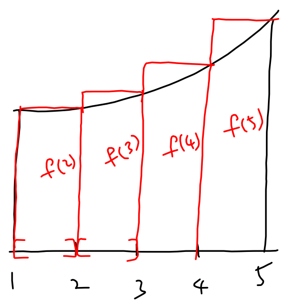
Ex.6) Estimate the area under the graph of $f(x) = x - 2\ln x$ on $[1, 5]$

$n=4$

(a) using four approximating rectangles of equal width and right endpoints.



base: $\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$

$\Rightarrow$ Approximating area

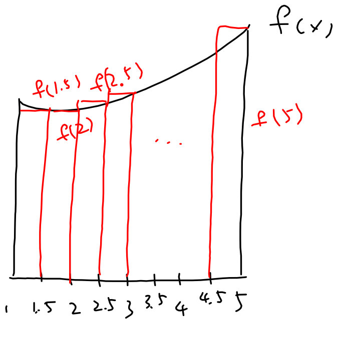
$= 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) + 1 \cdot f(5)$

$= (2 - 2\ln(2)) + (3 - 2\ln(3)) + (4 - 2\ln(4)) + (5 - 2\ln(5))$

≈ 4.425

$n=8$

(b) using eight approximating rectangles of equal width and right endpoints.



base: $\Delta x = \frac{b-a}{n} = \frac{5-1}{8} = \frac{1}{2}$

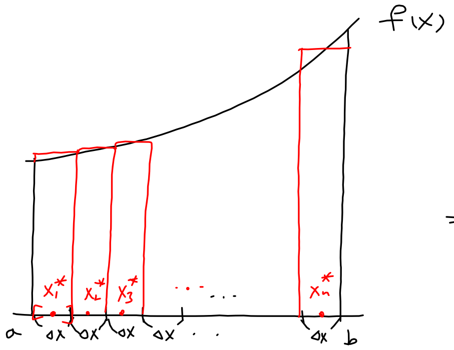
$\rightarrow$ Approximating area

$= \frac{1}{2} \cdot f(1.5) + \frac{1}{2} \cdot f(2) + \frac{1}{2} \cdot f(2.5) + \dots + \frac{1}{2} f(5)$

$= \frac{1}{2}\left[f(1.5) + f(2) + f(2.5) + \dots + f(5)\right]$

≈ 4.1329

Definition. The **area** of the region that lies under the graph of the continuous and positive function f is the limit of the sum of the areas of approximating rectangles.



$\Rightarrow$ Area

$$\approx \Delta x \cdot f(x_1^*) + \Delta x \cdot f(x_2^*) + \Delta x \cdot f(x_3^*) + \dots + \Delta x \cdot f(x_n^*)$$

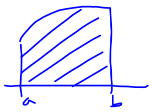
base : $\Delta x = \frac{b-a}{n}$

$$= \sum_{i=1}^{n} \Delta x \cdot f(x_i^*) \quad : \text{Riemann Sum}$$

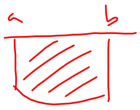
Then

$$\lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i^*) = \int_a^b f(x)\, dx$$

Section 6.4 The Definite Integral



$\int_a^b f(x)\,dx = \text{area}$



$-\int_a^b f(x)\,dx = \text{area}$

Definition. The Definite Integral

If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 = a, x_n = b$, and $x_1^*, x_2^*, \cdots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then **the definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

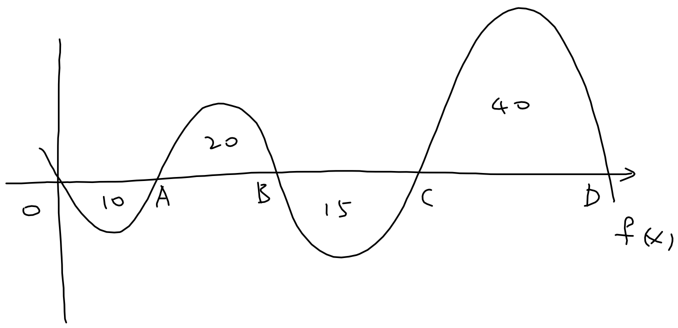
provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

We refer to a as the **lower limit** of integration and to b as the **upper limit** of integration and the interval $[a, b]$ as the **interval of integration**.

In the event $f(x)$ is positive on the interval $[a, b]$, then the definite integral is the same as the area bounded by $f(x)$, the x-axis, $x = a$ and $x = b$. If $f(x)$ is not always positive on the interval $[a, b]$, then the definite integral is the net area.

The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called **Riemann sum.**

Ex.1) Compute the following definite integrals using the graph given below.



(a) $\int_0^A f(x)\, dx = -10$ (b) $\int_0^C f(x)\, dx = -10 + 20 - 15 = -5$ (c) $\int_A^B f(x)\, dx = 20$

(d) $\int_0^D f(x)\, dx$ (e) $\int_C^C f(x)\, dx = 0$

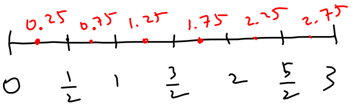
$= -10 + 20 - 15 + 40$

$= 35$

$f(x) = 2x^2 - x - 2 \ , \ [0, 3]$

$n = 6$

Ex.2) Approximate $\int_0^3 (2x^2 - x - 2)\, dx$ by using the Riemann sum with 6 equal subintervals, taking the sample points to be the midpoints of each subinterval.



base : $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$

$$\Rightarrow \int_0^3 (2x^2 - x - 2)\, dx$$

$$\approx \frac{1}{2} \cdot f(0.25) + \frac{1}{2} f(0.75) + \frac{1}{2} \cdot f(1.25)$$

$$+ \frac{1}{2} \cdot f(1.75) + \frac{1}{2} \cdot f(2.25) + \frac{1}{2} \cdot f(2.75)$$

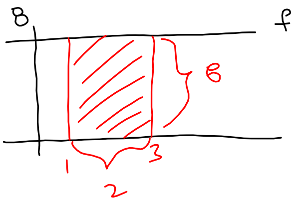
$$= 7.375$$

: estimated area under $f(x)$, $[0, 3]$

Ex.3) Use Geometry to evaluate the following integrals.

$f(x) = 8$

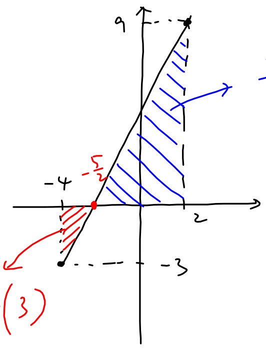
(a) $\displaystyle\int_1^3 8\,dx$



$$\Rightarrow \int_1^3 8\,dx = 2 \cdot 8 = 16$$

$f(x) = 2x+5$

(b) $\displaystyle\int_{-4}^2 (2x+5)\,dx$



$\frac{1}{2} \cdot \left(\frac{9}{2}\right)(9)$

$= \frac{81}{4}$

$\frac{1}{2} \cdot \left(\frac{3}{2}\right) \cdot (3)$

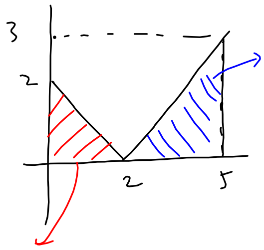
$= \frac{9}{4}$

$$\Rightarrow \int_{-4}^{2} (2x+5)\,dx = -\frac{9}{4} + \frac{81}{4} = \frac{72}{4}$$

$$= 18$$

$f(x) = |x-2|$ vertex: $(2, 0)$

(c) $\displaystyle\int_0^5 |x-2|\, dx$



$\frac{1}{2}\cdot(3)\cdot(3)$
$= \frac{9}{2}$

$\frac{1}{2}\cdot(2)\cdot(2)$
$= 2$

$$\Rightarrow \int_0^5 |x-2|\, dx = 2 + \frac{9}{2} = \frac{13}{2}$$

Order Properties of Definite Integrals

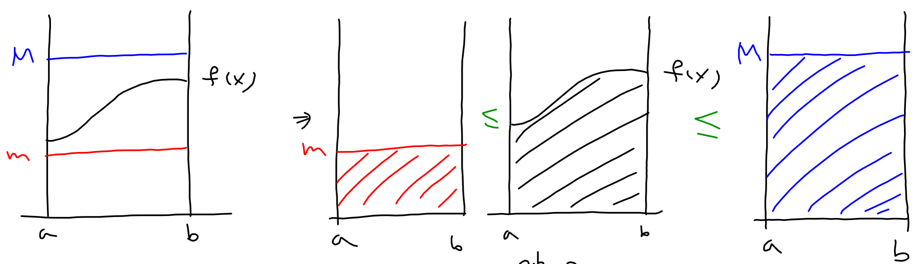
Assume that all the integrals given below exist and that $a \leq b$. Then

(a) If $f(x) \geq 0$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$.

(b) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

(c) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$



$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

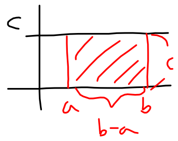
~~Ex.4) Find bounds of the function $y = e^{-x^2}$ to estimate $\int_0^1 e^{-x^2}\,dx$~~

Section 6.5 The Fundamental Theorem of Calculus

Properties of Definite Integrals

If f and g are continuous functions on $[a, b]$,

(a) $\displaystyle\int_a^b c\,dx = c(b-a)$



$\Rightarrow c(b-a)$

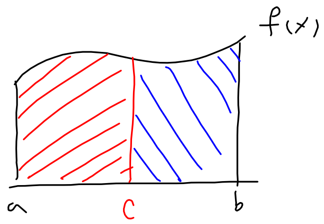
(b) $\displaystyle\int_a^a f(x)\,dx = 0$

(c) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(d) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(e) $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(f) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Ex.5) Given $\int_1^4 x\ dx = 7.5$, $\int_1^4 x^2\ dx = 21$, and $\int_4^5 x^2\ dx = \frac{61}{3}$, calculate the following

(a) $\int_1^4 (4x^2 - 9x)\ dx$ $= \int_1^4 4x^2 dx + \int_1^4 (-9x) dx$

$= 4\int_1^4 x^2 dx - 9\int_1^4 x\ dx$

$= 4\cdot(21) - 9\cdot(7.5) = 16.5$

(b) $\int_1^5 (-4x^2)\ dx$ $= -4\int_1^5 x^2 dx$

$= -4\left[\int_1^4 x^2 dx + \int_4^5 x^2 dx\right]$

$= -4\left[21 + \frac{61}{3}\right] = -\frac{496}{3}$

Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is an antiderivative of f, that is $F' = f$.

$\int_a^b f(x)\,dx = F(x)\Big|_a^b$

$= F(b) - F(a)$

Ex.6) Evaluate the following $f(x) = -x^2 + 4$

(a) $\int_{-2}^{3} (-x^2 + 4)\,dx = -\frac{1}{3}x^3 + 4x \Big|_{-2}^{3}$

$= \left(-\frac{1}{3}(3)^3 + 4(3)\right) - \left(-\frac{1}{3}(-2)^3 + 4(-2)\right)$

$= (-9 + 12) - \left(\frac{8}{3} - 8\right) = \frac{25}{3}$

(b) $\int_1^5 (e^x + x)\,dx = e^x + \frac{1}{2}x^2 \Big|_1^5$

$= \left(e^{(5)} + \frac{1}{2}(5)^2\right) - \left(e^{(1)} + \frac{1}{2}(1)^2\right)$

$= e^5 + \frac{25}{2} - e - \frac{1}{2}$

$= e^5 - e + \frac{24}{2} = e^5 - e + 12$

(c) $\int_0^1 (\underbrace{2x+3}_{=u})^6 \, \overset{=\frac{1}{2}du}{dx} = \int_3^5 u^6 \cdot \frac{1}{2} du = \frac{1}{2} \int_3^5 u^6 \, du$

Let $u = 2x + 3$

$\Rightarrow du = 2dx$

$\Rightarrow \frac{1}{2} du = dx$

$\begin{cases} x = 0 \Rightarrow u = 2(0) + 3 = 3 \\ x = 1 \Rightarrow u = 2(1) + 3 = 5 \end{cases}$

$= \frac{1}{2} \cdot \frac{1}{7} u^7 \Big|_3^5$

$= \frac{1}{14} \left[(5)^7 - (3)^7 \right]$

$= \frac{37969}{7}$

Estimating Definite Integrals on the Calculator. You can estimate the value of the definite integral $\int_a^b f(x)\,dx$ by using the following command from your homescreen.

$$< \text{MATH} > +9 : \text{fnInt}$$

$$\text{fnInt}(f(x), x, a, b)$$

Ex.7) A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

what we stated with

: total change in the bee population over the first 15 weeks period.

$= 100 + n(t)\Big|_0^{15}$

$= 100 + n(15) - \underbrace{n(0)}_{=100} = 100 - n(15) - 100 = n(15)$: represents the total # of bees at the end of 15 weeks period.

Ex.8) A forest fire covers 2000 acres at time $t = 0$. The fire is growing at a rate of $8\sqrt{t}$ acres per hour, where t is in hours. How many acres are covered 24 hours later?