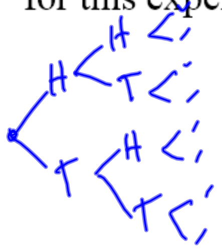
2.1 The Multiplication Principle and Permutations

Example

A fair coin is flipped three times. How many different outcomes are there for this experiment?



8 outcomes

$$\underset{1^{st}}{\underline{2}} * \underset{2^{nd}}{\underline{2}} * \underset{3^{rd}}{\underline{2}} = 8$$

Multiplication Principle:

Suppose a task T_1 can be done N_1 ways and a task T_2 can be done N_2 ways and so on until task T_k can be done N_k ways. Then the number of ways of performing the tasks T_1, T_2, ... T_k is given by the product $N_1 \times N_2 \times \ldots \times N_k$

Factorials: $n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$. Note that $0! = 1$

Example

How many different license plates are possible where

a) The first characters is a letter, the next two characters are digits and the last three characters are letters?

$$\underset{L}{\underline{26}} * \underset{D}{\underline{10}} * \underset{D}{\underline{10}} * \underset{L}{\underline{26}} * \underset{L}{\underline{26}} * \underset{L}{\underline{26}} = 45,697,600$$

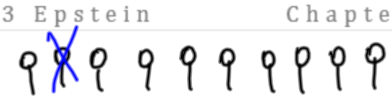
b) The first two characters are letters, the next character is a digit, the next a letter and the last three are digits?

$$\underset{L}{\underline{26}} \; \underset{L}{\underline{26}} \; \underset{D}{\underline{10}} \; \underset{L}{\underline{26}} \; \underset{D}{\underline{10}} \; \underset{D}{\underline{10}} \; \underset{D}{\underline{10}} = 175,760,000$$

c) No characters are duplicated in part (b)

$$\underset{L}{\underline{26}} * \underset{L}{\underline{25}} * \underset{D}{\underline{10}} * \underset{L}{\underline{24}} * \underset{D}{\underline{9}} * \underset{D}{\underline{8}} * \underset{D}{\underline{7}} = 78,624,000$$

Example



How many ways can 10 students be seated in a row of 10 chairs?

$$\underset{C\#1}{\underline{10}}\ \underset{C\#2}{\underline{9}}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underset{C\#10}{\underline{1}} = 10!$$

$$= 3,628,800$$

$$= P(10,10)$$

Example

How many ways can 4 of 10 students be seated in a row of 4 chairs?

$$\underline{10} * \underline{9} * \underline{8} * \underline{7} = 5040 = P(10,4) = 10\ {}_nP_r\ 4$$

Permutations: If we have a finite set of n elements and we want to place r of them in an arrangement, we say the number of permutations of n things arranged r at a time is $P(n, r)$.

Example

How many ways can gold, silver and bronze medals be awarded in a race of 12 people?

$$\frac{12}{G} * \frac{11}{S} * \frac{10}{B} = 1320 = P(12,3)$$

Example

A sales person has 6 prospects. How many ways can she arrange her schedule to see all 6?

$$\underline{6} * \underline{5} * \underline{4} * \underline{3} * \underline{2} * \underline{1} = 6! = 720 = P(6,6)$$

Example $15*14*13\ldots 3\times2\times1 = 15! =$

You have a group of 15 different books. Five are math books, four are chemistry and six are history books. How many different arrangements are possible if books of the same type are kept together?

M C H or M H C or . . .

$$\underbrace{3*2*1}_{\#\text{ arr groups}} * \underbrace{5*4*3*2*1}_{\text{arr M books}} * \underbrace{4*3*2*1}_{\text{arr chem books}} * \underbrace{6\times5\times4\times3\times2\times1}_{\text{arr H books}} = 12,441,600$$

Example

How many different 4 digit numbers can be made from the digits

1, 2, 3, 4, 5, 6, 7

a) If there are no restrictions? $\underset{1000's}{7} \cdot \underset{100's}{7} \cdot \underset{10's}{7} \cdot \underset{1's}{7} = 2401$

b) If the number must be even? $7 \cdot 7 \cdot 7 \cdot \underset{\text{even}}{3} = 1029$

c) If it is even and there are no repeats? $6 * 5 * 4 * \underset{\text{even}}{3}$ 1st

1211, 1121 are

1111 not ok 1112 ok

d) If four of the same digit is not allowed? $7*7*7*7 - 7 = 2394$

1111
2222
3333
4444
5555
6666
7777

2.2 Combinations

Example Group ABCD same as BADC same as .. .. .

How many ways can a **group** of 4 students be chosen from 10 students?

$$\frac{10*9*8*7}{4*3*2*1} = 210 = C(10,4) = 10 \text{ nCr } 4$$

Combinations: If we have a finite set of n elements and we want to take r of them in a group, we say the number of combinations of n things grouped r at a time is $C(n, r)$.

Example

How many ways can a hand of 6 clubs be chosen from a standard deck?

$$\underbrace{C(13,6)}_{\text{6 clubs}} = 1716 = \underbrace{C(13,6)}_{\text{6 clubs}} \; \underbrace{C(39,0)}_{\text{0 not clubs}}$$

$C(12,4) = 495$ w/no chair

Example

From a group of 12 people, how many ways can a committee of 4 be formed if one person is the chair of the committee?

$$\underbrace{12}_{\text{Chair}} \cdot \underbrace{C(11,3)}_{\text{rest}} = 1980$$

$$\underbrace{C(12,4)}_{\text{Committee}} \; \underbrace{C(4,1)}_{\text{Chair}} = 1980$$

Example

A bag contains 6 blue, 1 green and 3 pink jelly beans. You choose 3 at random. How many samples are possible in which

a) the jelly beans are all blue?

$\underbrace{C(6,3)}_{\text{Blue}} \; \underbrace{C(4,0)}_{\text{not blue}} = 20 = \frac{6*5*4}{3*2*1}$

b) the jelly beans are all green?

0

c) the jelly beans are all pink?

$1 = \underbrace{C(3,3)}_{\text{3 pink}} \; \underbrace{C(7,0)}_{\text{0 not pink}}$

$B_1 B_2 B_3$
$B_1 B_2 B_4$
$B_1 P\, G$ etc

d) there are 2 blue and 1 pink?

$\underbrace{C(6,2)}_{2B} \cdot \underbrace{C(3,1)}_{1P} \cdot \underbrace{C(1,0)}_{0G} = 15 * 3 * 1 = 45$

e) How many ways to choose 3 jelly beans?

$\underbrace{C(10,3)}_{\text{Pick 3}} = 120$

\# have 0 Blue
\+ 1
\+ 2
\+ 3

f) How many ways to choose no blue?

$\underbrace{C(6,0)}_{0B} \cdot \underbrace{C(4,3)}_{\text{3 not blue}} = 4$

$1B2B^c + 2B1B^c + 3B0B^c$

g) How many ways to choose at least one blue?

$\underbrace{C(6,1)}_{1B} \cdot \underbrace{C(4,2)}_{2B^c} + \underbrace{C(6,2)}_{2B} \cdot \underbrace{C(4,1)}_{1B^c} + \underbrace{C(6,3)}_{3B} \cdot \underbrace{C(4,0)}_{0B^c} = 116$

Example

A class of 12 students will divide into 3 teams of 4. How many ways can this be done?

$$\underbrace{C(12,4)}_{\text{team 1}} \cdot \underbrace{C(8,4)}_{\text{team 2}} \cdot \underbrace{C(4,4)}_{\text{team 3}} = 495 * 70 * 1 = 34,650$$

Example

A school is putting together a committee. The committee will have a chair and an assistant chair chosen from a group of 10 teachers, two parents chosen from a group of 15 parents and two students chosen from a group of 20 students. How many different committees are possible?

$$\underbrace{10}_{\text{Ch}}\ \underbrace{9}_{\text{ACh}}\ \underbrace{C(15,2)}_{\text{2Par}}\ \underbrace{C(20,2)}_{\text{2St}}$$

Example

You are dealt a hand of four cards from a well-shuffled standard deck of 52 cards.

(a) How many ways can you be dealt at least 3 spades? 3S 1S^c + 4S 0S^c

1 hand of 4 cards

$$\underbrace{C(13,3)}_{3S} \cdot \underbrace{C(39,1)}_{1S^c} + \underbrace{C(13,4)}_{4S} \cdot \underbrace{C(39,0)}_{0S^c} = 286 * 39 + 715 = 11,869$$

13 nCr 3 * 39 + 13 nCr 4

(b) How many ways can you be dealt exactly two diamonds or exactly two clubs?

$$\underbrace{C(13,2)}_{2D} \cdot \underbrace{C(39,2)}_{2D^c} + \underbrace{C(13,2)}_{2C} \cdot \underbrace{C(39,2)}_{2C^c} - \underbrace{C(13,2)}_{2C} \cdot \underbrace{C(13,2)}_{2D}$$

Example

Seven children stop at a restaurant where they have a choice of a cheeseburger, a hot dog, pizza or a burrito. How many different purchases are possible?

c c c c c c c
h h h h h h h
c c h h p p b
c c c c b b b
c c c p p p p

?→

c h p b

*******|||
|*******||
||**|*
****|||***
||*|

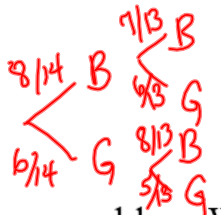
7 *'s
3 |'s

$$\frac{10!}{7!\,3!} = 120$$

4 items ⇒ 3 bars
7 orders ⇒ 7 stars

2.3 Probability Applications of Counting Principles

Recall that $P(E) = \dfrac{n(E)}{n(S)}$



Example

Suppose we have a jar with 8 blue and 6 green marbles. What is the probability that in a sample of 2, both will be blue?

$$\frac{C(8,2)\,C(6,0)}{C(14,2)} = \frac{28}{91}$$

Find the probability distribution table for the number of blue marbles in the sample of 2 marbles:

EVENT	PROB
0	$\frac{C(8,0)C(6,2)}{C(14,2)} = \frac{15}{91}$
1	$\frac{C(8,1)C(6,1)}{C(14,2)} = \frac{48}{91}$
2	$\frac{C(8,2)C(6,0)}{C(14,2)} = \frac{28}{91}$

$$\frac{48}{91} + \frac{28}{91} = \frac{76}{91}$$

What is the probability there is at least one blue marble?

Example

A lottery chooses 6 of 54 numbers be chosen and the order doesn't matter. What is the probability of choosing no winning numbers?

$$\frac{C(6,0)\,C(48,6)}{C(54,6)} = \frac{12,271,512}{25,827,165}$$

What is the probability of choosing at least 3 winning numbers?

$$\frac{\overbrace{C(6,3)\,C(48,3)}^{345,920} + C(6,4)\,C(48,2) + C(6,5)\,C(48,1) + C(6,6)\,C(48,0)}{C(54,6)} = \frac{363,129}{25,827,165}$$

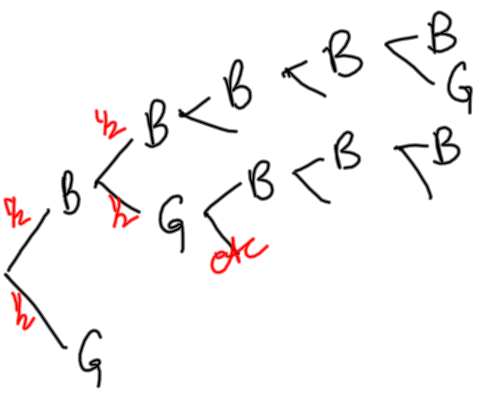
Example

A stack of 100 copies has 3 defective papers. What is the probability that in a sample of 10 there will be no defective papers?

$$\frac{\underset{\text{DEF}}{C(3,0)} * \underset{\text{GOOD}}{C(97,10)}}{C(100,10)} = \frac{1.2576\ldots E13}{1.731\ldots E13} = \frac{178}{245} \quad (\approx .73)$$

Example

A family has 5 children. What is the probability that exactly 3 of the children are girls? $n(S) = \underset{1st}{2} \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$



EVENT	PROB
0G	$\frac{1}{32} = \frac{C(5,0)}{32}$
1G	$\frac{5}{32} = \frac{C(5,1)}{32}$
2G	$\frac{C(5,2)}{32} = \frac{10}{32}$
3G	$\frac{C(5,3)}{32} = \frac{10}{32}$
4G	$\frac{C(5,4)}{32} = \frac{5}{32}$
5G	$\frac{1}{32} = \frac{C(5,5)}{32}$

Example

A box contains 7 red, 6 green, 5 black and 3 purple marbles. What is the probability that in a sample of 6,

all red or all green or all black or all purple

a) The balls are all the same color?

6R 0R^c 6G 6G^c

$$\frac{C(7,6)C(14,0)+C(6,6)C(15,0)+0+0-0}{C(21,6)}=\frac{8}{54{,}264}$$

UNION RULE

$n(A\cup B)=n(A)+n(B)-n(A\cap B)$

b) There are exactly three red and two purple balls?

$$\frac{\underset{3R}{C(7,3)} * \underset{2P}{C(3,2)} * \underset{1N}{C(11,1)}}{C(21,6)}=\frac{1155}{54{,}264}$$

c) There are exactly three red or two purple balls?

$$\left[\underset{3R}{C(7,3)}\,\underset{3R^c}{C(14,3)}+\underset{2P}{C(3,2)}\,\underset{4P^c}{C(18,4)}-\underset{3R}{C(7,3)}\cdot\underset{2P}{C(3,2)}\cdot\underset{1N}{C(11,1)}\right]\Big/C(21,6)$$

4 of the last 5 or 5 of the last 5

Example

An exam consists of 25 questions in which exactly ~~12 must be answered~~. What is the probability that a ~~student answered~~ exactly 2 of the first 5 questions and at least 4 of the last 5 questions?

(1st 5) (mid 15 a) (last 5)

$$\frac{\underset{1st\,5}{C(5,2)}\cdot\underset{mid\,15}{C(15,6)}\cdot\underset{last\,5}{C(5,4)}+\underset{1st\,5}{C(5,2)}\,\underset{mid\,15}{C(15,5)}\,\underset{last\,5}{C(5,5)}}{C(25,12)}=\frac{280{,}280}{5{,}200{,}300}$$

Example RRRR GGG B

We have 4 identical red marbles, 3 identical green marbles and one blue marble. How many distinguishable arrangements of the 8 marbles are there?

$$\frac{8!}{4!\,3!\,1!}$$

"words"

"MISSISSIPPI" problem

Distinguishable Permutations: If there are n_1 items of type 1 and n_2 items of type 2 and ... n_r items of type r, then the number of distinguishable permutations of the $n=n_1+n_2+\ldots+n_r$ items is

$$\frac{n!}{n_1!\times n_2!\times\ldots\times n_r!}$$

Example

How many "words" can be made from the letters *MANHATTAN*?

$$\frac{9!}{\underbrace{1!}_{M}\,\underbrace{3!}_{A}\,\underbrace{2!}_{N}\,\underbrace{1!}_{H}\,\underbrace{2!}_{T}} = 15{,}120$$

2.4 Bernoulli Trials

In a Bernoulli trial we have the following:

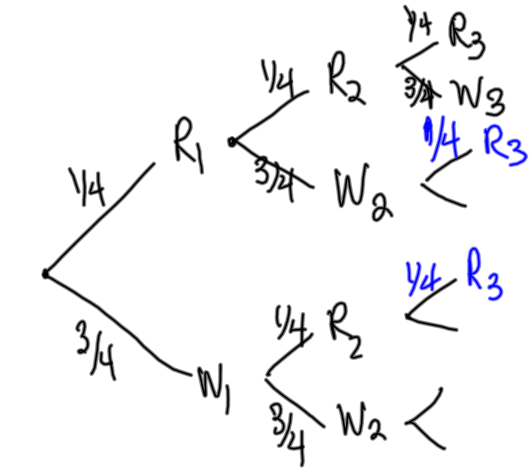
- The same experiment repeated several times.
- The only possible outcomes of these experiments are success or failure.
- The repeated trials are independent so the probability of success remains the same for each trial.

Example

A student takes a 3 question multiple choice test by guessing. Each question has 4 possible answers. Find the probability distribution table for the number of correct answers.

$P(R_1 \cap R_2 \cap R_3) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \left(\frac{1}{4}\right)^3$

$P(R_1 \cap R_2 \cap W_3) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)$



binompdf(N, p, x)

N = # of trials = 3

p = prob of success in a single trial = .25

x = # of successes

(score) EVENT	PROB
x = 3	$C(3,3)(1/4)^3(3/4)^0 = .015625$
x = 2	$C(3,2)(1/4)^2(3/4)^1 = .140625$
x = 1	$C(3,1)(1/4)^1(3/4)^2 = .421875$
x = 0	$C(3,0)(1/4)^0(3/4)^3 = .421875$

Example

During a fifteen day period in the summer, a town had a 20% chance of rain each day. If the rain chances are independent,

a) what is the probability of no rain?

Binomial => sucess = rain

$N = 15,\ p = .2,\ X = 0$

$\text{binompdf}(15, .2, 0) = 0.0352$

Binomial =>
success is no rain

$N = 15,\ p = .8,\ X = 15$

$\text{binompdf}(15, .8, 15) = 0.0352$

$\text{binomcdf}(N, p, x)$

find prob $X = 0, 1, 2, .. x$

b) what is the probability of at most two rain days?

success = rain, $N = 15,\ p = .2$

$X = 0, 1, 2$

$P(X \le 2) = .3980$

$\text{binomcdf}(15, .2, 2) = .3980$

c) what is the probability of between 2 and 6 rain days?

0, 1, 2, 3, 4, 5

$X = 3, 4, 5$

$P(2 < X < 6) = \text{binomcdf}(15, .2, 5) - \text{binomcdf}(15, .2, 2)$

$= \text{binompdf}(15, .2, 3) + b\ (15, .2, 4) + b\ (15, .2, 5)$

d) what is the probability of more than 4 rain days?

$X = 5, 6, \ldots\ 15$

$1 = \underbrace{P(X=0) + P(X=1) +}_{X=0,1,2,3,4} \underbrace{\ldots\ P(X=15)}_{X=5,6,\ldots}$

$P(X > 4) = P(X \ge 5) = 1 - P(X \le 4) = 1 - \text{binomcdf}(15, .2, 4)$

Example Binomial: success = side effect, $N=10$, $p=0.05$

A new drug being tested causes a serious side effect in 5% of the patients using the drug. What is the probability that in a group of 10 patients,

a) none will get the side effect?

$X=0$ binompdf(10, .05, 0) = .5987

b) all will get the side effect?

$X=10$ binompdf(10, .05, 10) = 9.766×10^{-14}

c) at least one person will get the side effect?

$X = 1, 2, \ldots, 10$

$P(X\geq 1) = 1 -$ binompdf(10, .05, 0) = 0.4013

d) between 1 and 4 people inclusive get the side effect?

$X = 1, 2, 3, 4$

binompdf(10, .05, 1) + ... + binompdf(10, .05, 4)

or binomcdf(10, .05, 4) − binompdf(10, .05, 0)

A flash drive has a 0.8% chance of being defective. A shipment has 1000 flash drives. What is the probability that

success = defective, N= 1000, p = 0.008 = $\frac{.8}{100}$

(a) None are defective

$X = 0$: binompdf(1000, .008, 0).

(b) fewer than 8 are defective

$X = 0, 1, 2, \ldots 7$: binomcdf(1000, .008, 7)

(c) At least 5 are defective

$X = 5, 6, \ldots 1000$: 1- binomcdf(1000, .008, 4)

(d) between 6 and 12 are defective

$X = 7, 8, 9, 10, 11$: binomcdf(1000, .008, 11)- binomcdf(1000, .008, 6)

(e) At least 7 but no more than 25

$X = 7, 8, \cdots 25$

binomcdf(1000, .008, 25) – binomcdf(1000, .008, 6)