightarrow & \mathcal{G}X \\
 \downarrow = & & \downarrow \text{sh} & & \\
 \mathcal{G}\mathbb{O}(*) & \xrightarrow{\Delta} & \mathcal{G}\mathbb{O}(*)\times \mathcal{G}X & \longrightarrow & |\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, \mathbb{O}(*)\times X)|
 \end{array}$$

where both rows are fibration sequences. The homotopy fiber of  $\text{incl}_1$  can be identified with  $\Omega\mathcal{G}X$ . The homotopy fiber of  $\Delta$  can be identified with  $\Omega|\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, \mathbb{O}(*)\times X)|$ . As both vertical arrows are homotopy equivalences, so is the induced map between the homotopy fibers. Since the above commutative diagram is a diagram of loop spaces, we conclude that  $\mathcal{G}X$  and  $|\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{O}, \mathbb{O}(*)\times X)|$  are homotopy equivalent.  $\square$

We can now state and prove our main theorem.

**Theorem 6.5.** *Suppose  $\mathcal{O}$  is an OHS. Then group completion defines a functor from  $\mathcal{O}$ -algebras to  $\Omega^{\infty}$ -spaces. In particular,  $\mathcal{G}\mathbb{O}(*)$  is equivalent to an  $\Omega^{\infty}$ -space and, for every  $\mathcal{O}$ -algebra  $X$ , there is an  $\Omega^{\infty}$ -map*



$$\mathcal{G}\mathbb{O}(\ast) \times \mathcal{G}X \rightarrow \mathcal{G}X,$$

compatible with  $\mathcal{O}$ -algebra maps, where the source is given the product  $\Omega^\infty$ -space structure.

**Proof.** By Corollary 4.10 we may assume without loss of generality that  $\mathcal{O}$  comes equipped with a map  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$  which is compatible with the multiplication  $\mu: \mathcal{A} \rightarrow \mathcal{O}$ . Hence, we may apply Proposition 6.1, to conclude that  $|\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, \mathbb{O}(\ast) \times X)|$  is an  $\Omega^\infty$ -space. By Proposition 6.4,

$$\mathcal{G}X \simeq |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, \mathbb{O}(\ast) \times X)|.$$

For a map of  $\mathcal{O}$ -algebras  $f: X \rightarrow Y$  we have an infinite loop map

$$|\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, \mathbb{O}(\ast) \times X)| \longrightarrow |\mathcal{G}B_\bullet(\mathbb{C}_\infty, \mathbb{O}, \mathbb{O}(\ast) \times Y)|.$$

The action of  $\mathcal{G}\mathbb{O}(\ast)$  on  $\mathcal{G}X$  is as defined in (6.3). The maps  $\mathbb{O}(\ast) \rightarrow \mathbb{O}(X) \rightarrow X$  are maps of  $\mathcal{O}$ -algebras and so  $\mathcal{G}\mathbb{O}(\ast) \rightarrow \mathcal{G}X$  is a map of  $\Omega^\infty$ -spaces. In (6.3) the first map is an infinite loop map and the second is loop sum. Since loop sum is an  $\Omega^\infty$ -map for  $\Omega^\infty$ -spaces, the result follows.  $\square$

We remark that, via the map  $\pi: \mathcal{O} \rightarrow \mathcal{C}_\infty$ , any infinite loop space is a group-complete  $\mathcal{O}$ -algebra. However, the induced action by  $\mathcal{G}\mathbb{O}(\ast)$  factors through  $\pi: \mathcal{G}\mathbb{O}(\ast) \rightarrow \mathcal{G}\mathbb{C}_\infty(\ast) \simeq \Omega^\infty S^\infty(\ast) \simeq \ast$ , and is thus trivial.

**Example 6.6 (Abelian monoids).** A topological abelian monoid  $A$  defines an operad  $\mathcal{O}$  where

$$\mathcal{O}(n) := A$$

with the trivial  $\Sigma_n$  action and  $\ast = 1 = 0_A$ . The structure maps in  $\mathcal{O}$  are given by the monoid multiplication and we consider  $\mathcal{O}$  a graded operad concentrated in degree 0. There is a canonical map of operads  $\mu: \text{Ass} \rightarrow \mathcal{O}$  which factors through  $\text{Com}$  and sends  $\text{Ass}(n)$  to the identity element of  $\mathcal{O}(n) = A$ . Thus  $\mu$  satisfies condition (4.2).

For this operad  $D: \mathcal{O}(n) \rightarrow \mathcal{O}(0)$  is the identity map on  $A$ , and so using Theorem 6.5 every group-like  $\mathcal{O}$ -algebra is an  $\Omega^\infty$ -space and comes equipped with an  $\Omega^\infty$ -space action by the group completion of  $A$ .

We can characterize  $\mathcal{O}$ -algebras as abelian monoids  $X$  with a monoid map  $f: A \rightarrow X$ . Then the consequences of Theorem 6.5 are familiar for this operad since any abelian monoid  $X$  is an  $E_\infty$ -algebra, and the group completion  $\Omega BX$  is an infinite loop space.

### 7. Higher dimensional manifolds

In [9] Galatius and Randal-Williams prove a general homological stability result for even dimensional manifolds. We utilize their result to provide new examples of operads with homological stability. We first recall the set-up from [9].

#### 7.1. Homological stability for spaces of manifolds

Let  $W$  be a  $2k$ -dimensional smooth, compact manifold with boundary  $P$ . Let  $\theta: B \rightarrow BO(2k)$  be a fibration and  $\theta^*(\gamma_{2k})$  be the pull-back of the canonical  $2k$ -bundle over  $BO(2k)$ . Fix a bundle map

$$\ell_P: \epsilon^1 \oplus TP \rightarrow \theta^*\gamma_{2k}$$

and let  $\text{Bun}_{k,\partial}^\theta(TW; \ell_P)$  denote the space of all bundle maps  $\ell: TW \rightarrow \theta^*\gamma_{2k}$  where the restriction of  $\ell$  to  $P$  is  $\ell_P$  and the underlying map  $\ell: W \rightarrow B$  is  $k$ -connected. We call a bundle map  $\ell_W: TW \rightarrow \theta^*\gamma_{2k}$  a  $\theta$ -structure on  $W$  and define a moduli space associated to  $W$  and a given  $\theta$ -structure  $\ell_W$  by letting

$$\mathcal{M}_k^\theta(W; \ell_W) \subset \text{Bun}_{k,\partial}^\theta(TW; \ell_P) // \text{Diff}(W; \partial)$$

be the connected component determined by  $\ell_W$ . Here  $\text{Diff}(W; \partial)$  is the topological group of diffeomorphisms of  $W$  that restrict to the identity on a neighborhood of  $\partial W$  and  $//$  denotes the homotopy orbit space (also called the Borel construction). We refer the reader to [9] for more details.

To construct maps between moduli spaces, let  $M$  be a cobordism between  $P$  and  $Q$  equipped with a  $\theta$ -structure  $\ell_M$  which restricts to  $\ell_P$  over  $P$  and to  $\ell_Q$  over  $Q$ . Then there is a map

$$\cup_P(M, \ell_M): \mathcal{M}_k^\theta(W; \ell_W) \rightarrow \mathcal{M}_k^\theta(W'; \ell_{W'})$$

where  $W' = W \cup_P M$  and  $\ell_{W'} = \ell_W \cup \ell_M$ .

**Theorem 7.1.** [9, 1.7] *If  $2k \geq 6$ ,  $B$  is simply connected, and  $(M, P)$  is  $(k - 1)$ -connected, the map  $\cup_P(M, \ell_M)$  induces an isomorphism on homology with constant coefficients in degrees  $* \leq (g - 4)/3$ .*

A  $g$ -fold connected sum of  $S^k \times S^k$  from which  $n$  disks have been removed is a manifold of type  $W_{g,n}$ . Then the **genus**  $g$  of  $(W, \ell_W)$  is the maximal number of disjoint copies of  $W_{1,1}$  in  $W$  where each  $W_{1,1}$  has a  $\theta$ -structure that is admissible in the sense of [9, 1.2].

**Remark 7.2** (Addendum to Theorem 7.1). The above statement is a finite genus homological stability statement. However, the crucial property for an OHS only requires an

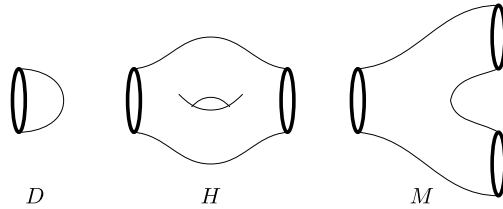


Fig. 1. Orientable atomic surfaces.

infinite genus homological stability property! This is proved in [9, 1.3] for all  $2k \geq 2$ . More concretely, the map  $\cup_P(M, \ell_M)$  induces isomorphisms in homology in any even dimension with any abelian system of local coefficients from the homotopy limit of  $\mathcal{M}_k^\theta(W; \ell_W)$  under gluing operation  $\cup_P(H_P, \ell_{H_P})$  to the homotopy colimit of  $\mathcal{M}_k^\theta(W'; \ell_{W'})$  under the gluing operation  $\cup_Q(H_Q, \ell_{H_Q})$ . Here  $H_P$  is the connected sum of  $S^k \times S^k$  with  $[0, 1] \times P$ , and the manifold  $H_Q$  is defined similarly. Again, the  $\theta$ -structures  $\ell_{H_P}$  and  $\ell_{H_Q}$  have to be admissible in the sense of [9, 1.2].

### 7.2. A manifold operad

The cobordism categories of embedded manifolds in Euclidean space in [10] and subsequent work by Galatius and Randal-Williams give rise to  $\Gamma$ -spaces in the sense of [27] but they are not symmetric monoidal. Therefore we cannot simply use an appropriate subcategory to define our operad. Instead we follow the approach for surface categories using atomic cobordisms first introduced in [30].

Fix a fibration  $\theta: B \rightarrow BO(2k)$  where  $B$  is  $k$ -connected. For example, take  $B$  to be the  $k$ -connective cover of  $BO(2k)$ . Let  $Q = S^{2k-1}$  and consider fixed atomic  $2k$ -cobordisms

- a handle  $H: Q \mapsto Q$ ,
- a sphere with three holes  $M: Q \sqcup Q \mapsto Q$ ,
- a disk  $D: \emptyset \mapsto Q$ ,

where  $H$  is of type  $W_{1,1+1}$ ,  $M$  of type  $W_{0,2+1}$  and  $D$  of type  $W_{0,1}$ , all with parametrized boundaries. See Fig. 1 for atomic cobordism for  $k = 1$  and  $B = BSO(2)$ . Let  $\ell_Q$  be a  $\theta$ -structure on  $Q$  such that  $\ell_Q \sqcup \ell_Q$  extends to a  $\theta$ -structure  $\ell_H$  on  $H$ ,  $\ell_Q \sqcup \ell_Q \sqcup \ell_Q$  extends to a  $\theta$ -structure  $\ell_M$  on  $M$ , and  $\ell_Q$  extends to a  $\theta$ -structure  $\ell_D$  on  $D$ . We will call the quadruple  $\{\ell_Q, \ell_H, \ell_M, \ell_D\}$  an **operadic**  $\theta$ -structure. Gluing the atomic pieces along their boundaries (as cobordisms) produces new cobordisms  $W$  of type

$$W_{g,n+1}: \coprod_n Q \mapsto Q$$

each equipped with a  $\theta$ -structure  $\ell_W$ .

For  $(g, n)$ , let  $\mathcal{C}_{g,n}$  be the groupoid whose objects are pairs  $(W, \ell)$  where

- $W$  is a cobordism of type  $W_{g,n+1}$  constructed from the atomic manifolds and equipped with an ordering of its  $n$  incoming boundary components, and
- $\ell: TW \rightarrow \theta^*(\gamma_{2k})$  is a  $\theta$ -structure in the path-connected component of  $\ell_W$  in the space of all bundle maps  $TW \rightarrow \theta^*(\gamma_{2k})$  or in the orbit of such a map under the action of the group  $\text{Diff}(W; \partial)$  of diffeomorphisms of  $W$  that fix a collar of the boundary.

By assumption  $B$  is  $k$ -connected and  $W$  is  $(k - 1)$ -connected and hence  $\ell$  is  $k$ -connected. A morphism between  $(W, \ell)$  and  $(W', \ell')$  is a diffeomorphism  $\phi: W \rightarrow W'$  that commutes with the parametrizations of the boundary components (and collars) and the ordering of the incoming boundary components. Furthermore, we require that

$$\ell = D\phi \circ \ell': TW \rightarrow \theta^*(\gamma_{2k}).$$

**Lemma 7.3.** *There is a homotopy equivalence*

$$BC_{g,n} \simeq \mathcal{M}_k^\theta(W_{g,n+1}, \ell_{W_{g,n+1}}).$$

**Proof.** For  $W \in \mathcal{C}_{g,n}$ , let  $X_W$  denote the space of all  $\theta$ -structures  $\ell$  which are in the path-connected component of  $\ell_W$  in the space of all bundle maps  $TW \rightarrow \theta^*(\gamma_{2k})$  or in the orbit of such a map under the action of the group  $\text{Diff}(W; \partial)$ . Then, for any other  $W' \in \mathcal{C}_{g,n}$ ,  $X_W$  is homotopy equivalent to  $X_{W'}$ , and given a diffeomorphism  $\phi: W \rightarrow W'$ , the  $\theta$ -structure  $D\phi \circ \ell'$  is in  $X_W$  whenever  $\ell' \in X_{W'}$ . Thus by construction  $\mathcal{C}_{g,n}$  is equivalent as a topological category to its full subcategory on objects of the form  $(W, \ell)$  for some fixed  $W$ . This subcategory is the translation category of  $X_W$  under the action of  $\text{Diff}(W; \partial)$ . Then the nerve of this subcategory has the homotopy type of  $X_W // \text{Diff}(W; \partial)$ . By definition this is  $\mathcal{M}_k^\theta(W, \ell_W)$ .  $\square$

We then form a manifold operad  $\mathcal{W}^{2k}$  by defining

$$\mathcal{W}^{2k}(n) = \coprod_{g \geq 0} BC_{g,n}.$$

The operad is graded by the genus and the structure maps  $\gamma$  are defined by gluing outgoing boundary spheres of one cobordism to incoming boundary spheres of others. In the next subsection we will identify an  $A_\infty$ -operad as a sub-operad of the genus zero sub-operad  $\mathcal{W}_0^{2k}$  of  $\mathcal{W}^{2k}$  so that condition (4.2) is satisfied.

Since the pair  $(D, Q) = (D^{2k}, S^{2k-1})$  is  $(k - 1)$ -connected, for  $2k \geq 2$  the results of Galatius and Randal-Williams quoted above, Theorem 7.1 and Remark 7.2, show that

$$D: \mathcal{W}_\infty(n) \longrightarrow \mathcal{W}_\infty(0)$$

is a homology isomorphism. Thus we have proved the following result.

**Theorem 7.4.** For  $2k \geq 2$ , the operad  $\mathcal{W}^{2k}$  associated to a fibration  $\theta: B \rightarrow BO(2k)$  with  $B$   $k$ -connected and an operadic choice of  $\theta$ -structures defines an operad with homological stability.

**Example 7.5.** Let  $\theta: BO(2k)\langle k \rangle \rightarrow BO(2k)$  be the  $k$ -connected cover of  $BO(2k)$ . Assume that our atomic cobordisms  $D, M$ , and  $H$  are embedded in  $[0, 1] \times \mathbb{R}^\infty$  as cobordisms, that is, the outgoing boundary  $\partial_+$  is contained in  $\{0\} \times \mathbb{R}^\infty$  and the incoming boundary  $\partial_-$  is contained in  $\{1\} \times \mathbb{R}^\infty$ ; each boundary component is equipped with a collar  $\partial_+ \times [0, \epsilon]$ , respectively  $\partial_- \times (\epsilon, 1]$ ; and the boundary spheres can be mapped to each other by rigid translations in  $\mathbb{R} \times \mathbb{R}^\infty$ . Then the Gauss map defines a map from the atomic cobordisms (and hence any  $W$  built from them) to  $BO(2k)$  which on (and near) the boundary restricts to a map  $\ell_Q: \epsilon^1 \oplus TQ \rightarrow \gamma_{2k}$ . As each atomic cobordism and each  $W$  built from them is  $k$  connected, the Gauss maps have compatible lifts to  $BO(2k)\langle k \rangle$ , their Moore–Postnikov  $k$ -stage. Thus, for this  $\theta$ -structure,  $\mathcal{W}^{2k} \simeq B \text{Diff}(W_{g,n+1}; \partial)$ .

As an immediate consequence of Theorem 7.4, we have that

$$\Omega B \left( \prod_{g \geq 0} \mathcal{M}_k^\theta(W_{g,1}, \ell_{W_{g,1}}) \right) = \Omega B \mathcal{W}^{2k}(0)$$

is homotopy equivalent to an infinite loop space. Furthermore, on application of the group completion theorem, we deduce that it is homotopy equivalent to

$$\mathbb{Z} \times \left( \text{hocolim}_{g \rightarrow \infty} \mathcal{M}_k^\theta(W_{g,1}, \ell_{W_{g,1}}) \right)^+$$

After restricting to a connected component, the analog of the Madsen–Weiss Theorem for higher dimensional manifolds and [9, 1.8], imply

$$\Omega B_0 \left( \prod_{g \geq 0} \mathcal{M}_k^\theta(W_{g,1}, \ell_{W_{g,1}}) \right) \simeq \left( \text{hocolim}_{g \rightarrow \infty} \mathcal{M}_k^\theta(W_{g,1}, \ell_{W_{g,1}}) \right)^+ \simeq \Omega_0^\infty \mathbf{MT}\theta. \tag{7.6}$$

**Remark 7.7.** Maps  $\mathcal{M}_k^\theta(W_{g,1+1}, \ell_{W_{g,1+1}}) \rightarrow \Omega^\infty \mathbf{MT}\theta$  which are compatible with gluing (a strictly monoidal structure on the disjoint union over all  $g$ ) were constructed in [10]. Following ideas in [29], a map of strict monoids can be constructed

$$\prod_{g \geq 0} \mathcal{M}_k^\theta(W_{g,1}, \ell_{W_{g,1}}) \longrightarrow \prod_{g \geq 0} \mathcal{M}_k^\theta(W_{g,1+1}, \ell_{W_{g,1+1}})$$

which on composition and group completion induces the homotopy equivalence on a connected component in (7.6). We do not know whether the evident infinite loop space structure on  $\Omega_0^\infty \mathbf{MT}\theta$  is compatible under this map with the infinite loop space structure

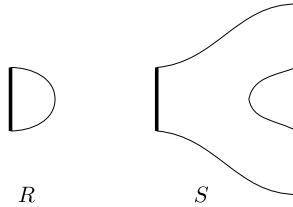


Fig. 2. Tiles.

produced by our methods on the left hand term in (7.6) though we note that for the case  $2k = 2$  this was affirmed by Wahl in [32].

### 7.3. Manifold models of $A_\infty$ -operads

For each dimension  $d \geq 2$  we will construct an  $A_\infty$ -operad  $\mathcal{A}^d$  build from  $d$ -dimensional spheres and diffeomorphisms.

Let  $S: I \sqcup I \rightarrow I$  and  $R: \emptyset \rightarrow I$  be fixed cobordisms from two copies of the unit interval  $I$  and the empty set, respectively, to  $I$ . See Fig. 2. By gluing copies of  $S$  and  $R$  as cobordisms along incoming and outgoing boundary intervals, and labeling their incoming boundary components, we can construct cobordisms  $T$  from  $n$  copies of  $I$  to  $I$ . We denote the union of the  $n + 1$  copies of  $I$  in the boundary by  $\partial_0$ .

Let  $\mathcal{T}_n$  be the groupoid whose objects are all such cobordisms and whose morphisms from  $T$  to  $\tilde{T}$  are the diffeomorphisms from  $T$  to  $\tilde{T}$  that commute with the parametrizations of the  $n$  labeled incoming and the outgoing copies of  $I$ :

$$\mathcal{T}_n(T, \tilde{T}) = \text{Diff}(T, \tilde{T}; \partial_0).$$

We note that each of these morphism spaces is homotopy equivalent to the space of diffeomorphisms of the two-dimensional disk  $D^2$  that fix the boundary, and hence is contractible. The symmetric group  $\Sigma_n$  acts on  $\mathcal{T}_n$  by relabeling the incoming boundary components. Gluing of incoming to outgoing copies of  $I$  makes  $\mathcal{T} = \{\mathcal{T}_n\}_{n \geq 0}$  an operad in categories (enriched in spaces), and its classifying space  $B\mathcal{T} = \{B\mathcal{T}_n\}_{n \geq 0}$  an operad in spaces. Because the morphism spaces  $\mathcal{T}_n(T, \tilde{T})$  are contractible, so is  $B\mathcal{T}_n$ . We summarize with the following lemma.

**Lemma 7.8.**  *$B\mathcal{T}$  is an  $A_\infty$ -operad.*  $\square$

Next, for  $d \geq 2$ , consider for each tile  $T \in \mathcal{T}_n$  its thickening, the space

$$T^d = T \times I^{d-1}.$$

As  $T$  is homeomorphic to the disk  $D^2$ ,  $T^d$  is homeomorphic to the  $d + 1$  dimensional disk, and its boundary  $\partial T^d$  is homeomorphic to the sphere  $S^d$ . Furthermore,

$$\partial T^d = \partial T \times I^{d-1} \cup T \times \partial I^{d-1}$$

comes with  $n$  incoming and one outgoing cubes  $I^d$  given by  $\partial_0 \times I^{d-1}$ . Define a new groupoid  $\mathcal{T}_n^d$  with objects  $\partial T^d$  for each tile  $T \in \mathcal{T}_n$ , and morphisms  $\mathcal{T}_n^d(\partial T^d, \partial \tilde{T}^d)$  to be all diffeomorphisms that are the restrictions to the boundary of diffeomorphisms from  $T^d$  to  $\tilde{T}^d$  that have the form  $\phi \times \text{Id}_{I^{d-1}}$  for some diffeomorphism of tiles  $\phi \in \mathcal{T}(T, \tilde{T})$ . Taking connected sum along the fixed cubes  $I^d$  defines the structure maps of the operad. By construction,  $\mathcal{T}_n$  is isomorphic to  $\mathcal{T}_n^d$  as a category enriched in spaces. Define

$$\mathcal{A}^d := B\mathcal{T}^d = \{B\mathcal{T}_n^d\}_{n \geq 0}.$$

**Corollary 7.9.**  $\mathcal{A}^d$  is an  $A_\infty$ -operad.  $\square$

Now let  $d = 2k$  and identify the thickening of the atomic tiles  $S$  and  $R$  with the atomic  $2k$ -cobordisms  $M$  and  $D$ . We want to extend this to a map of operads

$$\mu: \mathcal{A}^{2k} \longrightarrow \mathcal{W}^{2k}.$$

This can easily be done when the  $\theta$ -structure is as in Example 7.5. For more general  $\theta$ -structures the operad  $\mathcal{A}^{2k}$  needs to be adapted as follows. If  $\ell_Q$  extends to  $\ell_M$  and  $\ell_D$  on  $M$  and  $D$  respectively, then each cobordism  $W$  built from these will have an operadic  $\theta$ -structure glued together from these, and hence each of our objects  $\partial T^d$  has such an operadic  $\theta$ -structure. We thus need to thicken the object space in  $\mathcal{A}^d$ . For each tile  $T$  we have a family of objects  $(\partial T^d, \ell)$  where  $\ell$  is the pullback of the operadic  $\theta$ -structure along a diffeomorphism of  $\partial T^d$  in  $\mathcal{A}^d$ . But as the space of all such diffeomorphisms is contractible, the homotopy type of the operad is not affected, and in particular the resulting operad is still an  $A_\infty$ -operad. Finally we note that as  $\mathcal{W}^{2k}(2)$  is path-connected,  $\mu(\mathcal{A}^{2k}(2))$  is contained in one path-component and hence  $\mu$  satisfies the homotopy commutativity condition (4.2).

**Remark 7.10.** Strictly speaking, the thickening  $T^d$  is not smooth and care needs to be taken when identifying  $\partial T^d$  with  $S^d$ . In particular, we want the maps  $\phi \times \text{Id}_{I^d}$  to produce diffeomorphisms on  $S^d$ . This can easily be achieved by considering only those diffeomorphisms from  $T$  to  $\tilde{T}$  that restrict on a collar  $\partial_f \times [0, \epsilon)$  of the free boundary  $\partial_f = \partial T \setminus \partial_0$  to diffeomorphisms of the form  $\alpha \times \text{Id}_{[0, \epsilon)}$  for some smooth  $\alpha: \partial_f(T) \rightarrow \partial_f(\tilde{T})$ . We leave the details to the dedicated reader.

#### 7.4. Diffeomorphism groups for other tangential structures

We will show that the diffeomorphism and mapping class groups of  $W_{g,1}$  also give rise to infinite loop spaces. We first construct a  $\mathcal{W}^{2k}$ -algebra consisting of classifying spaces of diffeomorphisms groups.

We define a category  $\mathcal{D}_g$  that is entirely analogous to  $\mathcal{C}_{g,0}$  in §7.2. For each  $W$  built out of atomic  $2k$ -cobordisms of type  $W_{g,1}$  the category  $\mathcal{D}_g$  has one object, and its morphisms are diffeomorphisms that fix the parametrization of the boundary. Let  $F: \mathcal{C}_{g,0} \rightarrow \mathcal{D}_g$  be the functor that forgets the  $\theta$ -structure on objects. By the definition of  $\mathcal{D}_g$  this map is surjective. As  $\mathcal{D}_g$  is a groupoid, its classifying space is

$$B\mathcal{D}_g \simeq B\text{Diff}(W_{g,1}; \partial).$$

The forgetful functor induces a map

$$BF: BC_{g,0} \rightarrow B\mathcal{D}_g$$

which, under the above homotopy equivalence and Lemma 7.3, corresponds naturally to the bundle map

$$X_{W_{g,1}} // \text{Diff}(W_{g,1}; \partial) \rightarrow B\text{Diff}(W_{g,1}; \partial).$$

As  $F$  commutes with gluing, it gives  $\coprod_{g \geq 0} B\mathcal{D}_g$  the structure of an algebra over  $\mathcal{W}^{2k}$ .

**Corollary 7.11.** *The space  $\mathbb{Z} \times (\text{hocolim}_{g \rightarrow \infty} B\text{Diff}(W_{g,1}))^+$  is an infinite loop space.*

**Proof.** Pick an operadic  $\theta$ -structure on  $Q = S^{2k-1}$ , as in Example 7.5, and consider the associated operad  $\mathcal{W}^{2k}$ . We have argued above that

$$\coprod_{g \geq 0} B\mathcal{D}_g$$

is an algebra over  $\mathcal{W}^{2k}$ . By Theorem 7.4,  $\mathcal{W}^{2k}$  is an operad with homological stability (OHS) and hence by our main result, Theorem 6.5, the group completion of any  $\mathcal{W}^{2k}$ -algebra is an infinite loop space. By Remark 4.3, the group completion is given by the plus construction on the homotopy colimit, as the underlying monoid is homotopy commutative.  $\square$

More generally, the forgetful maps to diffeomorphism groups factor through the orientation preserving diffeomorphism groups. Thus we can also consider the oriented analog  $\mathcal{D}_g^+$  of  $\mathcal{D}_g$ . Galatius and Randal-Williams [9] identify the associated limit spaces in these cases with certain homotopy quotients

$$(\Omega^\infty \mathbf{MT}\theta) // \text{hAut}(u) \quad \text{and} \quad (\Omega^\infty \mathbf{MT}\theta) // \text{hAut}(u^+)$$

of  $\Omega^\infty \mathbf{MT}\theta$  by the monoid of weak homotopy equivalences of the  $k$ -connected cover  $B$  of  $BO(2k)$  which commute with the maps  $u: B \rightarrow BO(2k)$  and  $u^+: B \rightarrow BSO(2k)$  respectively. We note here that there is no a priori reason why these quotient spaces are infinite loop spaces.



Other tangential structures are also considered in [9] and could be considered similarly.

**Remark 7.12.** The category  $\mathcal{D}_g$  (and hence  $\pi_0\mathcal{D}_g$  considered below) could be enlarged by introducing further atomic manifolds  $A_\alpha: Q \rightarrow Q$  with  $\alpha$  from some countable set  $S$ . The union of the associated classifying spaces form  $\mathcal{W}^{2k}$ -algebras and hence group complete to infinite loop spaces.

### 7.5. Mapping class groups

Let  $\Gamma W_g := \pi_0(\text{Diff}(W_{g,1}; \partial))$  denote the mapping class group of  $W_{g,1}$ . By replacing the diffeomorphism groups by their connected components we can construct a new category  $\pi_0\mathcal{D}_g$  with a canonical forgetful functor  $\mathcal{D}_g \rightarrow \pi_0\mathcal{D}_g$  so that

$$\coprod_{g \geq 0} B\pi_0\mathcal{D}_g$$

is an algebra over  $\mathcal{W}^{2g}$ . We note the homotopy equivalence  $B\pi_0\mathcal{D}_g \simeq B\Gamma W_{g,1}$ .

**Corollary 7.13.** *The space  $\mathbb{Z} \times (\text{hocolim}_{g \rightarrow \infty} B\Gamma W_{g,1})^+$  is an infinite loop space.*

**Proof.** The argument is precisely as above.  $\square$

Mapping class groups of surfaces and three manifolds are well-studied and notoriously complicated. However, in higher dimensions these mapping class groups tend to be much better behaved. In the next section we will identify the above infinite loop space with a K-theory space when  $2k = 4$ . Nevertheless, as we will argue, the above corollary and Remark 7.12 provide us with many examples of infinite loop spaces where at the moment we do not have any other means of identifying them as such.

### 7.6. The map to K-theory

Our final goal is to construct the map from  $\Omega^\infty \mathbf{MT}\theta$  to K-theory as an infinite loop space map.

Diffeomorphisms act on homology. In particular, for  $W_{g,1}$  of dimension  $2k$  the diffeomorphisms act on the middle dimensional homology group

$$H_k(W_{g,1}) \simeq \mathbb{Z}^{2g} \simeq H_k(W_g)$$

preserving the intersection form. Define categories  $\mathcal{K}_g$  with objects the same as in  $\mathcal{D}_g$  and morphisms from  $W$  to  $W'$  given by linear isomorphisms  $H_k(W) \rightarrow H_k(W')$  that preserve the natural intersection forms. Thus when  $k$  is odd,  $B\mathcal{K}_g \simeq B\text{Sp}_{2g}\mathbb{Z}$ , and when  $k$  is even,  $B\mathcal{K}_g \simeq B\text{O}_{g,g}\mathbb{Z}$ . There is a chain of functors

$$\mathcal{C}_{g,0} \longrightarrow \mathcal{D}_g \longrightarrow \pi_0 \mathcal{D}_g \longrightarrow \mathcal{K}_g$$

giving rise to a chain of  $W^{2k}$ -algebra maps on the union over all  $g$  of the classifying spaces, and hence maps of infinite loop spaces on their group completions. We summarize with the following theorem.

**Theorem 7.14.** *For  $2k \geq 2$ , a fibration  $\theta: B \rightarrow BO(2k)$  with  $B$   $k$ -connected, and an operadic choice of  $\theta$ -structure on  $Q = S^{2k-1}$ , the action of the diffeomorphisms on the middle dimensional homology of the underlying manifolds induces a map of infinite loop spaces from*

$$\mathbb{Z} \times (\operatorname{hocolim}_{g \rightarrow \infty} \mathcal{M}_k^\theta(W_{g,1}, \ell_{W_{g,1}}))^+ \simeq \Omega^\infty \mathbf{MT}\theta$$

factoring through the infinite loop spaces

$$\mathbb{Z} \times (\operatorname{hocolim}_{g \rightarrow \infty} B \operatorname{Diff}(W_{g,1}, \partial))^+ \text{ and } \mathbb{Z} \times (\operatorname{hocolim}_{g \rightarrow \infty} B\Gamma W_{g,1})^+$$

to algebraic  $K$ -theory,  $KSp \simeq \mathbb{Z} \times BSp(\mathbb{Z})^+$  when  $k$  is odd, and  $KO \simeq \mathbb{Z} \times BO_{\infty, \infty}(\mathbb{Z})^+$  when  $k$  is even.  $\square$

We finish this section with a discussion of the map from the infinite loop space associated to the mapping class groups of the  $W_{g,1}^{2k}$  to  $K$ -theory.

For  $2k = 4$ : the stable mapping class group

$$\Gamma W_\infty^{2k} := \lim_g \Gamma W_{g,1}^{2k}$$

is isomorphic to the group  $O_{\infty, \infty}(\mathbb{Z})$ . Indeed, following [12], we can give the following argument. By a theorem of Wall [34], the map

$$\Gamma W_{g,1}^4 \rightarrow O_{g,g}(\mathbb{Z})$$

is surjective for all  $g \geq 1$ . This map factors through the group of pseudo-isotopy classes  $\mathcal{P}(W_{g,1}^4)$ , and by a theorem of Kreck [14] the map

$$\mathcal{P}(W_{g,1}^4) \rightarrow O_{g,g}(\mathbb{Z})$$

is injective. Finally, by a theorem of Quinn [24], pseudo-isotopic diffeomorphisms will be isotopic after stabilization with a number of copies of  $S^2 \times S^2$ .

For  $2k > 4$ : While both Wall’s and Quinn’s theorems are restricted to dimension four, a version of Kreck’s theorem holds in all even dimensions, and the mapping class group of  $W_{g,1}^{2k}$  is an extension of the automorphism group of the quadratic forms  $Q_{W_g^{2k}}$  associated to the bilinear intersection form on  $H_k(W_g^{2k}) \simeq \mathbb{Z}^{2g}$ :

$$1 \longrightarrow \mathcal{I}_{g,1}^{2k} \longrightarrow \Gamma W_{2g,1}^{2k} \longrightarrow \text{Aut}(Q_{W_g^{2k}}) \longrightarrow 1.$$

The difference from the case where  $2k = 2$  is that the analog of the Torelli group  $\mathcal{I}_{2g,1}^{2k}$  in higher dimensions is much simpler. According to Kreck [14], it is an extension of two abelian groups, one of which is the finite group  $\Theta_{2k+1}$  of exotic smooth structures on the  $2k + 1$  sphere and, depending on  $k \pmod{8}$ , the other is free abelian of size  $\mathbb{Z}^{2g}$  or all 2-torsion:

$$1 \longrightarrow \Theta_{2k+1} \longrightarrow \mathcal{I}_{g,1}^{2k} \longrightarrow \text{Hom}(H_k(W_{g,1}^{2k}), S\pi_k(SO(k))) \longrightarrow 1.$$

Here  $S\pi_k(SO(k))$  is the image of  $\pi_k(SO(k))$  under the map  $SO(k) \rightarrow SO(k + 1)$ . We refer to [8] for a recent study of these mapping class groups and more details including a table of the  $S\pi_k(SO(k))$ . In particular, it is proved in [8] that

$$\Gamma W_{g,1}^{12} \simeq \mathbb{Z}/3\mathbb{Z} \times O_{g,g}(\mathbb{Z}).$$

Thus in this case, the mapping class groups give rise to the infinite loop space

$$\mathbb{Z} \times (B\Gamma_\infty^{12})^+ \simeq B\mathbb{Z}/3\mathbb{Z} \times KO.$$

In general, the two extensions from which  $\Gamma W_{g,1}^{2k}$  is built are not well understood. Thus, the infinite loop space structure on

$$\mathbb{Z} \times (B\Gamma W_\infty^{2k})^+$$

may not be surprising but seems also non-obvious and new in most cases.

### Appendix A. Surface operads

In this appendix we collect examples of operads with homological stability arising from the geometry of surfaces. The relevant homological stability results are due to Harer [13], Wahl [33], and Randal-Williams [26].

There are also homological stability results related to graphs and automorphisms of free groups. The operads related to these groups tend to receive a map from an  $E_\infty$ -operad, and hence it is not surprising that their algebras group complete to infinite loop spaces. Therefore we have not treated these examples in detail here.

#### A.1. Surface operads

In [31], a version of our main theorem was proved for a particular surface operad. In that set-up it was necessary to have a strict multiplication, that is a map from  $Ass$  into the surface operad. To construct this map some surfaces had to be identified and diffeomorphisms were replaced by mapping class groups. The present set-up gives us more flexibility and the constructions can be significantly simplified.

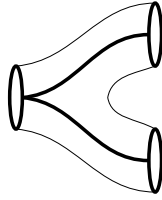


Fig. 3. Trees within surfaces.

**Example A.1** (*Oriented surfaces and diffeomorphisms*). We will now construct a variant of the surface operad studied in [31]. Like in the constructions of §7, we have three atomic surfaces: a disk  $D$ , a torus  $T = F_{1,2}$  and a pair of pants  $P = F_{0,3}$ . (In §7,  $T$  is denoted by  $H$ , and  $P$  is denoted by  $M$ .) Let  $F_{g,n+1}$  denote an oriented surface built out of atomic surfaces that has genus  $g$  with  $n + 1$  (collared) boundary components,  $n$  incoming and 1 outgoing.

We define a connected groupoid  $\mathcal{S}_{g,n+1}$  where the objects are surfaces of type  $F_{g,n+1}$  with a labeling of the incoming boundary components. Define

$$\mathcal{S}_{g,n+1}((F, \sigma), (F', \sigma')) := \text{Diff}(F, F'; \partial),$$

the diffeomorphisms that preserve the collars and the labels on the boundary components. We have a homotopy equivalence

$$B\mathcal{S}_{g,n+1} \simeq B\Gamma_{g,n+1},$$

where  $\Gamma_{g,n+1}$  is the mapping class group.

We can now define an operad  $\mathcal{S}$  with

$$\mathcal{S}(n) = \coprod_{g \geq 0} B\mathcal{S}_{g,n+1}.$$

The gluing of surfaces defines associative and equivariant operad structure maps. The operad is graded by the genus  $g \in \mathbb{N}$ . Consider the suboperad  $\mathcal{S}_0$  corresponding to the genus zero surfaces. To identify an  $A_\infty$ -suboperad  $\mathcal{A}$  in  $\mathcal{S}_0$ , connect the point corresponding to  $1 \in S^1$  in the outgoing boundary circle by a simple path to those in each of the incoming boundary circles of the pair of pants  $P$ . See Fig. 3. This way, after gluing, every surface of genus zero will be decorated by a tree connecting the outgoing boundary circle to each of its incoming circles. Form  $\mathcal{A} \subset \mathcal{S}_0$  by restricting the diffeomorphisms to those that also map trees to trees. The complement of a tree is a disk, so these diffeomorphism groups will be contractible, and hence  $\mathcal{A}$  is an  $A_\infty$ -operad.

By the main theorem of [15] the associated infinite loop space is

$$\mathcal{GS}(0) \simeq \mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega^\infty \mathbf{MTSO}(2),$$

and by [32] the explicit infinite loop space structure on the right is the same as the one induced by the operad structure on  $\mathcal{GS}(0)$ .

**Example A.2** (*Nonorientable surfaces and diffeomorphisms*). We now add another surface building block: a nonorientable surface of genus one  $N = \mathbb{R}P^2 \setminus (D^2 \amalg D^2)$ . Let  $N_{k,n+1}$  be a surface of nonorientable genus  $k$  with one outgoing and  $n$  incoming boundary components. As in Example A.1 we construct  $N_{k,n+1}$  from  $D, P, S^1$  and  $N$ , and associate a groupoid  $\mathcal{N}_{k,n+1}$  where the morphisms are given by diffeomorphisms.

Thus we define an operad  $\mathcal{N}$  with

$$\mathcal{N}(n) \simeq \coprod_{k \geq 0} BN_{k,n+1}.$$

The genus zero suboperad  $\mathcal{N}_0$  is the same as  $\mathcal{S}_0$ , so we have a map  $\mu: \mathcal{A} \rightarrow \mathcal{N}$  satisfying condition (4.2). By [33, A] we have  $H_*(\mathcal{N}_{k,n+1}) \cong H_*(\mathcal{N}_{k,1})$  for  $k \geq 4 * + 3$ . Hence,  $\mathcal{N}$  is an OHS. By [33], the associated infinite loop space is

$$\mathcal{GN}(0) \simeq \mathbb{Z} \times BN_{\infty}^+ \simeq \Omega^{\infty} \mathbf{MTO}(2),$$

where  $\mathcal{N}_{\infty} = \lim_{k \rightarrow \infty} \pi_0 \text{Diff}(N_{k,1}, \partial)$  denotes the infinite mapping class group.

**Example A.3** (*Mixed surfaces and diffeomorphisms*). We include this example as it has a grading different from  $\mathbb{N}$ . The procedure in Example A.1, applied to the atomic surfaces  $D, P, T, S^1$  and  $N$ , constructs an operad we will denote by  $\mathcal{SN}$ . Considering both orientable and nonorientable genus defines a  $\{+, -\} \times \mathbb{N}$  grading on  $\mathcal{SN}$  with addition defined by  $(+, g) + (+, g') = (+, g + g')$ ,  $(-, k) + (-, k') = (-, k + k')$  and  $(+, g) + (-, k) = (-, 2g + k)$ . The group completion of this monoid is  $\mathbb{Z}$  with generator  $(-, 1)$ .

Take  $N$  to be the propagator for  $\mathcal{SN}$  and observe that  $\gamma(T; N) = \gamma(N; \gamma(N; N))$ . In particular, under the action of this propagator, orientable and nonorientable surfaces conflate and

$$\mathcal{GSN}(0) \simeq \mathbb{Z} \times BN_{\infty}^+.$$

As a consequence, this operad gives a bridge between the operads of orientable surfaces and nonorientable surfaces. Comparing building blocks, there are operad maps

$$\mathcal{N} \rightarrow \mathcal{SN} \leftarrow \mathcal{S}$$

and the maps on connected components of group completions correspond to

$$\mathbb{Z} \xrightarrow{\text{Id}} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z}.$$

**Example A.4** (*Framed,  $r$ -spin, and pin surfaces*). In [26] Randal-Williams studies the homological stability for diffeomorphism groups of surfaces with general  $\theta$ -structures. In

particular, for  $r$ -spin structures  $\theta^r$  on oriented surfaces, he shows that reducing boundary components by gluing in disks induces isomorphisms in degrees increasing with the genus of the underlying surfaces, see [26, 2.14]. Hence, these diffeomorphism groups (or their associated mapping class groups) give rise to OHSs. Similarly, he shows that homological stability holds for the  $pin^+$  and  $pin^-$  mapping class groups of surfaces, see [26, 4.1 8, 4.19], and thus these too give rise to OHSs. In all these cases care has to be taken when defining the  $\theta$  structures on the atomic surfaces so that gluing is well defined and gives rise to an operad.

*A.2. Applications*

**Example A.5** (*Map to K-theory*). In [31] Tillmann constructs an infinite loop space map into  $K$ -theory using the operad associated to orientable surfaces. There is also an interesting example coming from the unorientable case. In particular, let  $N_{k,1}$  be a nonorientable surface of genus  $k$  with 1 boundary component. The action of the mapping class group  $\mathcal{N}_{k,1}$  on  $H_1(N_{k,1}) = \mathbb{F}_2^k$  induces a representation

$$\rho: \mathcal{N}_{k,1} \rightarrow GL_k \mathbb{F}_2.$$

Note that  $\mathcal{N}(0) = \coprod_{k \geq 0} B\mathcal{N}_{k,1}$ . Thus the map  $\rho: \mathcal{N}(0) \rightarrow \coprod_{k \geq 0} BGL_k \mathbb{F}_2$  is a map of  $\mathcal{N}$ -algebras, and hence the map on group completions is an  $\Omega^\infty$ -space map

$$\rho: \mathbb{Z} \times B\mathcal{N}_\infty^+ \simeq \Omega^\infty \mathbf{MTO}(2) \rightarrow \mathbb{Z} \times BGL(\mathbb{F}_2)^+ \simeq K(\mathbb{F}_2).$$

Note that the Madsen–Tillmann spectrum  $\Omega^\infty \mathbf{MTO}(2)$  is geometrically defined as a Thom spectrum, while  $K(\mathbb{F}_2)$  is algebraically defined. It would be difficult to define a map of infinite loop spaces directly without making use of the Madsen–Weiss Theorem which describes  $\Omega^\infty \mathbf{MTO}(2)$  in terms of classifying spaces of diffeomorphisms groups of surfaces.

**Example A.6** (*Detecting  $\Omega^\infty$ -spaces*). Let  $\tilde{\mathcal{N}}_{k,1}$  be defined as the split extension of  $\mathcal{N}_{k,1}$  by its  $\mathbb{F}_2$ -homology  $H_1(N_{k,1}, \mathbb{F}_2) = \mathbb{F}_2^k$ :

$$0 \longrightarrow \mathbb{F}_2^k \longrightarrow \tilde{\mathcal{N}}_{k,1} \longrightarrow \mathcal{N}_{k,1} \longrightarrow 0.$$

Then there exists an  $\mathcal{N}$ -algebra

$$X \simeq \coprod_{g \geq 0} B\tilde{\mathcal{N}}_{g,1}$$

and  $\mathbb{Z} \times B\tilde{\mathcal{N}}_\infty^+$  is an  $\Omega^\infty$ -space. At this point we do not know another description of this infinite loop space.

Similarly, as already noted in [31], in the oriented case we get an infinite loop space structure on the group completion of

$$Y = \coprod_{g \geq 0} B\tilde{\Gamma}_{g,1}$$

and Ebert and Randal-Williams show in [6]

$$\Omega B(Y) \simeq \Omega \text{MTSO}(2) \wedge BC_+^\times.$$

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