MATH 142 Business Calculus

Fall 2019, WEEK 12

JoungDong Kim

Week 12 Section 6.4 6.5 The Definite Integral, The Fundamental Theorem of Calculus

Section 6.4 The Definite Integral

Definition. The Definite Integral

If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0 = a, x_n = b$, and $x_1^*, x_2^*, \cdots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then **the definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

We refer to a as the **lower limit** of integration and to b as the **upper limit** of integration and the interval $[a, b]$ as the **interval of integration**.

In the event $f(x)$ is positive on the interval $[a, b]$, then the definite integral is the same as the area bounded by $f(x)$, the x-axis, $x = a$ and $x = b$. If $f(x)$ is not always positive on the interval $[a, b]$, then the definite integral is the net area.

The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called **Riemann sum.**

Ex.1) Compute the following definite integrals using the graph given below.

(a) $\int_0^A f(x)\ dx$ (b) $\int_0^C f(x)\ dx$ (c) $\int_A^B f(x)\ dx$

(d) $\int_0^D f(x)\ dx$ (e) $\int_C^C f(x)\ dx$

Ex.2) Approximate $\int_0^3 (2x^2 - x - 2)\ dx$ by using the Riemann sum with 6 equal subintervals, taking the sample points to be the midpoints of each subinterval.

Ex.3) Use Geometry to evaluate the following integrals.

(a) $\int_{1}^{3} 8 \, dx$

(b) $\int_{-4}^{2} (2x + 5) \, dx$

(c) $\int_{0}^{5} |x - 2| \, dx$

Order Properties of Definite Integrals

Assume that all the integrals given below exist and that $a \leq b$. Then

(a) If $f(x) \geq 0$, then $\int_a^b f(x)\, dx \geq 0$.

(b) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$.

(c) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$$

Ex.4) Find bounds of the function $y = e^{-x^2}$ to estimate $\int_0^1 e^{-x^2}\, dx$

Section 6.5 The Fundamental Theorem of Calculus

Properties of Definite Integrals

If f and g are continuous functions on $[a, b]$,

(a) $\int_a^b c\ dx = c(b-a)$

(b) $\int_a^a f(x)\ dx = 0$

(c) $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

(d) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

(e) $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

(f) $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

Ex.5) Given $\int_1^4 x\ dx = 7.5$, $\int_1^4 x^2\ dx = 21$, and $\int_4^5 x^2\ dx = \frac{61}{3}$, calculate the following

(a) $\int_1^4 (4x^2 - 9x)\ dx$

(b) $\int_1^5 (-4x^2)\ dx$

Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a),$$

where F is an antiderivative of f, that is $F' = f$.

Ex.6) Evaluate the following

(a) $\displaystyle\int_{-2}^{3} (-x^2 + 4)\ dx$

(b) $\displaystyle\int_{1}^{5} (e^x + x)\ dx$

(c) $\displaystyle\int_{0}^{1} (2x + 3)^6\ dx$

Estimating Definite Integrals on the Calculator. You can estimate the value of the definite integral $\int_a^b f(x)\,dx$ by using the following command from your homescreen.

$$< \text{MATH} > +9 : \text{fnInt}$$

$$\text{fnInt}(f(x), x, a, b)$$

Ex.7) A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

Ex.8) A forest fire covers 2000 acres at time $t = 0$. The fire is growing at a rate of $8\sqrt{t}$ acres per hour, where t is in hours. How many acres are covered 24 hours later?

Ex.9) An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the second hour?

Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\ dt$$

is continuous on $[a, b]$ and differentiable on (a, b) and

$$g'(x) = f(x)$$

Ex.10) Find the derivative of the following functions.

(a) $g(x) = \displaystyle\int_1^x t^3\ dt$

(b) $h(x) = \displaystyle\int_3^x e^{t^2-t}\ dt$

Ex.11) If $f(6) = 11$, f' is continuous, and $\int_6^7 f'(x)\,dx = 19$, what is the value of $f(7)$?

Definition. Average Value of $f(x)$ over $[a,b]$.

If $f(x)$ is continuous on $[a,b]$, we define the **average value of $f(x)$ on $[a,b]$** to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex.12) Find the average value of the given functions on the given intervals.

(a) $f(x) = 5$ on $[0, 2]$

(b) $f(x) = -2x^3 + x$ on $[-2, 1]$

Ex.13) Given the supply function $p = 10(e^{0.02x} - 1)$, find the average price (in \$) over the interval $[20, 30]$.