Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

where F is an antiderivative of f, that is $F' = f$.

Ex.6) Evaluate the following

(a) $\displaystyle\int_{-2}^{3} (-x^2 + 4)\, dx$

(b) $\displaystyle\int_{1}^{5} (e^x + x)\, dx$

(c) $\int_0^1 (2x+3)^6 \; dx$

Estimating Definite Integrals on the Calculator. You can estimate the value of the definite integral $\int_a^b f(x)\,dx$ by using the following command from your homescreen.

$$< \text{MATH} > +9 : \text{fnInt}$$

$$\text{fnInt}(f(x), x, a, b)$$

Ex.7) A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

Ex.8) A forest fire covers 2000 acres at time $t = 0$. The fire is growing at a rate of $8\sqrt{t}$ acres per hour, where t is in hours. How many acres are covered 24 hours later?

$$= 2000 + \int_0^{24} 8\sqrt{t}\,dt$$

$$= 2000 + \text{fnInt}(\sim\sim)$$

$$\approx 2627.07$$

or

$$2000 + 8\int_0^{24} t^{\frac{1}{2}}\,dt$$

$$= 2000 + 8 \cdot \frac{1}{\frac{1}{2}+1} t^{\frac{1}{2}+1} \Big|_0^{24}$$

$$= 2000 + 8 \cdot \frac{1}{\frac{3}{2}} t^{\frac{3}{2}} \Big|_0^{24}$$

$$= 2000 + 8 \cdot \frac{2}{3} t^{\frac{3}{2}} \Big|_0^{24}$$

$$= 2000 + \frac{16}{3} \cdot \left[(24)^{\frac{3}{2}} - (0)^{\frac{3}{2}} \right]$$

$$\approx 2627.07$$

Ex.9) An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the second hour?

60 min → 120 min

$$= \int_{60}^{120} r(t)\,dt = \int_{60}^{120} 100 \cdot e^{-0.01t}\,dt$$

$$\approx 2476.02 \text{ liters}$$

Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on (a, b) and

$$g'(x) = f(x)$$

or $\left(\int_a^x f(t)\, dt \right)' = f(x)$

$F' = f$

proof) Use the F.T.C, II.

$$\left(\int_a^x f(t)\, dt \right)' = \left(F(t) \Big|_a^x \right)'$$

$$= \left(F(x) - F(a) \right)'$$

$$= F'(x) - F(a)'$$

$$= f(x) - 0$$

$$= f(x)$$

Ex.10) Find the derivative of the following functions.

(a) $g(x) = \int_1^x t^3\, dt$

$g'(x) = x^3$

(b) $h(x) = \int_3^x e^{t^2-t}\, dt$

$h'(x) = e^{x^2-x}$

Ex.11) If $f(6) = 11$, f' is continuous, and $\int_6^7 f'(x)\,dx \overset{=19}{}$, what is the value of $f(7)$?

$$\int_6^7 f'(x)\,dx = 19$$

F.T.C II

$$\Rightarrow f(x)\Big|_6^7 = 19$$

$$\Rightarrow f(7) - f(6) = 19$$

$$\Rightarrow f(7) - 11 = 19$$

$$\therefore f(7) = 30$$

Definition. Average Value of $f(x)$ over $[a,b]$.

If $f(x)$ is continuous on $[a,b]$, we define the **average value of $f(x)$ on $[a,b]$** to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex.12) Find the average value of the given functions on the given intervals.

(a) $f(x) = 5$ on $[0, 2]$

$$\text{ave.} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-0}\int_0^2 5\,dx$$

$$= \frac{1}{2}\left[10\right] = 5$$

(b) $f(x) = -2x^3 + x$ on $[-2, 1]$

$$\text{ave.} = \frac{1}{1-(-2)}\int_{-2}^1 (-2x^3 + x)\,dx$$

$$= \frac{1}{3}\int_{-2}^1 (-2x^3 + x)\,dx$$

$$= 2$$

Ex.13) Given the supply function $p = 10(e^{0.02x} - 1)$, find the average price (in \$) over the interval $[20, 30]$.

$$\text{ave.} = \frac{1}{b-a}\int_a^b p(x)\,dx$$

$$= \frac{1}{30-20}\int_{20}^{30} 10\cdot\left(e^{0.02x} - 1\right)dx$$

$$\approx \$6.51$$