Exponential Growth and Decay  (Section 4.5)

Suppose a quantity \( y(t) \) is growing or decaying at a rate \( y'(t) \). If \( y'(t) \) is proportional to \( y(t) \), we have the differential equation

\[
\frac{dy}{dt} = ky
\]

The solution is \( y(t) = y_0 e^{kt} \) where \( y_0 \) is the initial quantity and \( k \) is a constant. When \( k > 0 \), this is exponential growth, when \( k < 0 \), this is exponential decay.

**EXAMPLE 1**
A bacteria culture starts with \( 4000 \) bacteria and the population triples every 30 minutes.

(a) Find an expression for the number of bacteria after \( t \) hours.

\[
y(t) = y_0 e^{kt} = 4000 e^{kt} \]

\[
y(\frac{1}{2}) = 3 \times 4000 = 12000 = 4000 e^{k \times \frac{1}{2}}
\]

\[
\ln 3 = \ln e^{k \times \frac{1}{2}} = \frac{k}{2} \Rightarrow k = 2 \ln 3
\]

\[
y(t) = 4000 e^{2 \ln 3 \times t} = 4000 (e^{\ln 3})^t = 4000 e^{(2 \ln 3) t}
\]

(b) Determine the number of bacteria after 20 minutes.

\[
y(\frac{1}{3}) = 4000 \times 9^{\frac{1}{3}} \approx 9320 \text{ bacteria}
\]

Note: The half-life of a substance is the amount of time it takes for half of the substance to disintegrate.
EXAMPLE 2
Iodine-131 (or $^{131}\text{I}$) is a major uranium fission product (nearly 3%) and it has a half-life of 8 days. How long will it take for an initial quantity of 100 mg to decrease to 10 mg?

\[
y(0) = 100 e^{kt}
\]

\[
y(8) = 50 = 100 e^{8k}
\]

\[
\frac{1}{2}e^{8k} = e^{8k/8} = e^{k/8}
\]

\[
y(8) = 100 e^{k/8} = \frac{100}{100} = 10
\]

\[
\ln(0.1) = \ln(1/10) = \ln(2)^{t/8} = \frac{t}{8} \ln(2)
\]

\[
t = 2 \ln(0.1) \approx 26.6 \text{ days}
\]

EXAMPLE 3
After 3 days a sample of an unknown radioactive element is found to have decayed to 58% of its original amount. What is the half-life of this element? How long until the sample is 10% of its original amount? What is this element? FYI

http://ie.lbl.gov/education/isotopes.htm
EXAMPLE 4
A curve passes through the point (0, 7) and has the property that the slope of the curve at every point \( p \) is half the \( y \)-coordinate of \( p \). Find the equation of the curve.

\[ y = y(x) \quad \text{Given that} \quad \frac{dy}{dx} = \frac{1}{2} y \]

\[ k = \frac{1}{2} \quad \text{so use} \quad \frac{dy}{dx} = \frac{1}{2} y \]

\[ y = y_0 e^{\frac{1}{2}x} \quad \text{known} \quad (0,7) \quad \text{is on this} \]

\[ 7 = y_0 e^0 \Rightarrow y_0 = 7 \Rightarrow y(x) = 7e^{\frac{x}{2}} \]

EXAMPLE 5
A tank contains 1500 liters of a brine with a concentration of 35 g of salt per liter. Pure water enters the tank at a rate of 20 liters per minute. The solution is kept mixed and the tank drains at a rate of 20 liters per minute.
(a) How much salt will be in the tank after 30 minutes?
(b) How long will it take for the concentration of salt to fall to 9 g of salt per liter?

\[ y(t) = \text{amount of salt in the tank at time} \ t \ \text{in minutes} \]

\[ y(0) = 1500 \text{ L} \times 35 \text{ g/L} = 52,500 \text{ g of salt} \]

\[ \frac{dy}{dt} = \text{rate of salt entering} - \text{rate of salt leaving} \]

\[ = 0 \text{ g/min} \times 20 \text{ L/min} - \frac{y}{1500} \text{ g/L} \times 20 \text{ L/min} = -\frac{1}{75} \frac{y}{\text{min}} - \frac{dy}{dt} \]

\[ \frac{dy}{dt} = Ky = -\frac{1}{75} y \Rightarrow k = -\frac{1}{75} \]

\[ y(t) = y_0 e^{kt} = 52,500 \text{ g of salt} \]

(a) \[ y(30) = 52,500 e^{-30 \times \frac{1}{75}} \approx 85.92 \text{ g of salt} \]

(b) \[ 9 \text{ g/L} \times 1500 \text{ L} = 13,500 \text{ g of salt} \]

\[ 52,500 e^{\frac{t}{75}} \]

\[ t = -75 \ln(\frac{9}{85.92}) \approx 102 \text{ min} \]
Note: If $A_0$ dollars are invested at $r\%$ annual interest compounded $n$ times per year, then the amount of money after $t$ years is

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

**EXAMPLE 6**
If $4000$ is invested at $8\%$ annual interest compounded monthly, how much money is in the account at the end of $6$ years? How long will it take for the money to triple in value?

1. \[ A(t) = 4000 \left(1 + \frac{0.08}{12}\right)^{6 \times 12} = 6454 \]
2. \[ 12000 = 4000 \left(1 + \frac{0.08}{12}\right)^{12t} \Rightarrow 3 = \left(1 + \frac{0.08}{12}\right)^{12t} \]
   \[ \ln 3 = 12t \ln \left(1 + \frac{0.08}{12}\right) \Rightarrow t = \frac{\ln 3}{12 \ln \left(1 + \frac{0.08}{12}\right)} = 13.8 \text{ years} \]

Note: If $P$ dollars are invested in an account that pays $r\%$ annual interest compounded continuously, the amount after $t$ years is

$$A(t) = Pe^{rt}$$

**EXAMPLE 7**
How much money should be invested now at $6\%$ annual interest compounded monthly so that you will have $30,000$ in $18$ years?

$$30000 = P e^{0.06 \times 18}$$

$$P = \frac{30000}{e^{0.06 \times 18}} \approx 10,188$$

(c) Janice L. Epstein
Newton’s Law of Cooling
The rate of cooling of an object is proportional to the temperature difference between the object and the surrounding temperature.

\[ y(t) = \text{temperature of the object at time } t \]
\[ y_0 \text{ is the initial temperature of the object} \]
\[ T = \text{temperature of surrounding environment} \]
\[ \frac{dy}{dt} = k(y - T) \quad \rightarrow \quad y(t) = (y_0 - T)e^{kt} + T \]

**EXAMPLE 8**
A thermometer is taken from a cake at temperature 350°F and left on a countertop in a room at 75°F. After 1 minute the thermometer reads 300°F.
(a) What will the thermometer read after 2 minutes?
(b) When will the thermometer read 120°F?

(a) \[ y(t) = (y_0 - T)e^{kt} + T \]
\[ y(t) = (350 - 75)e^{kt} + 75 = 275e^{kt} + 75 \]
\[ y(1) = 300 = 275e^{k} + 75 \Rightarrow 275e^{k} = 225 \]
\[ e^{k} = \frac{225}{275} = \frac{9}{11} \Rightarrow \ln \left( \frac{9}{11} \right) = k \]
\[ y(t) = 275e^{\ln \left( \frac{9}{11} \right) \cdot t} + 75 = 275 \left( \frac{9}{11} \right)^{t} + 75 = y(t) \]
\[ y(2) = 275 \left( \frac{9}{11} \right)^{2} + 75 \approx 259°F \text{ (dropped } 2°F \text{ more)} \]

(b) \[ 120 = 275 \left( \frac{9}{11} \right)^{t} + 75 \quad \Rightarrow \quad t = \frac{\ln \left( \frac{965}{275} \right)}{\ln \left( \frac{9}{11} \right)} \approx 9 \text{ min} \]