Section 5.8 Implicit Differentiation and Related Rates

Why we differentiate Implicitly?

Suppose $y = f(x)$. In this case, we say y is an explicit function of x and we can therefore differentiate as usual: $\dfrac{dy}{dx} = f'(x)$. In this section, we investigate how to differentiate if y cannot be written as an explicit function of x, that is y is implicitly defined as a function of x. Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation.** This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

$y = x + 4$: explicit.

$y - x = 4$: implicit.

$x^2 + y^2 = 4y$: implicit

ex) $\left(\underbrace{(x^2+2x)^4}_{y^4}\right)' = 4\underbrace{(x^2+2x)}_{y}{}^3 \cdot \underbrace{(2x+2)}_{y' \text{ or } \frac{dy}{dx}}$

$\Rightarrow \quad (y^4)' = 4y^3 \cdot \dfrac{dy}{dx}$

cf) $(x^4)' = 4x^3$

Ex.1) Find $\dfrac{dy}{dx}$ for the following equations.

(a) $x^2 + y^2 = 4$

$\overset{\frac{d}{dx}}{\Rightarrow} \quad 2x + 2y \cdot \dfrac{dy}{dx} = 0$

$\Rightarrow \quad 2y \cdot \dfrac{dy}{dx} = -2x$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{-2x}{2y}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

(b) $y^3 = 2x^2 + y^4$

$$\stackrel{\frac{d}{dx}}{\Rightarrow} 3y^2 \cdot \frac{dy}{dx} = 4x + 4y^3 \cdot \frac{dy}{dx}$$

$$\Rightarrow 3y^2 \cdot \frac{dy}{dx} - 4y^3 \cdot \frac{dy}{dx} = 4x$$

$$\Rightarrow \left(3y^2 - 4y^3\right) \cdot \frac{dy}{dx} = 4x$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x}{3y^2 - 4y^3}$$

(c) $x^4 - 2xy^2 + 2y^3 = 32$

$$\stackrel{\frac{d}{dx}}{\Rightarrow} 4x^3 \overbrace{-2 \cdot y^2 + (-2x) \cdot 2y \cdot \frac{dy}{dx}} + 6y^2 \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow -4xy \cdot \frac{dy}{dx} + 6y^2 \cdot \frac{dy}{dx} = -4x^3 + 2y^2$$

$$\Rightarrow \left(-4xy + 6y^2\right) \frac{dy}{dx} = -4x^3 + 2y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4x^3 + 2y^2}{-4xy + 6y^2} \quad \text{or} \quad \frac{-2x^3 + y^2}{-2xy + 3y^2}$$

(d) $(x^2+y^2)^3 = 2y^4 + 6x^2$

$$\overset{\frac{d}{dx}}{\Rightarrow} 3\left(x^2+y^2\right)^2 \cdot \left(2x + 2y \cdot \frac{dy}{dx}\right) = 8y^3 \cdot \frac{dy}{dx} + 12x$$

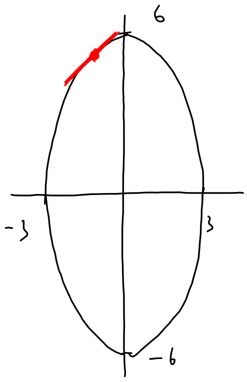
$$\Rightarrow 6x\left(x^2+y^2\right)^2 + 6y \cdot (x^2+y^2)^2 \cdot \frac{dy}{dx} = 8y^3 \cdot \frac{dy}{dx} + 12x$$

$$\Rightarrow 6y \cdot (x^2+y^2)^2 \cdot \frac{dy}{dx} - 8y^3 \cdot \frac{dy}{dx} = 12x - 6x\left(x^2+y^2\right)^2$$

$$\Rightarrow \left(6y\left(x^2+y^2\right)^2 - 8y^3\right) \cdot \frac{dy}{dx} = 12x - 6x\left(x^2+y^2\right)^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{12x - 6x\left(x^2+y^2\right)^2}{6y\left(x^2+y^2\right)^2 - 8y^3} \quad \text{or} \quad \frac{6x - 3x\left(x^2+y^2\right)^2}{3y\left(x^2+y^2\right)^2 - 4y^3}$$

Ex.2) Find the tangent line to the ellipse $\frac{x^2}{9}+\frac{y^2}{36}=1$ at the point $(-1, 4\sqrt{2})$.



① point : $(-1, 4\sqrt{2})$

② slope of tangent line : $\left.\frac{dy}{dx}\right|_{(-1,4\sqrt{2})}$

$$\frac{1}{9}x^2+\frac{1}{36}y^2=1$$

$$\frac{d}{dx} \Rightarrow \frac{2}{9}x+\frac{\cancel{2}^{1}}{\cancel{36}_{18}}y\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{18}{y}\cdot\frac{y}{18}\cdot\frac{dy}{dx}=-\frac{2x}{\cancel{9}_{1}}\cdot\frac{\cancel{18}^{2}}{y}$$

$$\Rightarrow \frac{dy}{dx}=\left.\frac{-4x}{y}\right|_{(-1,4\sqrt{2})}=\frac{-4(-1)}{4\sqrt{2}}=\frac{4}{4\sqrt{2}}=\frac{1\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

$$y-y_1=m(x-x_1)$$

$$\Rightarrow y-4\sqrt{2}=\frac{\sqrt{2}}{2}(x-(-1))$$

$$\Rightarrow y-4\sqrt{2}=\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}$$

$$\therefore y=\frac{\sqrt{2}}{2}x+\frac{9\sqrt{2}}{2}$$

Related Rates

In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides (or implicitly) with respect to time. In this section, we have two or more quantities that are changing with respect to time. Our first goal is to find a formula that relates the quantities. Then we will solve for the desired rate of change. If x is changing with respect to time t, then $x = x(t)$ and $\frac{dx}{dt}$ is the rate of change x with respect to time.

eg. of x and y

$x(t)$ $y(t)$

$\frac{dx}{dt}$: given $\frac{dy}{dt}$: ?

$\frac{dx}{dt} = -2$

Ex.3) Suppose both x and y are changing size with respect to time t (assume x and y are both positive). At the instant when $x = 4$, the rate of change of x with respect to t is -2. If x and y are related by the equation $x^3 + 4y^2 = 84$, find the corresponding change in y at this same instant.

bridge eg.

$\frac{dy}{dt} = ?$

$x^3 + 4y^2 = 84$

$x(t)$ $y(t)$: positive

$x = 4$

$\frac{dx}{dt} = -2$

Q: $\frac{dy}{dt} = ?$

$\frac{d}{dt}$

$x^3 + 4y^2 = 84$

since $x = 4$

$\Rightarrow (4)^3 + 4y^2 = 84$

$\Rightarrow 64 + 4y^2 = 84$

$\Rightarrow 4y^2 = 20$

$\Rightarrow y^2 = 5$

$\Rightarrow y = \pm\sqrt{5}$

$\therefore y = \sqrt{5}$

$$3x^2 \cdot \frac{dx}{dt} + 8y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow 3(4)^2 \cdot (-2) + 8(\sqrt{5}) \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow -96 + 8\sqrt{5} \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow 8\sqrt{5} \cdot \frac{dy}{dt} = 96$$

$$\therefore \frac{dy}{dt} = \frac{96}{8\sqrt{5}} = \frac{12}{\sqrt{5}} \text{ or } \frac{12\sqrt{5}}{5}$$

Ex.4) The cost equation is given by $C = 8 + 0.2x^2$, where C is the total cost when x units are produced. If the number of items manufactured is increasing at the rate of 20 per week, find the rate of change of C with respect to time when 4 items are manufactured.

$$\frac{dC}{dt} = ? \qquad x = 4 \qquad \frac{dx}{dt} = 20$$

$$C = 8 + 0.2x^2$$

$$\Downarrow \frac{d}{dt}$$

$$\frac{dC}{dt} = 0 + 0.4x \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dC}{dt} = 0.4 \cdot (4) \cdot (20)$$

$$\therefore \frac{dC}{dt} = \$32 \text{ per week when } x = 4$$

Ex.5) A stone is thrown into a still pond and circular ripples move out. The radius of the disturbed region increases at a rate of 3 ft/sec. Find the rate at which the area of the disturbed region is increasing when the farthest ripple is 20 ft from the place at which the rock struck the pond.

$\frac{dr}{dt} = 3 \ ft/sec$ $\qquad \frac{dA}{dt} = ?$, $r = 20 \ ft$

$\Downarrow$

$A = \pi r^2$

$\Downarrow \frac{d}{dt}$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

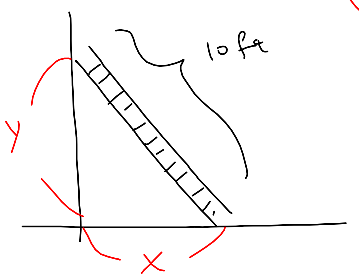
$$\Rightarrow \frac{dA}{dt} = 2\pi(20)\cdot(3) = 120\pi \ ft^2/sec$$

Ex.6) A 10 ft ladder leans against a wall and slides down, with the foot of the ladder observed to be moving away from the wall at 5 ft/sec when it is 8 ft from the wall. At what speed is the top of the ladder moving downward?

$x = 8$

$\frac{dx}{dt} = 5$ ft/sec

$\frac{dy}{dt} = ?$



$100 = x^2 + y^2$

since $x = 8$

$\Rightarrow 100 = 8^2 + y^2$

$\Rightarrow 100 = 64 + y^2$

$\Rightarrow 36 = y^2$

$\therefore y = 6$

$$10^2 = x^2 + y^2$$

$$\Downarrow \frac{d}{dt}$$

$$0 = \cancel{2}x \cdot \frac{dx}{dt} + \cancel{2}y \cdot \frac{dy}{dt}$$

$$\Rightarrow 0 = (8) \cdot (5) + (6) \cdot \frac{dy}{dt}$$

$$\Rightarrow 0 = 40 + 6 \cdot \frac{dy}{dt}$$

$$\Rightarrow -40 = 6 \cdot \frac{dy}{dt}$$

$$\therefore \frac{dy}{dt} = -\frac{40}{6} = -\frac{20}{3} \text{ ft/sec}$$