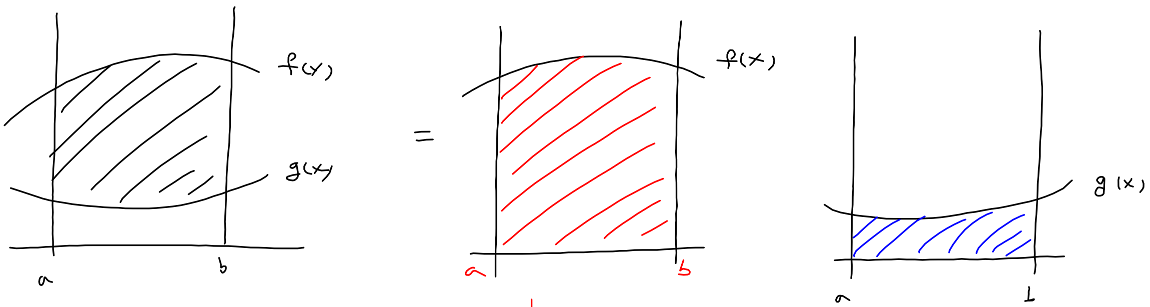
Section 6.6 Area Between Two Curves

Area Between Two Curves

Let $y = f(x)$ and $y = g(x)$ be two continuous functions with $f(x) \geq g(x)$ on $[a, b]$. Then the area between the graphs of the two curves on $[a, b]$ is given by the definite integral

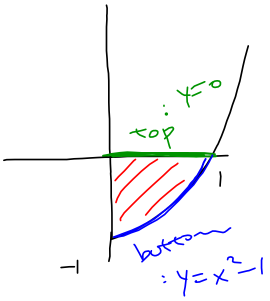
$$\int_a^b [f(x) - g(x)]\, dx$$



$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [\text{top} - \text{bottom}]\,dx$$

Ex.1) Find the area between $y = x^2 - 1$ and the x-axis on $[0, 1]$.



$$\text{Area} = -\int_0^1 (x^2-1)\,dx$$

$$= -\left(-\frac{2}{3}\right) = \frac{2}{3}$$

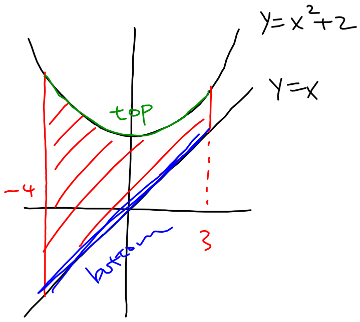
or

$$\text{Area} = \int_0^1 [\text{top} - \text{bottom}]\,dx$$

$$= \int_0^1 [\,0 - (x^2-1)\,]\,dx$$

$$= \frac{2}{3}$$

Ex.2) Find the area between $y = x$ and $y = x^2 + 2$ on $[-4, 3]$.

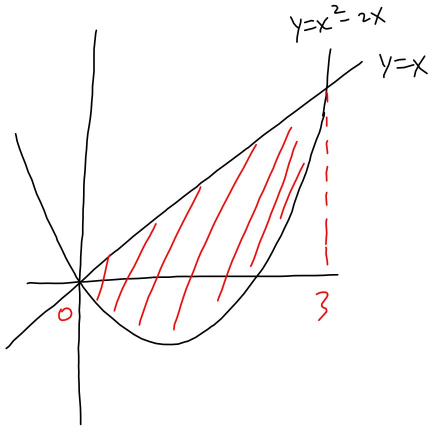


$$\text{Area} = \int_{-4}^{3} [\text{top} - \text{bottom}]\,dx$$

$$= \int_{-4}^{3} [\,(x^2+2) - x\,]\,dx$$

$$= 47.83$$

Ex.3) Find the area bounded by the curves $y = x^2 - 2x$ and $y = x$.



$= x(x-2)$

Intersection: $x^2 - 2x = x$

$\Rightarrow x^2 - 3x = 0$

$\Rightarrow x(x-3) = 0$

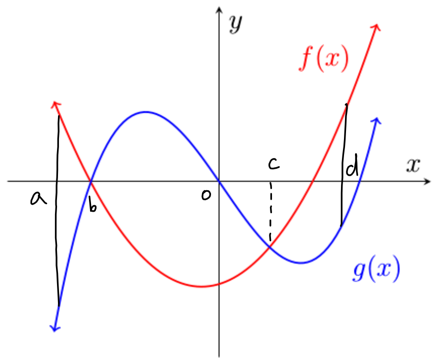
$\therefore x = 0,\ 3$

Or use calculator

$$\text{Area} = \int_0^3 \left[x - (x^2 - 2x) \right] dx$$

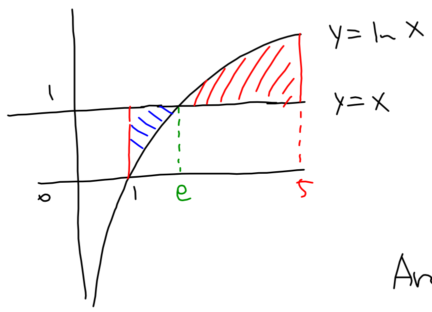
$$= 4.5$$

Ex.4) Given the graph below write definite integrals to represent the total area bounded by $f(x)$ and $g(x)$ on $[a, d]$.



$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - f(x)]\,dx + \int_c^d [f(x) - g(x)]\,dx$$

Ex.5) Find the area bounded by $y = \ln x$ and $y = 1$ on $[1, 5]$.



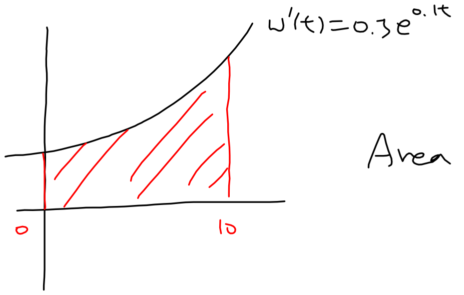
Intersection: $\ln x = 1$

$\therefore x = e$

$$\text{Area} = \int_1^e [1 - \ln x]\,dx + \int_e^5 [\ln x - 1]\,dx$$

$$= 1.484$$

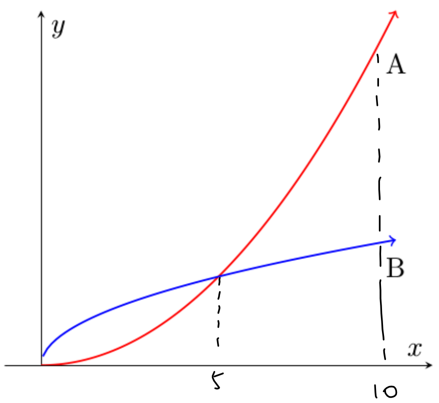
Ex.6) A yeast culture is growing at a rate of $w'(t) = 0.3e^{0.1t}$ grams per hour. Find the area between the graph of $w'(t)$ and t-axis on the interval $[0, 10]$ and interpret the results.



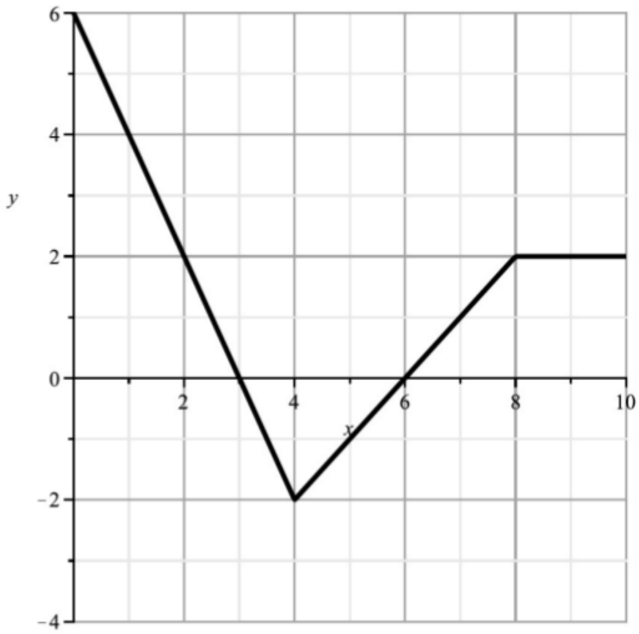
$$\text{Area} = \int_0^{10} 0.3e^{0.1t}\,dt$$

$$\approx 5.1548 \text{ grams}$$

Ex.7) The figure below shows the rate of growth of two trees. If the two trees are the same height at time $t = 0$, which tree is taller after 5 years? After 10 years?



Ex.8) If $g(x) = \int_0^x f(t)dt$, where the graph of $f(t)$ is given below, where $0 \leq x \leq 10$, evaluate $g(0)$, $g(3)$, $g(6)$ and $g(10)$.

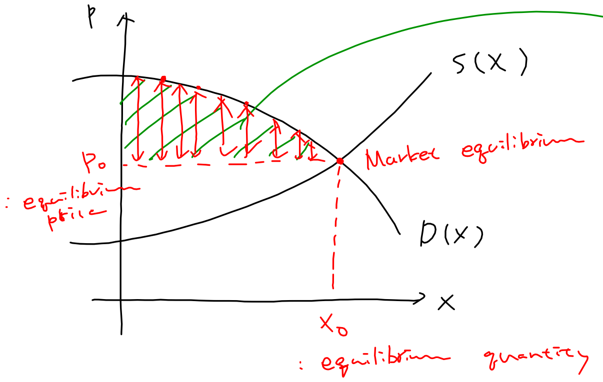


Section 6.7 Additional Applications of the Integral

Consumers' Surplus

Definition. If $p = D(x)$ is the demand equation, p_0 is the equilibrium price of the commodity, and x_0 is the equilibrium demand, then the **comsumers' surplus** is given by

$$\int_0^{x_0} [D(x) - p_0]\, dx$$



Consumers' Surplus

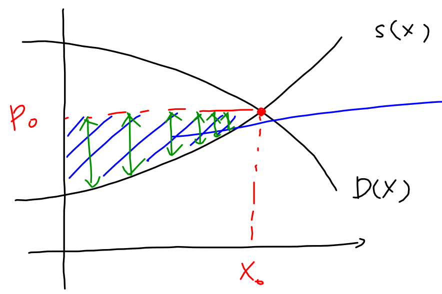
$$= \int_0^{x_0} [D(x) - P_0]\, dx$$

The **consumers' surplus** represents the total savings to consumers who are willing to pay more than p_0 for the product but are still able to buy the product for p_0.

Producers' Surplus

Definition. If $p = S(x)$ is the supply equation, p_0 is the equilibrium price of the commodity, and x_0 is the equilibrium demand, then the **producers' surplus** is given by

$$\int_0^{x_0} [p_0 - S(x)]\, dx$$

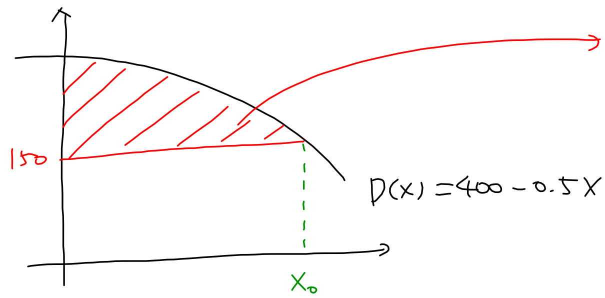


Producers' Surplus

$$= \int_0^{x_0} [P_0 - S(x)]\, dx$$

The **producers' surplus** represents the total gain to producers who are willing to supply units at a lower price than p_0 but are still able to supply units at p_0.

Ex.9) Find the consumers' surplus at a price level of \$150 for the price-demand equation $p = 400 - 0.5x$.



Consumers' surplus

$$= \int_0^{500} \left[400 - 0.5x - 150 \right] dx$$

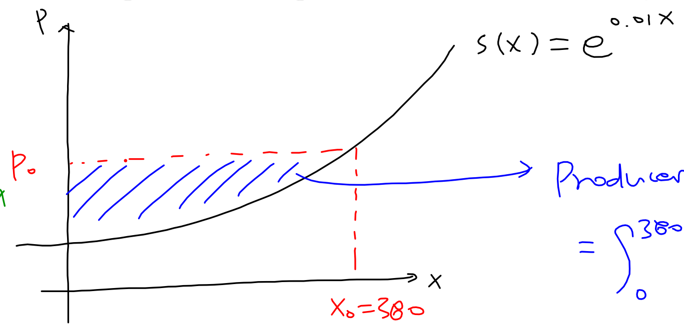
$$\approx \$62,500$$

$\Rightarrow 400 - 0.5x = 150$

$\Rightarrow -0.5x = -250$

$\therefore x_0 = 500$

Ex.10) Find the producers' surplus when 380 items are sold, if the supply equation is given by $p = e^{0.01x}$.



Producers' Surplus

$$= \int_0^{380} \left[44.70 - e^{0.01x} \right] dx$$

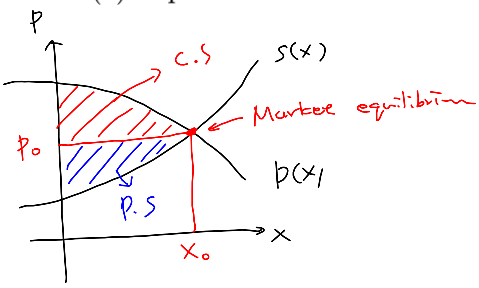
$$\approx \$12,616$$

$$P_0 = e^{0.01(380)}$$

$$\approx 44.70$$

Ex.11) If $p = D(x) = 80e^{-0.001x}$ and $p = S(x) = 30e^{0.001x}$, find the follwing:

(a) Equilibrium Point



Intersection : $(D(x) = S(x))$

$$80 \cdot e^{-0.001x} = 30 \cdot e^{0.001x}$$

$x_0 = 490$, $P_0 = 48.99$

(b) Consumers' surplus at the equilibrium price level.

$$C.S = \int_0^{490} \left[80 e^{-0.001x} - 48.99 \right] dx$$

$$\approx \$6,984$$

(c) Producers' surplus at the equilibrium price level.

$$P.S = \int_0^{490} \left[48.99 - 30 \cdot e^{0.001x} \right] dx$$

$$\approx \$5,035$$