

Setting Up Linear Programming Problems

Example

A company produces handmade skillets in two sizes, big and giant. To produce one big skillet requires 3 lbs of iron and 6 minutes of labor. To produce one giant skillet requires 4 lbs of iron and 3 minutes of labor. The profit for each big skillet is \$20 and the profit for each giant skillet is \$25. If 1000 lbs of iron and 20 hours of labor are available each day, how many each size skillet should be made to maximize profits?

$x = \#$ of big skillets

$y = \#$ of giant skillets

$P =$ profit in \$

OBJ: Max $P = 20x + 25y$ objective function

SUBJECT TO

$$3x + 4y \leq 1000 \text{ lb of iron}$$

$$6x + 3y \leq 1200 \text{ min of labor}$$

$$x \geq 0, y \geq 0$$

Example

Annika has \$3000 available to invest, with which she buys shares of stock in the Bass Company and Forte Inc. A share of Bass stock costs \$10 and pays dividends of \$3.00 per share each year. Forte stock costs \$50 per share and pays dividends of \$4.00 per share each year. Annika wishes to purchase no more than 80 total shares of stock in order to avoid paying higher trading fees to her broker. Due to the volatility of the Bass stock, her broker recommends that she invest in at most 50 shares of that company's stock. Determine the number of shares Annika should buy of each kind of stock in order to maximize the amount from dividends that she will receive at the end of the first year.

$$\begin{aligned} x &= \# \text{ of sh of Bass} \\ y &= \# \text{ of sh of Forte} \\ D &= \text{div. in } \$ \text{ from these stocks} \\ \text{Max } D &= 3x + 4y \end{aligned}$$

SUBJECT TO

$$\begin{aligned} 10x + 50y &\leq 3000 && (\text{total } \$ \text{ available}) \\ x + y &\leq 80 && (\text{share bought}) \\ x &\leq 50 && (\text{limit on Bass}) \\ x \geq 0, y &\geq 0 \end{aligned}$$

$$\begin{aligned} x &= \$ \text{ in Bass} \\ y &= \$ \text{ in Forte} \\ D &= \text{div in } \$ \\ \text{Max } D &= 3\left(\frac{x}{10}\right) + 4\left(\frac{y}{50}\right) \\ \text{SUB TO} \\ x + y &\leq 3000 && (\text{total } \$) \\ \frac{x}{10} + \frac{y}{50} &\leq 80 && (\text{sh bought}) \\ \frac{x}{10} &\leq 50 && (\text{limit on Bass}) \\ x \geq 0, y &\geq 0 \end{aligned}$$

Example

A dietitian is to prepare two foods in order to meet certain requirements. Each ounce of food I contains 100 units of vitamin C, 40 units of vitamin D and 20 units of vitamin E and costs 25 cents. Each ounce of food II contains 10 units of vitamin C, 80 units of vitamin D and 15 units of vitamin E and costs 15 cents. The mixture of the two foods is to contain at least 260 units of vitamin C, 320 units of vitamin D and 120 units of vitamin E. How many ounces of each type of food should be used in order to minimize the cost?

$x = \# \text{ of oz of food I}$
 $y = \# \text{ of oz of food II}$
 $C = \text{cost in } \$$
 OBJ MIN $C = .25x + .15y$

SUB TO
 $100x + 10y \geq 260 \text{ units of C}$
 $40x + 80y \geq 320 \text{ units of D}$
 $20x + 15y \geq 120 \text{ units of E}$
 $x \geq 0, y \geq 0$

Example

A craftsman has 150 units of wood, 90 units of glue and 150 units of paint. A small picture frame requires 1 unit of wood, 1 unit of glue and 2 units of paint while a large picture frame requires 5, 2 and 1 respectively. If a small frame sells for \$175 and a large frame for \$400, how many of each should be made to maximize the revenue?

$x = \#$ of sm frames

$y = \#$ of lg frames

$R =$ Rev. in \$ from frames

$$\text{MAX } R = 175x + 400y$$

SUBJECT TO

$$1x + 5y \leq 150 \text{ units of wood}$$

$$1x + 2y \leq 90 \text{ units of glue}$$

$$2x + 1y \leq 150 \text{ units of paint}$$

$$x \geq 0, y \geq 0$$

Graphing Systems of Linear Inequalities

The general forms for linear inequalities are

$$ax + by + c \geq 0$$

$$ax + by + c > 0$$

$$ax + by + c \leq 0$$

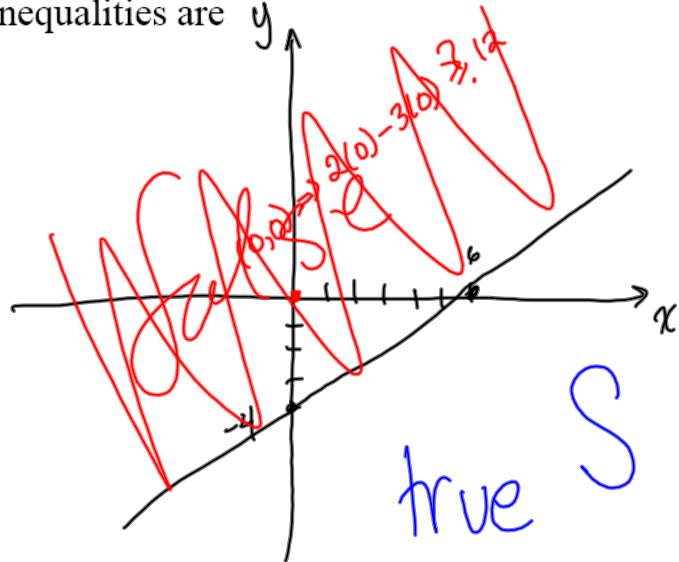
$$ax + by + c < 0$$

Example

Graph $2x - 3y \geq 12$

$$2x - 3y = 12$$

$$(0, -4) \text{ and } (6, 0)$$



NOTE - if your line passes through the origin, you must take a different point for a test point.

If our inequality had \geq or \leq we draw the bounding line as a solid.

If our inequality had $>$ or $<$ we draw the bounding line as DASHED .

The region that satisfies our inequality is called the *feasible region*. This is the region that is white (unshaded). Please label it with an *S*.

What if we have two inequalities (a system)? The feasible region (solution) will be where they are both true at the same time.

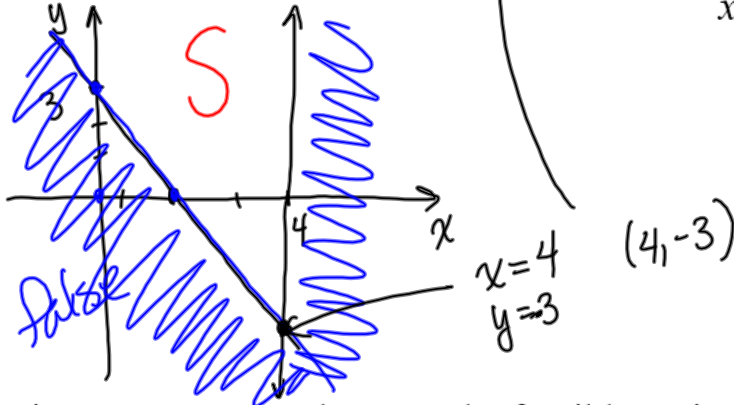
Example

Find the feasible region for the system

$$3x + 2y \geq 6$$

$$x \leq 4$$

$(0, 3), (2, 0)$



It is *UNBOUNDED* because the feasible region cannot be enclosed in a circle.

If S can be enclosed by a circle, it is called *BOUNDED*.

Example

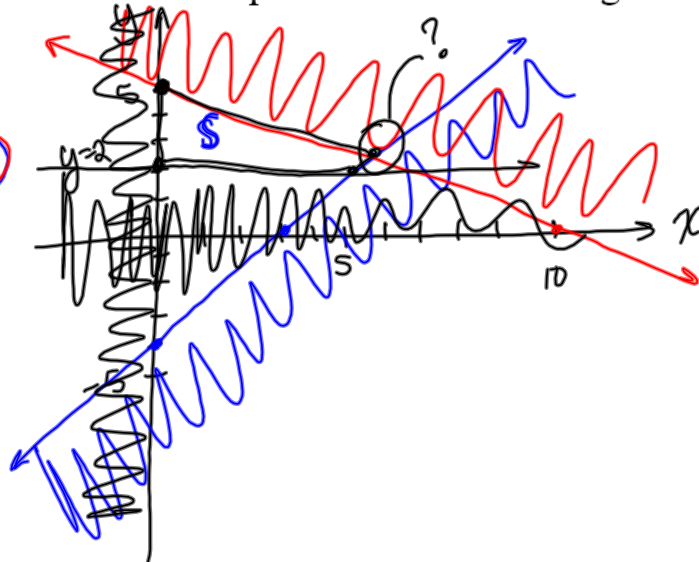
Find the feasible region and label the corner points for the following system of linear inequalities:

$$4x - 3y \leq 12 \quad (0, -4), (3, 0)$$

$$x + 2y \leq 10 \quad (0, 5), (10, 0)$$

$$x \geq 0$$

$$y \geq 2$$



Corners

$(0, 2)$ $(0, 5)$

$(4, 5)$ $(\frac{54}{11}, \frac{28}{11})$

try yourself

Principles of Linear Programming

Annika has \$3000 available to invest, with which she buys shares of stock in the Bass Company and Forte Inc. (...) Determine the number of shares Annika should buy of each kind of stock in order to maximize the amount from dividends that she will receive at the end of the first year.

Let x = the number of shares of Bass stock

Let y = the number of shares of Forte stock

Let D = the total amount of dividends earned

OBJECTIVE: Maximize $D = 3x + 4y$

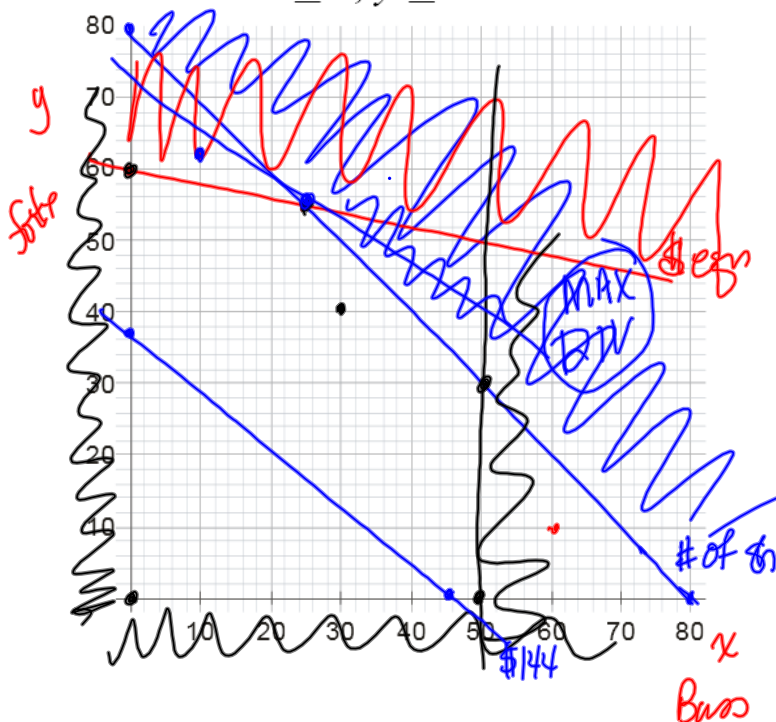
SUBJECT TO:

$$x + y \leq 80 \quad \text{Total number of shares}$$

$$10x + 50y \leq 3000 \quad \text{Total \$ invested}$$

$$x \leq 50 \quad \text{Limit on Bass stock}$$

$$x \geq 0, y \geq 0$$



$(60, 10) \Rightarrow$ NO, TOO MUCH BAS

$(10, 65) \Rightarrow$ NO, TOO MUCH \$

$(40, 50) \Rightarrow$ NO, TOO MANY SHARES

$(30, 40) \Rightarrow$ YES

$$DIV? \quad 3(30) + 4(40) = \$250$$

$$(48, 0) \quad DIV = 3(48) + 4(0) = 144$$

$$(0, 36) \quad DIV = 3(0) + 4(36) = 144$$

$$D = 3x + 4y$$

Can she buy 60 shares of Bass and 10 shares of Forte?

Can she buy 10 shares of Bass and 65 shares of Forte?

Can she buy 40 shares of Bass and 50 shares of Forte?

Can she buy 30 shares of Bass and 40 shares of Forte?

If Annika buys 48 shares of Bass and 0 shares of Forte, how much does she earn in dividends?

If Annika buys 0 shares of Bass and 36 shares of Forte, how much does she earn in dividends?

Solving Linear Programming Problems

Every linear programming problem has a feasible region associated with the constraints of the problem. These feasible regions may be bounded, unbounded or the empty set.

To find the solution (that is, where the maximum or minimum value occurs), we will use the two theorems below.

Theorem 1 If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set S , associated with the problem. Furthermore, if the objective function P is optimized at two adjacent vertices of S , then it is optimized at every point on the line segment joining these vertices, in which case there are infinitely many solutions to the problem.

Theorem 2 Suppose we are given a linear programming problem with a feasible set S and an objective function $P = ax + by$.

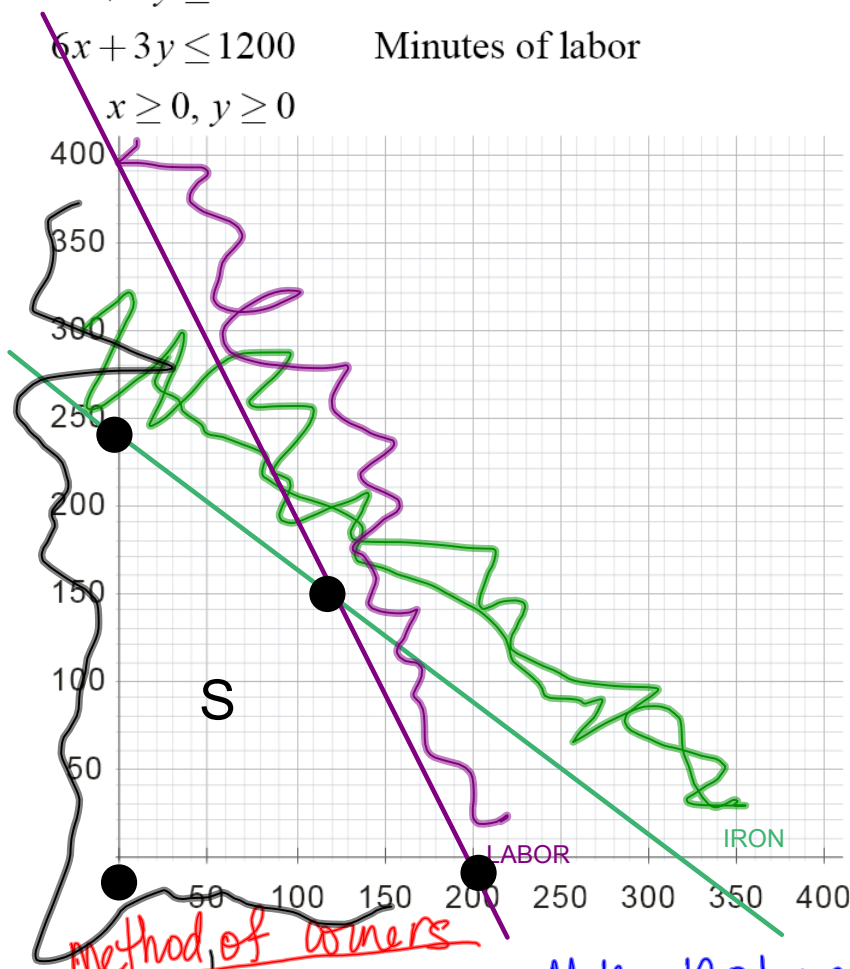
- **Case 1** If S is bounded, then P has both a maximum and a minimum value on S .
- **Case 2** If S is unbounded and both a and b are nonnegative, then P has a minimum value on S provided that the constraints defining S include the inequalities $x \geq 0$ and $y \geq 0$.
- **Case 3** If S is the empty set, then the linear programming problem has no solution; that is, P has neither a maximum nor a minimum value.

Example

A company produces handmade skillets in two sizes, big and giant. (...) how many each size skillet should be made to maximize profits?

x = the number of big skillets produced
 y = the number of giant skillets produced
 P = the profits (in \$) from selling skillets
OBJECTIVE: Maximize $P = 20x + 25y$

SUBJECT TO:
 $3x + 4y \leq 1000$ Pounds of iron
 $6x + 3y \leq 1200$ Minutes of labor
 $x \geq 0, y \geq 0$



method of corners

Vertex	$P = 20x + 25y$
(0,0)	0
(0,250)	6250
(200,0)	4000
(120,160)	6400

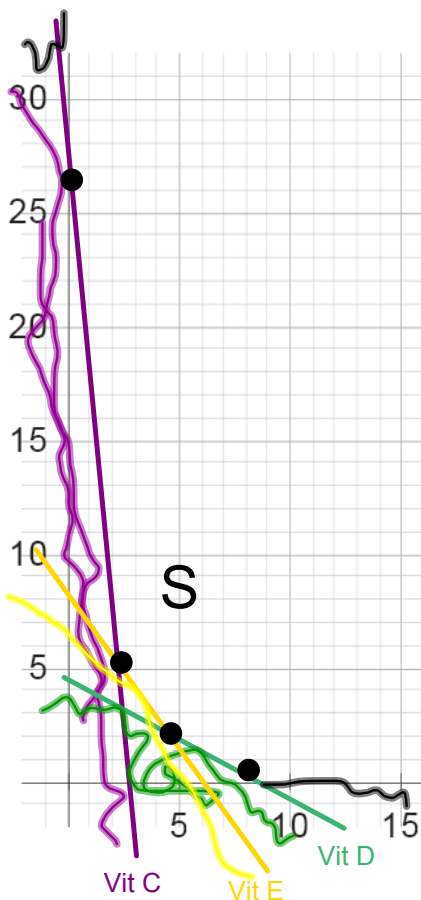
Make 120 big and 160 giant skillets with a max profit of \$6400
 Leftovers?

IRON: $3(x) + 4(y) \rightarrow 3(120) + 4(160) = 1000$
 Labor $6x + 3y \rightarrow 6(120) + 3(160) = 1200$
 No iron or labor left over

Example

A dietitian is to prepare two foods in order to meet certain requirements. (...) How many ounces of each type of food should be used in order to minimize the cost?

x = the number of ounces of food I
 y = the number of ounces of food II
 C = the cost (in \$) for the food
OBJECTIVE: Minimize $C = 0.25x + 0.15y$
SUBJECT TO:
 $100x + 10y \geq 260$ Units of Vit. C
 $40x + 80y \geq 320$ Units of Vit. D
 $20x + 15y \geq 120$ Units of Vit. E
 $x \geq 0, y \geq 0$



Vertex	$C = .25x + .15y$
$(0, 26)$	3.9
$(\frac{27}{13}, \frac{68}{13})$	$339/260 \approx 1.3$ min cost
$(4.8, 1.6)$	1.44
$(8, 0)$	2

Serve $\frac{27}{13}$ oz of food I and $\frac{68}{13}$ oz of food II for a min cost of \$1.30.
 "Leftovers" ?
 Vit C and E req. are just met
 but $40(\frac{27}{13}) + 80(\frac{68}{13}) = 50\frac{7}{13}$ units of D
 so patient got an extra $18\frac{7}{13}$ units of D

Example

A craftsman has 150 units of wood, 90 units of glue and 150 units of paint. (...) how many of each should be made to maximize the revenue?

x = the number of small picture frames produced

y = the number of large picture frames produced

C = the from selling picture frames (in dollars)

OBJECTIVE: Maximize $R = 175x + 400y$

SUBJECT TO:

$1x + 5y \leq 150$ Units of wood

$1x + 2y \leq 90$ Units of glue

$2x + 1y \leq 150$ Units of paint

$x \geq 0, y \geq 0$

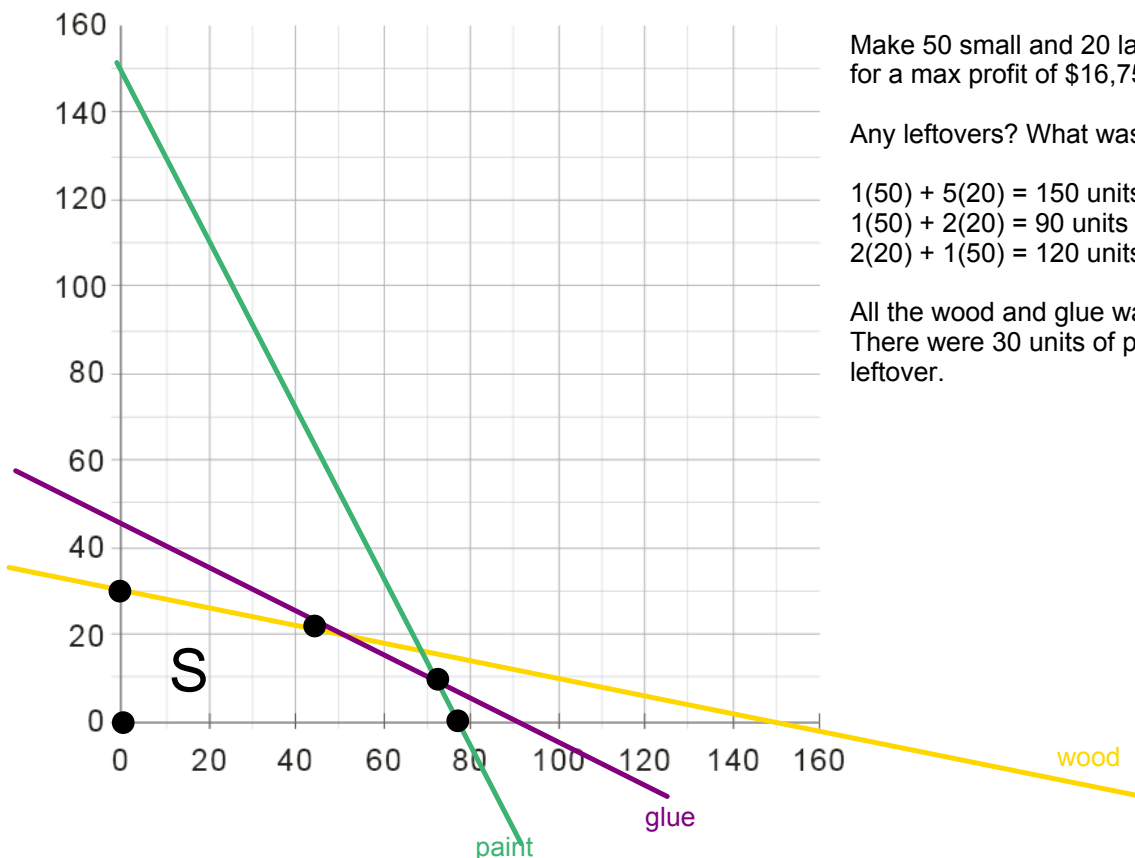
Vertex	$R=175x + 400y$
(0, 0)	0
(0, 30)	12,000
(50, 20)	16,750
(70, 10)	16,250
(75, 0)	13,125

Make 50 small and 20 large frames for a max profit of \$16,750.

Any leftovers? What was used?

$1(50) + 5(20) = 150$ units of wood
 $1(50) + 2(20) = 90$ units of paint
 $2(20) + 1(50) = 120$ units of paint

All the wood and glue was used. There were 30 units of paint leftover.



Example

A company produces handmade skillets in two sizes, big and giant. (...) how many each size skillet should be made to maximize profits if big skillets have a profit of \$30 each and giant skillets have a profit of \$40 each?

x = the number of big skillets produced

y = the number of giant skillets produced

P = the profits (in \$) from selling skillets

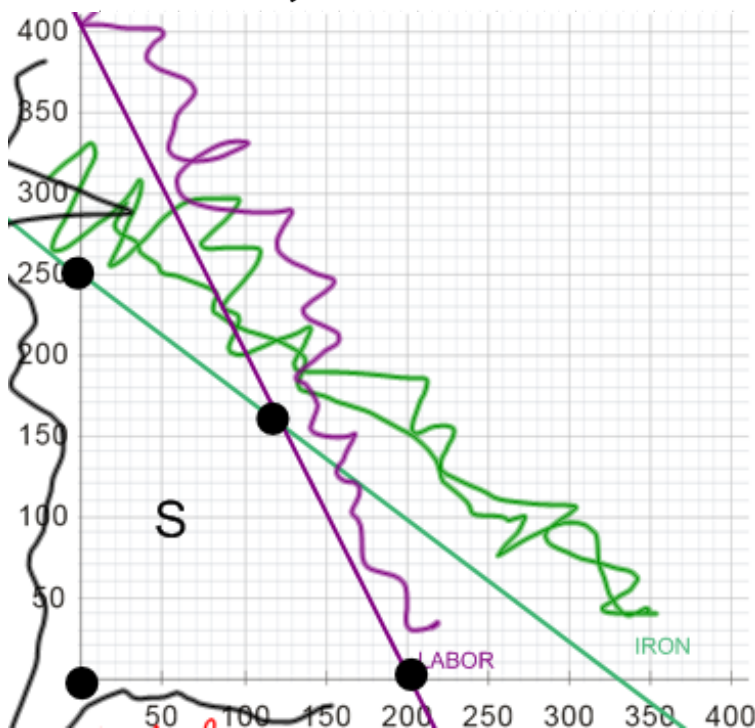
OBJECTIVE: Maximize $P = 30x + 40y$

SUBJECT TO:

$3x + 4y \leq 1000$ Pounds of iron

$6x + 3y \leq 1200$ Minutes of labor

$x \geq 0, y \geq 0$



Vertex $P = 30x + 40y$

(0, 0)

(0, 250)

(120, 160)

(200, 0)

Example

A dietitian is to prepare two foods in order to meet certain requirements. (...) How many ounces of each type of food should be used in order to minimize the cost if an ounce of food I costs 20 cents and an ounce of food II costs 15 cents?

x = the number of ounces of food I

y = the number of ounces of food II

C = the cost (in \$) for the food

OBJECTIVE: Minimize $C = 0.20x + 0.15y$

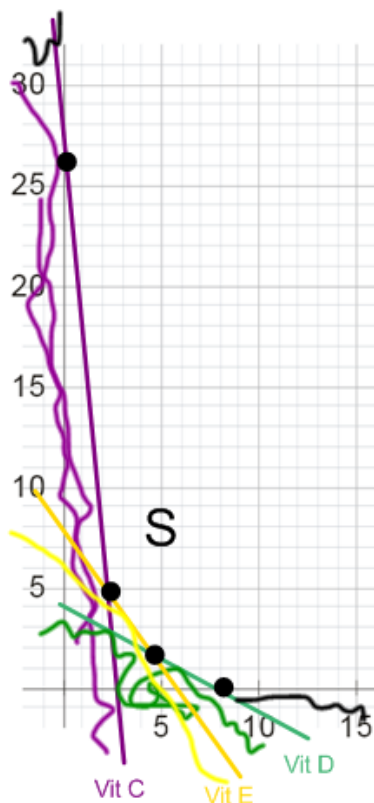
SUBJECT TO:

$$100x + 10y \geq 260 \quad \text{Units of Vit. C}$$

$$40x + 80y \geq 320 \quad \text{Units of Vit. D}$$

$$20x + 15y \geq 120 \quad \text{Units of Vit. E}$$

$$x \geq 0, y \geq 0$$



Vertex $C = 0.2x + .15y$

(0, 26)

(8, 0)

(4.8, 1.6)

(27/13, 68/13)