§ 10.2. Green's Theorem

Let $D$ be a closed, bounded region in $\mathbb{R}^2$ whose boundary $C = \partial D$ consists of finitely many simple closed curves that orient the curve $C$ so that $D$ is on the left. Let $F(x, y) = M(x, y)i + N(x, y)j$ be a $C^1$ vector field. Then

$$\oint_C M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.$$ 

Ex. Let $F = xy \, i + y^2 \, j$, $D$ bounded by $y = x^2$.

LHS = $\int_0^1 \left( x^3 \, dt + x^4 \, dt \right) + \int_0^1 \left( x^2 \, dt + x^2 \, dt \right) = \frac{1}{4} + 2 \frac{1}{6} - \frac{2}{3} = -\frac{1}{2}$

RHS = $\iint_D \left( 0 - x \right) \, dx \, dy = \int_0^1 \left( x - y \right) \, dy = \int_0^1 \left( -\frac{x^2}{2} \right) \, dy = -\frac{1}{4} + \frac{1}{6} = -\frac{1}{12}$.

For $F = -y \, i + x \, j$, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 + 1 = 2$. Then

$$\oint_C M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy$$

$$= 2 \iint_D \, dx \, dy = 2 \text{ area of } D.$$
Ex. Find the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)
\( x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi \)

Area = \( \frac{1}{2} \int_C y \, dl - x \, dy \)
\[ = \frac{1}{2} \int_0^{2\pi} (-bsin(t)\,(-asint) + acost\,(bcost)) \, dt \]
\[ = \frac{1}{2} \int_0^{2\pi} \, ab \, dt = ab \pi. \]

**n** = the outward unit normal vector.

Let \( D \) be a region bounded by \( C = \partial D \) s.t. Green's theorem applies. Let \( n \) be the outward unit normal vector at \( C = \partial D \), and \( F(x,y) = M(x,y)i + N(x,y)j \) be \( C \) vector field on \( D \). Then

\[ F \cdot n = \text{particles crossing } C \text{ out} = \text{Flux} \]

\( \nabla \cdot F = \text{rate of particles leaving a point} \)

\[ \oint_C (F \cdot n) \, ds = \iint_D \nabla \cdot F \, dA \quad \text{(divergence theorem)} \]

Total particles crossing \( C \text{ out} = \text{total particles left } D \).
when $C: x(t) = (x(t), y(t))$. 
$T(t) = (x'(t), y'(t)) \perp \vec{N}(t) = (-y'(t), x'(t))$, $n(t) = \frac{\vec{N}(t)}{\|\vec{N}(t)\|}$.

§10.3 Path Independence.

A vector field $\mathbf{F}$ is said to have a path-independence line integral if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

for any simple, piecewise $C'$ curves with the same initial and terminal points.

THEOREM: A vector field $\mathbf{F}$ has a path-indep line integral if and only if

$$\oint_{C} \mathbf{F} \cdot d\mathbf{s} = 0$$

for any piecewise $C'$ simple, closed curve.

Proof:

$$\oint_{C} \mathbf{F} \cdot d\mathbf{s} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{s} - \oint_{C_2} \mathbf{F} \cdot d\mathbf{s} = 0.$$
THM. Let $F$ be a continuous vector field on a connected open region $D$ of $\mathbb{R}^n$. Then $F = \nabla f$ if and only if $F$ has a path-independent line integral over curves in $D$. Moreover, if $C$ is any piecewise $C^1$, oriented curve in $D$ with initial point $A$ and terminal point $B$, then

$$\int_C F \cdot ds = f(B) - f(A), \quad (F = \nabla f).$$

**Proof:** $C: x(\alpha), \quad A = x(a), \quad B = x(b)$

$$\int_C F \cdot ds = \int_a^b \alpha(\alpha) \cdot x'(t) \cdot dt = \int_a^b \frac{d}{dt} f(x(\alpha)) \cdot dt$$

$$= f(x(\alpha)) |_a^b = f(B) - f(A).$$

**Ex.** $F = M \hat{i} + N \hat{j} = x \hat{i} + y \hat{j}$. Note $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.

$$\int_C F \cdot ds = \int_M dx + N dy = \int(M(\alpha) + N(\alpha)) \cdot dt$$

1. $x = t, \quad y = t, \quad 0 \leq t \leq 1, \quad \int_C F \cdot ds = \int_0^1 (t + t) dt = 1$.
2. $x = t, \quad y = t^2, \quad 0 \leq t \leq 1, \quad \int_C F \cdot ds = \int_0^1 (t + 2t^2) dt = \frac{1}{2} + \frac{1}{4} = 1$.
3. $x = 0, \quad y = t, \quad 0 \leq t \leq 1, \quad \int_C F \cdot ds = \int_0^1 (0 + 0dt) + \int_0^1 (t dt + 0) = 1$.
4. $f = \frac{1}{2}(x^2 + y^2), \quad F = \nabla f = (f_x, f_y) = (x, y)$,

$$\int_C F \cdot ds = f(1, 1) - f(0, 0) = \frac{1}{2}(1+1) + 0 = 1.$$
When $F = \nabla f$, $f$ is called a conservative vector field scalar potential.

For given $F$,
1) How to know if $F$ is conservative?
2) Assume $F$ is conservative, how to find $f$ s.t. $F = \nabla f$?

**Def.** A region $D$ in $\mathbb{R}^2$ is simply connected if any simple closed curve in $D$ can be shrunk to a point.

Yes: \begin{tabular}{c}
\end{tabular}

\begin{tabular}{c}
No: \end{tabular}

**Thm.** Let $F = M \mathbf{i} + N \mathbf{j}$ be a $C^1$ vector field in a simply connected region $D$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $F = \nabla f$ for some $f$ if and only if $\nabla \times \mathbf{F} = 0$ in $D$.

**Remark.** $\nabla \times \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & D \\ \end{vmatrix} = \frac{\partial N}{\partial z} i - \frac{\partial M}{\partial z} j + \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) k = 0$

$\Rightarrow N = N(x, y), M = M(x, y)$ and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.

**Ex.** Let $F = x^2 y \mathbf{i} - 2xy \mathbf{j}$.

then $\frac{\partial N}{\partial x} = -2y + \frac{\partial M}{\partial y} = x^2 \Rightarrow$ not conservative.
Ex. \( F = (2xy + \cos 2y) \mathbf{i} + (x^2 - 2x \sin 2y) \mathbf{j} = M \mathbf{i} + N \mathbf{j} \)

calculate \( \frac{\partial M}{\partial y} = 2x - 25 \cos 2y = \frac{\partial N}{\partial x} = 2x - 25 \cos 2y \)

\( \Rightarrow F \) is conservative. How to find \( f \) s.t. \( F = \nabla f \)?

\( F = M \mathbf{i} + N \mathbf{j} = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \).

\( M = \frac{\partial f}{\partial x} \Rightarrow f = \int M \, dx + \alpha(y) = x^2y + x \cos 2y + \alpha(y) \)

\( N = \frac{\partial f}{\partial y} \Rightarrow f = \int N \, dy + \beta(x) = x^2y + x \cos 2y + \beta(x) \)

\( f = \int M \, dx = \alpha(y) = 0 \Rightarrow f = x^2y + x \cos 2y \).

**THM.** If \( D \) is simply connected domain, then

\( F = \nabla f \) in \( D \) if and only if \( \nabla \times F = 0 \)

Ex. \( F = (e^{x \sin y} - yz) \mathbf{i} + (e^{xy} - xz) \mathbf{j} + (z - xy) \mathbf{k} \)

check \( \nabla \times F = \left| \begin{array}{ccc}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
i & j & k \\
e^{x \sin y} - yz & e^{xy} - xz & z - xy
\end{array} \right| = 0. \)

\( \Rightarrow F = M \mathbf{i} + N \mathbf{j} + PK = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right). \) To find \( f \).

\( f = \int M \, dx + \alpha(y,z) = \int (e^{x \sin y} - yz) \, dx + \alpha(y,z) = e^{x \sin y} - xyz + \alpha(y,z) \)

\( f = \int N \, dy + \beta(x,z) = \int (e^{xy} - xz) \, dy + \beta(x,z) = e^{xy} - xyz + \beta(x,z) \)

\( f = \int P \, dz + \gamma(x,y) = \int (z - xy) \, dz + \gamma(x,y) = z^2/2 - xyz + \gamma(x,y) \)

\( f = \int f = \alpha(y,z) = \beta(x,z) = \gamma(x,y) = e^{x \sin y} \)

\( \Rightarrow f = e^{x \sin y} - xyz + z^2/2 (\text{ const.}) \)
Next to compute $\int F \cdot dS$ along a curve from $(0,0,0)$ to $(1, \frac{\pi}{2}, 2)$, we have

$$\int F \cdot dS = f(1, \frac{\pi}{2}, 2) - f(0,0,0) = e - 1 - 1 \cdot \frac{\pi}{2} \cdot 2 + \frac{4}{2} = e - \pi + 2$$