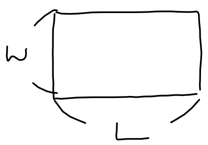
$\frac{dL}{dt} = 2$

Ex.7) The length of a rectangle is increasing at the rate of 2 feet per second, while the width is increasing at the rate of 1 foot per second. How fast is the area increasing when the length is 5 feet and the area is 50 squarefeet?



$\frac{dW}{dt} = 1$ $\frac{dA}{dt} = ?$, $L = 5$, $A = 50$

$A = L \cdot W$

$\Downarrow \frac{d}{dt}$

$$\frac{dA}{dt} = \frac{dL}{dt} \cdot W + L \cdot \frac{dW}{dt}$$

$A = L \cdot W$

since $L = 5$, $A = 50$

$\Rightarrow 50 = 5 \cdot W$

$\therefore W = 10$

$$\Rightarrow \frac{dA}{dt} = (2) \cdot (10) + (5) \cdot (1)$$

$$= 25 \text{ ft/sec.}$$

Chapter 6 Integration

Section 6.1 Antiderivative

Definition. Antiderivative

If $F'(x) = f(x)$, then $F(x)$ is called an antiderivative of $f(x)$.

$$(x^2)' = 2x$$

$\therefore$ $2x$ is a derivative of x^2

$\Leftrightarrow$ x^2 is an antiderivative of $2x$

$(x^2)' = 2x$

$(x^2+1)' = 2x$

$(x^2-10)' = 2x$

$\cdots$

$(x^2 + C)' = 2x$

Definition.

If F is an antiderivative of f, then the most general antiderivative of f if $F(x) + C$, where C is any arbitrary constant.

$$\boxed{\frac{d}{dx}(F(x) + C) = f(x)}$$

Definition. The Indefinite Integral

The collection of all antiderivatives of a function $f(x)$ is called the **indefinite integral** and is denoted by $\int f(x)\,dx$.

If we know one function $F(x)$ for which $F'(x) = f(x)$, then

$$\boxed{\int f(x)\,dx = F(x) + C}$$

where C is an arbitrary constant and called the **constant of integration**.

$$\int \underbrace{3x^2}_{f(x)}\,dx = \underbrace{x^3}_{F(x)} + C$$

check) $F' = f$

Integration Rules

(a) If $n \neq -1$, then

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

ex) $\int x^2 dx = \frac{1}{2+1} x^{2+1} + C$

$= \frac{1}{3} x^3 + C$

(b)

$$\int k \, dx = kx + C$$

ex) $\int 4 \, dx = 4x + C$

(c) For any constant k,

$$\int k f(x) \, dx = k \int f(x) \, dx$$

(d)

$$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

(e)

$$\int e^x \, dx = e^x + C$$

(f)

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

Ex.8) Find the following

(a) $\int 8\ dx = 8x + C$

(b) $\int x^5\ dx = \frac{1}{5+1} x^{5+1} + C = \frac{1}{6} x^6 + C$

(c) $\int \frac{1}{3} x^4\ dx = \frac{1}{3} \int x^4\, dx = \frac{1}{3} \cdot \frac{1}{4+1} x^{4+1} + C$

$= \frac{1}{15} x^5 + C$

(d) $\int \frac{2}{t^9}\,dt \quad = 2\int t^{-9}\,dt \quad = 2\cdot \frac{1}{-9+1} t^{-9+1} + C$

$= -\frac{1}{4} t^{-8} + C$

(e) $\int (5x^4 + x^3 - 2)\,dx \quad = 5\cdot \frac{1}{4+1} x^{4+1} + \frac{1}{3+1} x^{3+1} - 2x + C$

$= x^5 + \frac{1}{4}x^4 - 2x + C$

(f) $\int \frac{3}{x}\,dx \quad = 3\cdot \ln|x| + C$

(g) $\int (e^x + x^3)\, dx \quad = e^x + \frac{1}{4}x^4 + c$

(h) $\int \left(3\sqrt{x} - \frac{1}{x^2} - x^{\frac{3}{2}}\right)\, dx \quad = \int \left(3x^{\frac{1}{2}} - x^{-2} - x^{\frac{3}{2}}\right) dx$

$= 3 \cdot \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} - \frac{1}{-2+1} x^{-2+1} - \frac{1}{\frac{3}{2}+1} x^{\frac{3}{2}+1} + C$

$= 3 \cdot \frac{1}{\frac{3}{2}} x^{\frac{3}{2}} - \frac{1}{-1} x^{-1} - \frac{1}{\frac{5}{2}} x^{\frac{5}{2}} + C$

$= 3 \cdot \frac{2}{3} x^{\frac{3}{2}} + x^{-1} - \frac{2}{5} x^{\frac{5}{2}} + C$

$= 2x^{\frac{3}{2}} + x^{-1} - \frac{2}{5} x^{\frac{5}{2}} + C$

(i) $\int \frac{4x^2+x^3}{8x}\,dx = \int \left(\frac{4x^2}{8x} + \frac{x^3}{8x} \right) dx = \int \left(\frac{1}{2}x^1 + \frac{1}{8}x^2 \right) dx$

$$= \frac{1}{2} \cdot \frac{1}{1+1} x^{1+1} + \frac{1}{8} \cdot \frac{1}{2+1} x^{2+1} + C$$

$$= \frac{1}{4}x^2 + \frac{1}{24}x^3 + C$$

(j) $\int (x-2)(x+3)\,dx = \int (x^2+x-6)\,dx$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + C$$

Ex.9) Find y if $y(1) = 1$ and $\frac{dy}{dx} = \frac{3}{x} + \frac{1}{x^2}$

$$\Rightarrow \quad dy = \left(\frac{3}{x} + \frac{1}{x^2}\right) dx$$

take $\int$ on both sides

$$\Rightarrow \quad \int 1 \, dy = \int \left(\frac{3}{x} + \frac{1}{x^2}\right) dx$$

$$\Rightarrow \quad \underbrace{\int 1 \, dy}_{=y} = \int \left(\frac{3}{x} + x^{-2}\right) dx$$

$$\Rightarrow \quad y = 3 \cdot \ln|x| + \frac{1}{-2+1} x^{-2+1} + C$$

$$\Rightarrow \quad y = 3 \ln|x| - x^{-1} + C$$

and since $y(1) = 1$

$$\Rightarrow \quad 1 = 3 \overbrace{\ln|1|}^{=0} - (1)^{-1} + C$$

$$\Rightarrow \quad 1 = -1 + C$$

$$\therefore \quad C = 2$$

$$\boxed{\therefore \quad y = 3 \ln|x| - \frac{1}{x} + 2}$$

$R'(x)$

Ex.10) The daily marginal revenue function for the Black Day Sunglasses Company is given by $MR(x) = 30 - 0.0003x^2$ for $0 \leq x \leq 540$, where x represents the number of sunglasses produced and sold.

(a) Knowing that $R(50) = 1487.50$, find the revenue function.

$\int R'(x) = \int 30 - 0.0003x^2$

$\Rightarrow \int R'(x)\,dx = \int (30 - 0.0003x^2)\,dx$

$\Rightarrow R(x) = 30x - 0.0003 \cdot \frac{1}{2+1} x^{2+1} + C$

$\therefore R(x) = 30x - 0.0001x^3 + C$

since $R(50) = 1487.50$

$1487.50 = 30(50) - 0.0001(50)^3 + C$

$\Rightarrow 1487.50 = 1487.50 + C$

$\therefore C = 0$

$\boxed{\therefore R(x) = 30x - 0.0001x^3}$

(b) Find the price demand function for the sunglasses.

since $R(x) = p \cdot x$

$R(x) = 30x - 0.0001x^3$

$= (\underbrace{30 - 0.0001x^2}_{=p})x$

$\therefore p = 30 - 0.0001x^2$

(c) What will the price be when the demand is 250 sunglasses?

$p = 30 - 0.0001(250)^2$

$= \$23.75$ / sunglass

Section 6.2 Substitution

The method of substitution is based on reversing the chain rule. Recall that if $h(x) = g(f(x))$, then

$$\int \frac{d}{dx}(h(x)) = \int \frac{d}{dx}(g(f(x))) = \int g'(f(x)) \cdot f'(x)$$

$\Rightarrow g(f(x)) = \int g'(f(x)) \cdot f'(x)\,dx$

Thus,

$$\int g'(f(x))f'(x)\,dx = g(f(x)) + C$$

ex) $\int e^{\overbrace{x^3-1}^{=f(x)}} \cdot (\underbrace{3x^2}_{=f'(x)})\,dx = e^{x^3-1} + C$

Integration by u-substitution.

1. Select u (look for function where you normally have x)
2. Take the derivative of u using $\frac{du}{dx}$ notation.
3. Bring dx to the right hand side.
4. Bring any constant multiples to the left-hand side.
5. Substitute to replace all terms with x's.
6. Integrate with u's.
7. Return x's into the problem.

ex) $\int e^{\overbrace{x^3-1}^{=u}} \cdot \overbrace{(3x^2)\,dx}^{=du}$

Let $u = x^3-1$

$\Rightarrow \frac{du}{dx} = 3x^2$

$\Rightarrow du = 3x^2 \cdot dx$

$= \int e^u \cdot du$

$= e^u + C$

$= e^{x^3-1} + C$

Ex.1) Evaluate the following.

(a) $\int 7(\underbrace{8x+3}_{=u})^{10}\,\underbrace{dx}_{=\frac{1}{8}du} = 7\int u^{10}\cdot\frac{1}{8}\,du = \frac{7}{8}\int u^{10}\,du$

$= \frac{7}{8}\cdot\frac{1}{10+1}u^{10+1} + C$

$= \frac{7}{88}u^{11} + C$

$= \frac{7}{88}(8x+3)^{11} + C$

Let $u = 8x+3$

$\Rightarrow \frac{du}{dx} = 8$

$\Rightarrow du = 8dx$

$\Rightarrow \frac{1}{8}du = dx$

(b) $\int \frac{12x}{3x^2+5}\,dx$, with $3x^2+5 = u$ and $x\,dx = \frac{1}{6}du$: $= 12\int \frac{1}{u}\cdot\frac{1}{6}\cdot du = \frac{12}{6}\int \frac{1}{u}\,du$

$= 2\int \frac{1}{u}\,du$

$= 2\cdot\ln|u| + C$

$= 2\ln|3x^2+5| + C$

Let $u = 3x^2+5$

$\Rightarrow \frac{du}{dx} = 6x$

$\Rightarrow du = 6x\cdot dx$

$\Rightarrow \frac{1}{6}du = x\cdot dx$

(c) $\int (2x^6 e^{\underbrace{x^7+1}_{=u}})\,dx \quad = 2\int e^{x^7+1} \cdot \underbrace{x^6 \cdot dx}_{=\frac{1}{7}du}$

Let $u = x^7 + 1$

$\Rightarrow \frac{du}{dx} = 7x^6$

$\Rightarrow du = 7x^6 \cdot dx$

$\Rightarrow \frac{1}{7} du = x^6 \cdot dx$

$$= 2\int e^u \cdot \frac{1}{7} du$$

$$= \frac{2}{7}\int e^u \, du$$

$$= \frac{2}{7} e^u + C$$

$$= \frac{2}{7} e^{x^7+1} + C$$

(d) $\int \underbrace{2x^2 \sqrt[4]{x^3+2}\,dx}_{=\frac{1}{3}du}$, $x^3+2 = u$

$$= 2\int \sqrt[4]{u} \cdot \frac{1}{3} du$$

Let $u = x^3 + 2$

$\Rightarrow \frac{du}{dx} = 3x^2$

$\Rightarrow du = 3x^2 \cdot dx$

$\Rightarrow \frac{1}{3} du = x^2 \cdot dx$

$$= \frac{2}{3}\int u^{\frac{1}{4}} du$$

$$= \frac{2}{3} \cdot \frac{1}{\frac{1}{4}+1} u^{\frac{1}{4}+1} + C$$

$$= \frac{2}{3} \cdot \frac{1}{\frac{5}{4}} u^{\frac{5}{4}} + C$$

$$= \frac{2}{3} \cdot \frac{4}{5} \cdot u^{\frac{5}{4}} + C$$

$$= \frac{8}{15} u^{\frac{5}{4}} + C = \frac{8}{15}(x^3+2)^{\frac{5}{4}} + C$$