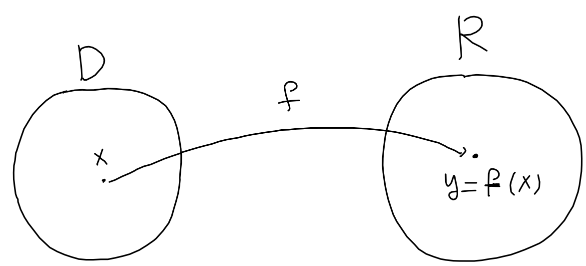
http://www.math.tamu.edu/~jdkim/math142summer2015

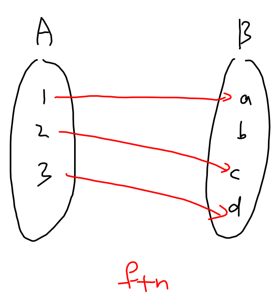
Chapter 1. Functions

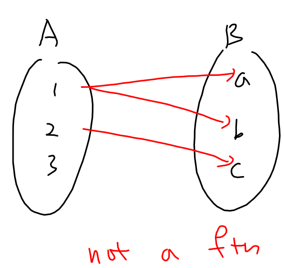
Section 1.1 and A.8

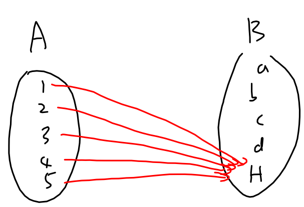
Definition. A **function** is a rule that assigns to each element x in the domain exactly one element in the range.

The **domain** is the set of all values of the *independent variable* (typically x) that produce real values for the *dependent variable* (typically y). The **range** is the set of all possible y values as x varies throughout the domain. If the correspondence is a function we use the notation $f(x)$ read "f of x". Ordered pair of the form (x, y) can be written as $(x, f(x))$.









Ex1) Find the domain of the following functions.

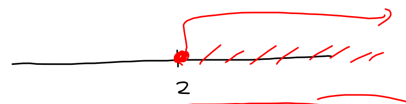
1. $f(x) = \sqrt{2x - 4}$ ≥ 0

$2x - 4 \geq 0$

$+4 \quad +4$

$\frac{2x}{2} \geq \frac{4}{2}$

$x \geq 2$



$\boxed{[2, \infty)}$

2. $g(x) = \dfrac{x-1}{x^2 - 5x + 6}$ $\neq 0$

$x^2 - 5x + 6 = 0$

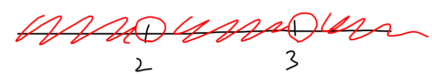
A = -5 : -2, -3
M = 6

$\Rightarrow (x-2)(x-3) = 0$

$x = 2 \qquad x = 3$

$\therefore$ Domain : all real number except $x = 2$, $x = 3$

or $\mathbb{R}$, $x \neq 2, 3$



$\boxed{(-\infty, 2) \cup (2, 3) \cup (3, \infty)}$

3. $h(x) = \dfrac{\sqrt{5x - 6}}{\sqrt[3]{3x - 5}}$

$3x - 5 \neq 0$

$+5 \quad +5$

$\frac{3x}{3} \neq \frac{5}{3}$

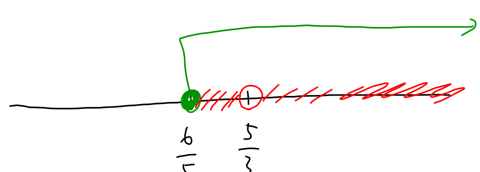
$x \neq \frac{5}{3}$

$5x - 6 \geq 0$

$+6 \quad +6$

$\frac{5x}{5} \geq \frac{6}{5}$

$x \geq \frac{6}{5}$



$\boxed{\left[\frac{6}{5}, \frac{5}{3}\right) \cup \left(\frac{5}{3}, \infty\right)}$

In Sum,

The domain of most algebraic functions is the set of all real numbers EXCEPT:

1. Any real numbers that cause the denominator of a function to be 0.
2. Any real numbers that cause us to take the square root (or any radical with an even index) of a negative number.

and more...

ex) $f(x) = \sqrt[7]{3x^2+4x-7}$: domain : all real #.

Definition. Functions with domains divided into two or more parts with a different rule applied to each part are called **piecewise-defined functions**.

The absolute value function, $|x|$, is such an example.

ex) $|4| = 4$

$|-2| = 2$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$\Uparrow$

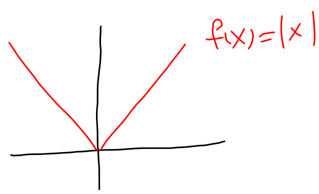
$|-2| = -(-2) = 2$

$$\Rightarrow |x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

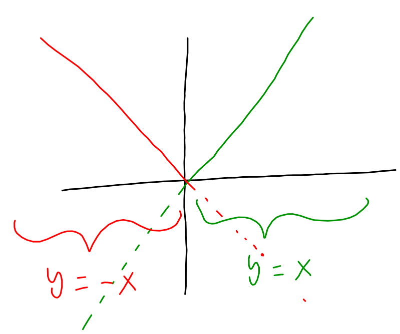
Graph of a functions

The **graph** of a function f consists of all points (x, y) such that x is in the domain of f and $y = f(x)$.

Ex2) Graph the function $f(x) = |x|$.



$$f(x) = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

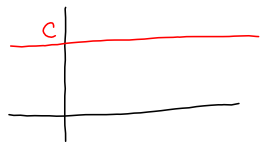


Catalog of Basic Functions

ex) $y=2$

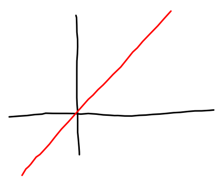
1. Constant function

$f(x) = c, \quad c:$ real constant

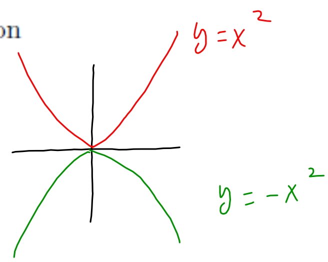


2. Identity function

$f(x) = x$



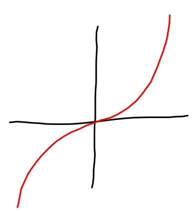
3. Square function



$f(x) = x^2$ domain: $\mathbb{R}$

range: $[0,\infty)$

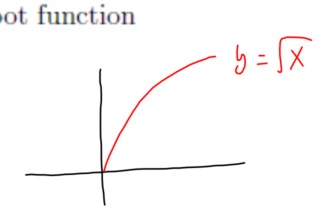
4. Cubing function



$f(x) = x^3$ domain: $\mathbb{R}$ or $(-\infty,\infty)$

range: $\mathbb{R}$ or $(-\infty,\infty)$

5. Square root function



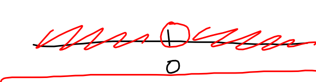
$f(x) = \sqrt{x}$ domain $x \geq 0$ or $[0,\infty)$

range $y \geq 0$ or $[0,\infty)$

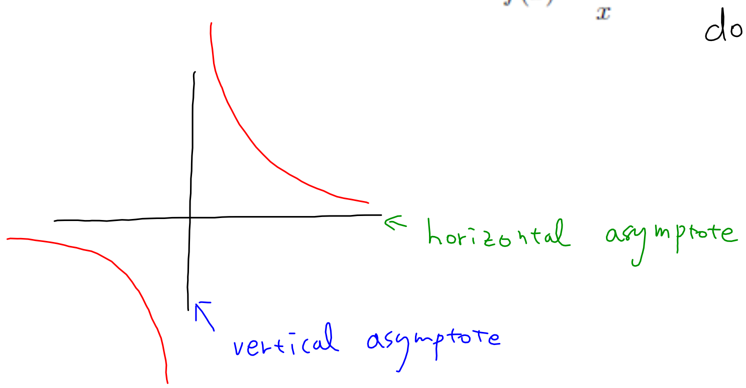
6. Reciprocal function

$f(x) = \dfrac{1}{x}$

domain: $\mathbb{R}$ except $x=0$



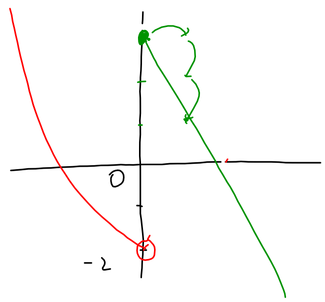
$\boxed{(-\infty,0) \cup (0,\infty)}$



Ex3) Graph the following piecewise defined function:

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x < 0 \\ -2x + 3 & \text{if} x \geq 0 \end{cases}$$

slope $= -2 = \frac{-2}{1}$



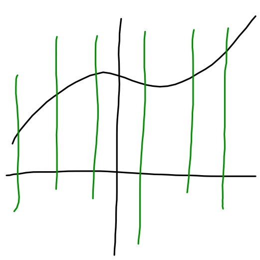
Note

$y = mx + b$

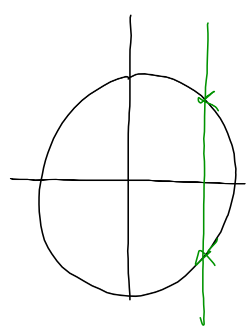
Vertical line test

A graph in the xy-plane represents a function of x, if and only if, every vertical line intersects the graph in at most one place.

ex)



fcn

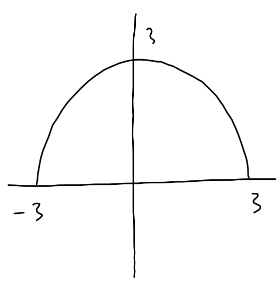


not fcn.

Finding domain and range from graphs

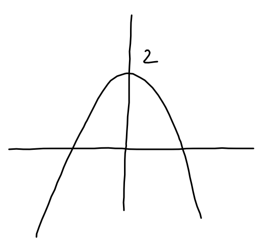
- To find the domain of f, vertical line scanning the x-axis.
- To find the range of f, horizontal line scanning the y-axis.

ex)



Domain : $[-3, 3]$

range : $[0, 3]$

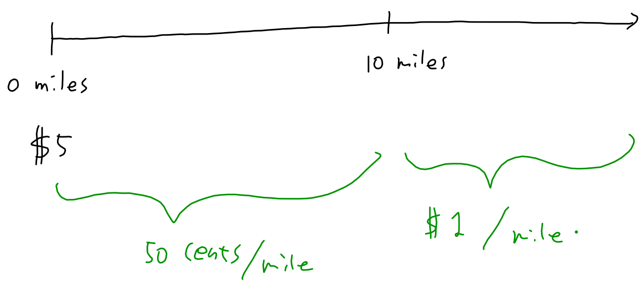


Domain : $\mathbb{R}$ or $(-\infty, \infty)$

range : $(-\infty, 2]$

Ex4) A taxi-cab company in a certain town charges all customers a base fee of \$5 per ride. They then charge an additional 50 cents per mile for the first 10 miles traveled and \$1/miles. Write a piecewise function, $C(x)$, for the cost of a cab ride if x represents the number of miles traveled.

Let X : mile. $C(X)$: total cost

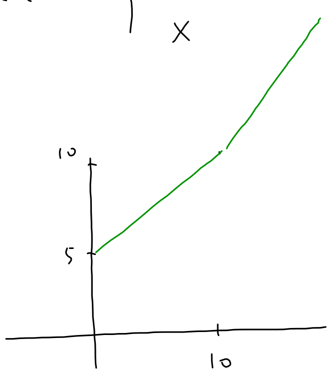


$\Rightarrow$ ~~total~~ cost

$$C(x) = \begin{cases} 5 + 0.5 \cdot X & \text{if } x < 10 \\ 5 + 0.5(10) + 1 \cdot (X - 10) & \text{if } x \geq 10 \end{cases}$$

$$= 5 + 5 + X - 10 = X$$

$$\therefore C(X) = \begin{cases} 5 + 0.5X & \text{if } X < 10 \\ X & \text{if } X \geq 10 \end{cases}$$



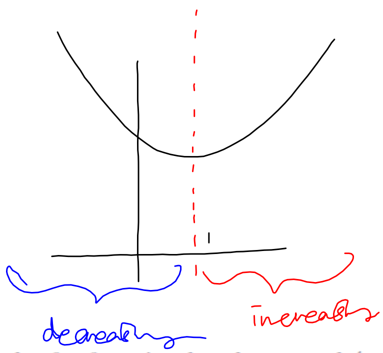
Increasing, Decreasing, Concavity, and Continuity

- A function $y = f(x)$ is said to be **increasing** (denoted by $\nearrow$) on the interval I if the graph of the function rises while moving left to right or, equivalently, if

$$f(x_1) < f(x_2) \quad \text{when} \quad x_1 < x_2 \quad \text{on } I$$

- A function $y = f(x)$ is said to be **decreasing** (denoted by $\searrow$) on the interval I if the graph of the function falls while moving left to right or, equivalently, if

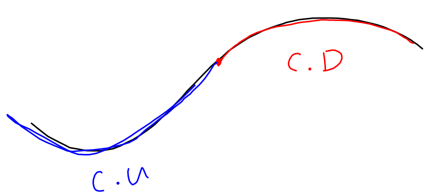
$$f(x_1) > f(x_2) \quad \text{when} \quad x_1 < x_2 \quad \text{on } I$$



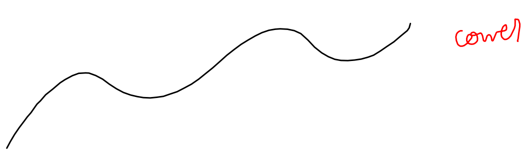
increasing : $(1, \infty)$

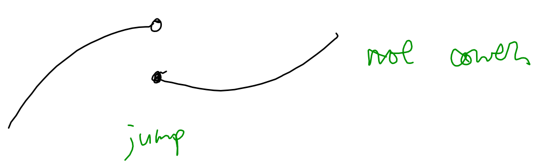
decreasing : $(-\infty, 1)$

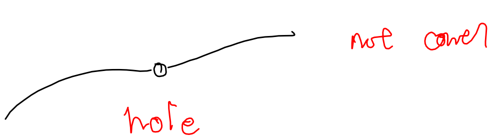
- If the graph of a function bends upward ($\smile$), we say that the function is **concave up**.
- If the graph of a function bends downward ($\frown$), we say that the function is **concave down**.
- A straight line is neither concave up nor concave down.



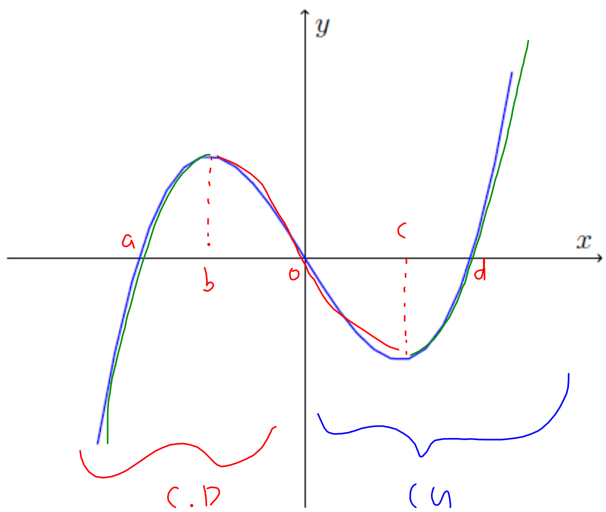
- A function is said to be **continuous** on an interval if the graph of the function does not have any breaks, gaps, or holes in that interval.







Ex5) Given the graph of $f(x)$ below, determine the interval(s) on which $f(x)$ is



1. increasing $(-\infty, b) \cup (c, \infty)$

2. decreasing (b, c)

3. cancave up $(0, \infty)$

4. cancave down $(-\infty, 0)$

5. continuous $(-\infty, \infty)$

Ex6) Let $f(x) = \dfrac{1}{x-2}$, find and simplify $\dfrac{f(x+h)-f(x)}{h}$.

$$f(x) = \frac{1}{x-2}$$

$$f(x+h) = \frac{1}{(x+h)-2} = \frac{1}{x+h-2}$$

LCD : $(x+h-2)(x-2)$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{\left(\frac{1}{x+h-2} - \frac{1}{x-2}\right)}{h} \quad \frac{(x+h-2)(x-2)}{(x+h-2)(x-2)}$$

$$= \frac{(x-2)-(x+h-2)}{h(x+h-2)(x-2)}$$

$$= \frac{\cancel{x}\cancel{-2}\cancel{-x}-h\cancel{+2}}{h(x+h-2)(x-2)}$$

$$= \frac{-\cancel{h}}{\cancel{h}(x+h-2)(x-2)} = \boxed{\frac{-1}{(x+h-2)(x-2)}}$$

Special Functions

ex) $f(x) = 2x^3 - 4x^2 + x - 5$

Polynomial function

A **polynomial function** of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (a_n \neq 0)$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$, are constants and n is a nonnegative integer. The domain is all real numbers. The coefficient a_n is called the **leading coefficient**.

A polynomial function of degree 1 ($n = 1$), $f(x) = a_1 x + a_0$, ($a_1 \neq 0$) is a **linear function**.

A polynomial function of degree 2 ($n = 2$), $f(x) = a_2 x^2 + a_1 x + a_0$, ($a_2 \neq 0$) is a **quadratic function**.

Note.

- Domain of polynomial is all real numbers.
- Polynomials are continuous for all real numbers.
- Graph is smooth.

Power function

A **power function** is a function of the form

$$f(x) = kx^r$$

where k and r are any real numbers.

- $r = n$, where n is a positive integer.

ex) $x^1, x^2, x^3, \cdots$: polynomial

- $r = \frac{1}{n}$, where n is a positive integer.

ex) $x^{\frac{1}{2}} = \sqrt{x}, \quad x^{\frac{1}{3}} = \sqrt[3]{x}$

- $r = -1$

ex) $x^{-1} = \frac{1}{x}, \quad x^{-3} = \frac{1}{x^3}$

Note

$(\sqrt{2})^2 = 2$

$\because \left((2)^{\frac{1}{2}}\right)^2$

$= 2^{\frac{1}{2}\cdot 2} = 2^1 = 2$

$\sqrt[3]{8} = 2$

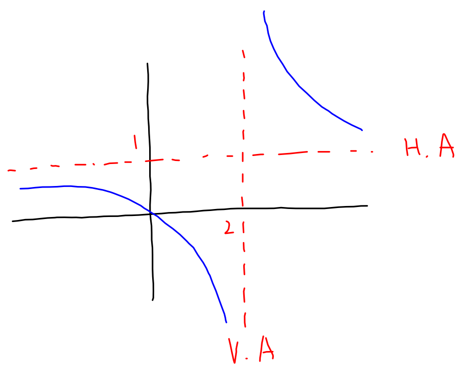
$\because (8)^{\frac{1}{3}} = (2^3)^{\frac{1}{3}}$

$= 2^{3\cdot\frac{1}{3}} = 2^1 = 2$

Rational function

A **rational function** $R(x)$ is the quotient of two polynomial functions and thus is of the form $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions. the domain of $R(x)$ is all real numbers for which $g(x) \neq 0$.

ex) $f(x) = \dfrac{x-3}{x-2}$ domain : $\mathbb{R}$ except $x=2$



Graphing Techniques

Transformations

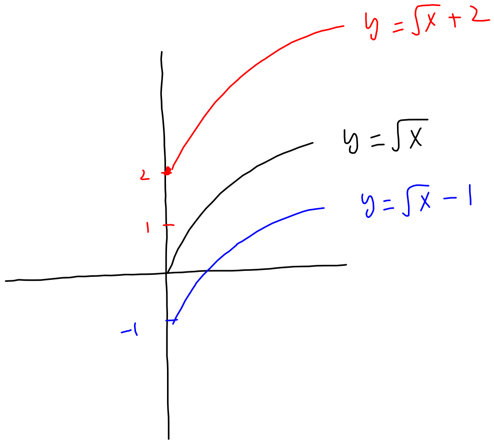
Vertical and Horizontal Shifts: Suppose $c > 0$. To obtain the graph of

- $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units **upward**. ex) $y = x^2 \xrightarrow{\uparrow 1 \text{ unit}} y = x^2 + 1$
- $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units **downward**. $y = x^2 \xrightarrow{\downarrow 1 \text{ unit}} y = x^2 - 1$
- $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units **to the right**. $y = x^2 \xrightarrow{\rightarrow 1 \text{ unit}} y = (x-1)^2$
- $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units **to the left**. $y = x^2 \xrightarrow{\leftarrow 1 \text{ unit}} y = (x+1)^2$

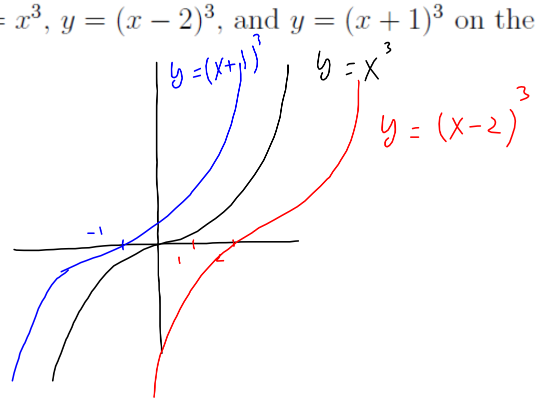
Expansion, Contraction and Reflecting: Suppose $c > 1$. To obtain the graph of

- $y = cf(x)$, **stretch** the graph of $y = f(x)$ **vertically** by a factor of c.
- $y = \frac{1}{c}f(x)$, **shrink** the graph of $y = f(x)$ **vertically** by a factor of c.
- $y = -f(x)$, **reflect** the graph of $y = f(x)$ x**-axis**.

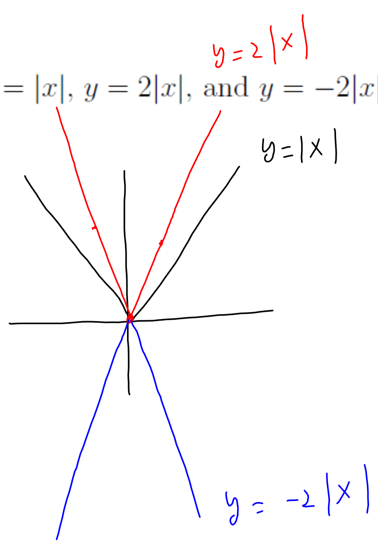
Ex7) Graph $y = \sqrt{x}$, $y = \sqrt{x} + 2$, and $y = \sqrt{x} - 1$ on the same graph.



Ex8) Graph $y = x^3$, $y = (x-2)^3$, and $y = (x+1)^3$ on the same graph.



Ex9) Graph $y = |x|$, $y = 2|x|$, and $y = -2|x|$ on the same graph.



Ex10) If $g(x) = -2f(x+1) - 3$, describe how $f(x)$ was transformed to get $g(x)$.

$g(x) = -2f(x+1) \cancel{-3}$

$y = -2f(x\cancel{+1})$ — $\downarrow$ 3 units

$y = \cancel{-}2f(x)$ — $\leftarrow$ 1 unit

$y = \cancel{2}f(x)$ — reflect about x-axis

$y = f(x)$ — vertical stretch by factor 2

ex)

Graph of $f(x) = -2(x+1)^2 \cancel{+2}$

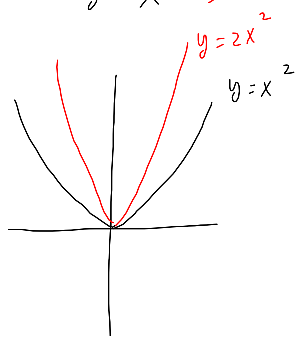
$y = -2(x\cancel{+1})^2$ — ③ $\uparrow$ 2 units, $\leftarrow$ 1 unit

$y = \cancel{-}2x^2$ — ② reflect about x-axis

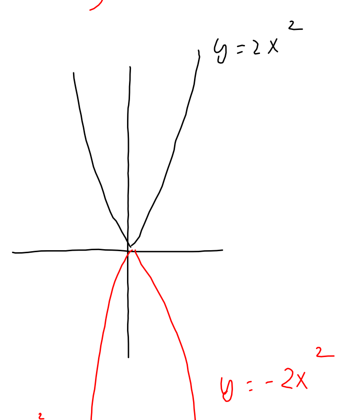
$y = \cancel{2}x^2$ — ① vertical stretch by factor 2

$y = x^2$

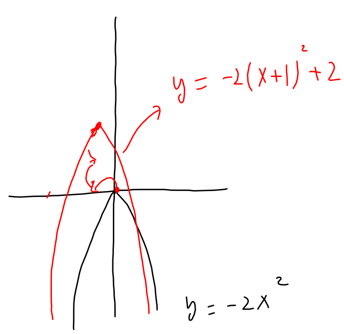
①



$\Rightarrow$ ②



$\Rightarrow$ ③



Section 1.2 Mathematical Models

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Mathematical Models of Cost, Revenue, and Profits

In the **linear cost model** we assume that the cost m of manufacturing one unit is the same no matter how many units are produced. Thus, the variable cost is the number of units produced times the cost of each unit:

$$\boxed{\begin{aligned} \text{variable cost} &= (\text{cost per unit}) \times (\text{number of units produced}) \\ &= mx \end{aligned}}$$

where x is the number of units.

If b is the fixed cost and $C(x)$ is the cost, then

$$\boxed{\begin{aligned} C(x) &= \text{cost} \\ &= (\text{variable cost}) + (\text{fixed cost}) \\ &= mx + b \end{aligned}}$$

In the **linear revenue model** we assume that the price p of a unit sold by a firm is the same no matter how many units are sold. Thus, the revenue is the price per unit times the number of units sold. If we denote the revenue by $R(x)$,

$$\begin{aligned} R(x) &= \text{revenue} \\ &= (\text{price per unit}) \times (\text{number sold}) \\ &= px \end{aligned}$$

Thus the **Profit** P is

$$\begin{aligned} P(x) &= \text{profit} \\ &= (\text{revenue}) - (\text{cost}) \\ &= R(x) - C(x) \\ &= px - (mx + b) \\ &= px - mx - b \\ &= (p - m)x - b \end{aligned}$$

Ex11) A manufacturer of garbage disposals, has a monthly fixed cost of \$10,000 and a production cost of \$20 for each garbage disposal manufactured. The unit sell for \$60 each.

1. What is the cost function?

$C(x) = (\text{variable cost}) + (\text{fixed cost})$

$= 20x + 10000$

2. What is the revenue function?

$R(x) = (\text{price per unit}) \cdot (\text{number sold})$

$= 60x$

3. What is the profit function?

$P(x) = R(x) - C(x)$

$= 60x - (20x + 10000)$

$= 60x - 20x - 10000 = 40x - 10000$

Break-Even Quantity: The value of x at which the profit is zero. (*i.e.*, the point of intersection of the revenue and cost functions) $P(x) = 0$ or $R(x) = C(x)$

Ex12) Find the break-even point for the garbage disposals in Ex11.

$P(x) = 0$

$\Rightarrow 40x - 10000 = 0$

$+10000 \quad +10000$

$\Rightarrow \frac{40x}{40} = \frac{10000}{40}$

$\therefore x = 250$: break even quantity.

Thus I have to sell more than 250.

Section 1.2 Mathematical Models

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Mathematical Models of Cost, Revenue, and Profits

In the **linear cost model** we assume that the cost m of manufacturing one unit is the same no matter how many units are produced. Thus, the variable cost is the number of units produced times the cost of each unit:

$$\begin{aligned} \text{variable cost} &= (\text{cost per unit}) \times (\text{number of units produced}) \\ &= mx \end{aligned}$$

where x is the number of units.

If b is the fixed cost and $C(x)$ is the cost, then

$$\begin{aligned} C(x) &= \text{cost} \\ &= (\text{variable cost}) + (\text{fixed cost}) \\ &= mx + b \end{aligned}$$

In the **linear revenue model** we assume that the price p of a unit sold by a firm is the same no matter how many units are sold. Thus, the revenue is the price per unit times the number of units sold. If we denote the revenue by $R(x)$,

$$\begin{aligned} R(x) &= \text{revenue} \\ &= (\text{price per unit}) \times (\text{number sold}) \\ &= px \end{aligned}$$

Thus the **Profit** P is

$$\begin{aligned} P(x) &= \text{profit} \\ &= (\text{revenue}) - (\text{cost}) \\ &= \boxed{R(x) - C(x)} \end{aligned}$$

Ex11) A manufacturer of garbage disposals, has a monthly fixed cost of \$10,000 and a production cost of \$20 for each garbage disposal manufactured. The unit sell for \$60 each.

1. What is the cost function?

2. What is the revenue function?

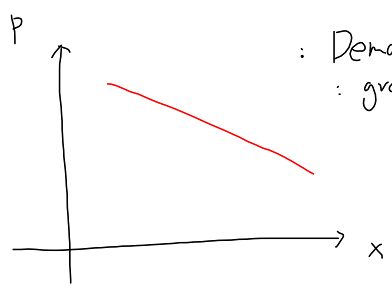
3. What is the profit function?

Break-Even Quantity: The value of x at which the profit is zero. (*i.e.*, the point of intersection of the revenue and cost functions)

Ex12) Find the break-even point for the garbage disposals in Ex11.

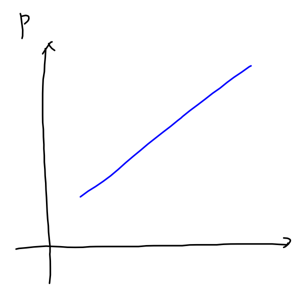
Mathematical Models of Supply and Demand

If the price is assumed to be linear, then the graphs of demand equation and supply equation are straight lines.



: Demand curve
: graph of demand equation

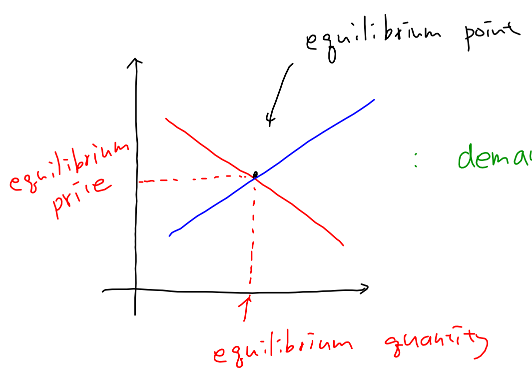
A typical demand curve slopes downward



supply curve
: graph of supply eq.

A typical supply curve slopes upward

Market Equilibrium: The point at which the consumer and supplier agree upon. (*i.e.*, the point of intersection of the supply and demand curves)



: demand curve = supply curve

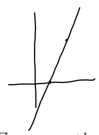
Note Linear equation : $y = mx + b$ $\swarrow$ $(x_1, y_1),\ (x_2, y_2)$

① slope : $m = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$

② point : (x_1, y_1)

$\Rightarrow y - y_1 = m(x - x_1)$

ex) $\overset{x_1\ \ y_1}{(1, 0)},\ \overset{x_2\ \ y_2}{(2, 3)}$



① slope: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 0}{2 - 1} = \frac{3}{1} = 3$

② point: $(1, 0)$

$\Rightarrow y - 0 = 3(x - 1) \Rightarrow \boxed{y = 3x - 3}$

Ex13) For a particular commodity, it is found that 9 units will be supplied at a unit price of \$10.50 whereas 3 units will be supplied at a unit price of \$5.50. For this commodity, it is found that the consummers are willing to consume 8 units at a unit price of \$7 but are only willing to consume 1 unit at a price of \$10.50. Assuming that both supply and demand functions are linear, find the

a) supply equation

$(\overset{x_1}{9}, \overset{P_1}{10.50}),\ (\overset{x_2}{3}, \overset{P_2}{5.50})$

① slope : $m = \frac{P_2 - P_1}{x_2 - x_1} = \frac{5.50 - 10.50}{3 - 9} = \frac{-5}{-6} = \frac{5}{6}$

② point : $(3, 5.50)$

$\Rightarrow P - P_1 = m(x - x_1)$

$\Rightarrow P - 5.50 = \frac{5}{6}(x - 3)$

$P - 5.50 = \frac{5}{6}x - 2.5$

$+5.50 \qquad +5.50$

$\boxed{\therefore P = \frac{5}{6}x + 3}$: supply curve

b) demand equation

$(8, 7),\ (1, 10.50)$

① slope : $m = \frac{\Delta P}{\Delta x} = \frac{P_2 - P_1}{x_2 - x_1} = \frac{10.50 - 7}{1 - 8} = \frac{3.50}{-7} = -0.5$

② point : $(8, 7)$

$\Rightarrow P - P_1 = m(x - x_1)$

$\Rightarrow P - 7 = -0.5(x - 8)$

$P - 7 = -0.5x + 4$

$+7 \qquad +7$

$\boxed{\therefore P = -0.5x + 11}$ demand curve

c) market equilibrium

$\frac{5}{6}x + 3 = -0.5x + 11$

$+0.5x \qquad +0.5x$

$-3 \qquad -3$

$\Rightarrow \frac{5}{6}x + \frac{1}{2}x = 8$

$\Rightarrow \frac{5}{6}x + \frac{3}{6}x = 8$

$\Rightarrow \cancel{\frac{6}{8}} \cdot \frac{\cancel{8}}{\cancel{6}}x = \cancel{8} \cdot \frac{6}{\cancel{8}}$

$\therefore x = 6$: equilibrium quantity

$\Rightarrow P = -\frac{1}{2}(6) + 11$

$= -3 + 11 = 8$: equilibrium price

$\therefore$ marke equilibrium : $(6, 8)$

Use calculator

$y_1 = \frac{5}{6}x + 3$

$y_2 = -\frac{1}{2}x + 11$

$\boxed{\text{2nd}}$ $\boxed{\text{trace}}$

$\boxed{\text{5 - intersection}}$

Practice1) For the quadratic function $y = \underset{=a}{2}x^2 \underset{=b}{-8}x \underset{=c}{+6}$, what is the coordinate of the vertex point?

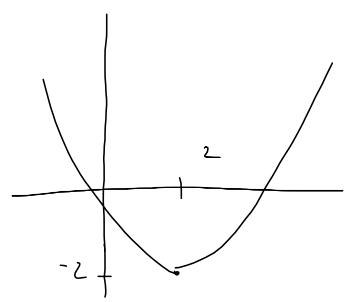
X- coordinate of vertex point (axis of symmetry)

$$: \boxed{X = -\frac{b}{2a}} = -\frac{(-8)}{2(2)} = -\frac{-8}{4} = \frac{8}{4} = 2$$

$$\therefore X = 2$$

$$\Rightarrow y = 2(2)^2 - 8(2) + 6 = -2$$

$\therefore$ vertex : $(2, -2)$



Ex14) It is found that the consumers of a particular toaster will demand 64 toaster ovens when the unit price is \$35 whereas they will demand 448 toaster ovens when the unit price is \$5. Assuming that the demand function is linear, and the selling price is determined by the demand function,

a) Find the demand equation.

② $(64, 35)$, $(448, 5)$

① slope: $m = \frac{P_2 - P_1}{X_2 - X_1} = \frac{5-35}{448-64} = \frac{-30}{384} = -\frac{5}{64}$

$p - p_1 = m(x - x_1)$

$\Rightarrow p - 35 = -\frac{5}{64}(x - 64)$

$\Rightarrow p - 35 = -\frac{5}{64}x + 5$

$\boxed{\therefore p = -\frac{5}{64}x + 40}$

b) Find the revenue function.

$R(x) = p \cdot x$

$= \left(-\frac{5}{64}x + 40\right)x = -\frac{5}{64}x^2 + 40x$

$a = -\frac{5}{64}$, $b = 40$

c) Find the number of items sold that will give the maximum revenue. What is the maximum revenue?

$x = \frac{-b}{2a} = -\frac{(40)}{2\left(-\frac{5}{64}\right)} = \frac{-40}{-\frac{5}{32}} = \frac{40}{\frac{5}{32}} = \cancel{40}^{8} \cdot \frac{32}{\cancel{5}_1} = 256$

$\Rightarrow R(256) = -\frac{5}{64}(256)^2 + 40(256) = \$5120$

$\therefore$ 256 toaster will yield a maximum revenue of \$5120.

d) If the company has a fixed cost of \$1,000 and a variable cost of \$15 per toaster, find the company's linear cost function.

$C(x) = (\text{variable cost}) + (\text{fixed cost})$

$= 15 \cdot x + 1000$

e) What is the company's maximum profit?

$P(x) = R(x) - C(x)$

$= -\frac{5}{64}x^2 + 40x - (15x + 1000)$

$= -\frac{5}{64}x^2 + 40x - 15x - 1000$

$P(x) = -\frac{5}{64}x^2 + 25x - 1000$

$x = -\frac{b}{2a} = -\frac{25}{2\left(\frac{5}{64}\right)} = \frac{-25}{-\frac{5}{32}} = \frac{25}{\frac{5}{32}} = \cancel{25}^{5} \cdot \frac{32}{\cancel{5}_1}$

$= 160$

$\Rightarrow P(160) = -\frac{5}{64}(160)^2 + 25(160) - 1000$

$= \$1000$

$\therefore$ A maximum profit of \$1000 occurs when 160 toaster ovens are sold

f) How many toasters should be sold for the company to break even?

$P(x) = 0$

$-\frac{5}{64}x^2 + 25x - 1000 = 0$

$\therefore x \approx 47, 273$

use calculator

$y_1 = -\frac{5}{64}x^2 + 25x - 1000$

$y_2 = 0$

$\boxed{\text{2nd}} + \boxed{\text{trace}} + 5$ - intersection.

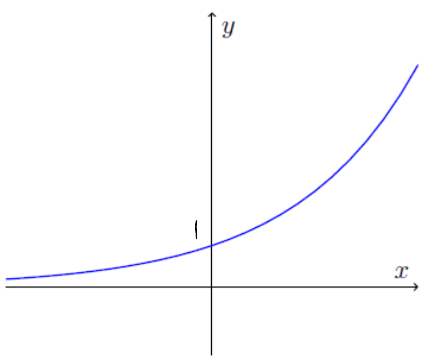
Section 1.3 Exponential Models

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant with $a \neq 1$.

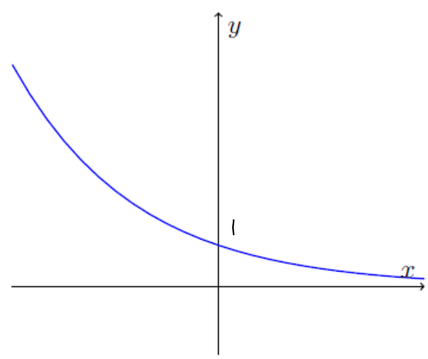
Properties of the Graphs of $f(x) = a^x$

1. Domain is the set of all real numbers.
2. Range is the set of all positive real numbers.
3. All graphs pass through the point $(0, 1)$.
4. The graph is continuous (no holes or jumps).
5. The x axis is a horizontal asymptote (but only in one direction).
6. If $a > 1$, the graph is increasing (exponential growth).
7. If $0 < a < 1$, the graph is decreasing (exponential decay).

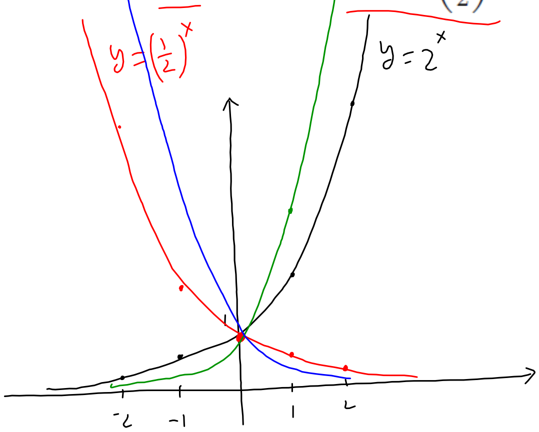
$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$



Ex15) Graph $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = \left(\frac{1}{2}\right)^x$, and $k(x) = \left(\frac{1}{3}\right)^x$ on the same graph.



$2^{-1} = \frac{1}{2}$ $\left(\frac{1}{2}\right)^{-1} = 2$

$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$ $\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$

Law of exponents

If a and b are positive numbers and x and y are any real numbers, then

1. $a^x \cdot a^y = a^{x+y}$ ex) $2^3 \cdot 2^7 = 2^{3+7} = 2^{10}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$ ex) $\dfrac{2^4}{2^2} = 2^{4-2} = 2^2$
3. $a^{-x} = \dfrac{1}{a^x}$
4. $a^0 = 1$ for $a \neq 0$
5. $(ab)^x = a^x \cdot b^x$ ex) $(3x)^4 = 3^4 \cdot x^4$
6. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
7. $(a^x)^y = a^{xy}$
8. $a^x = a^y$ if and only if $x = y$
9. For $x \neq 0$, $a^x = b^x$ if and only if $a = b$

The Natural Exponential Function

Any positive number can be used as the base for an exponential function, but some bases are used more frequently then others. For example, 2, 10 and e.

The number e is defined as the value that $\left(1+\frac{1}{n}\right)^n$ approaches as n becomes large, written as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.718281\cdots$$

The **Natural Exponential Function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$.

Ex16) Simplify $\dfrac{e^{x+4}}{e^{4-x}}$

$$= e^{(x+4)-(4-x)}$$

$$= e^{x+\cancel{4}-\cancel{4}+x} = e^{2x}$$

Note

$$\frac{a^m}{a^n} = a^{m-n}$$

Ex17) Solve each equation for x

a) $9^{x-1} = 3^{1+x}$

$$\Rightarrow (3^2)^{x-1} = 3^{1+x}$$

Note

$$(a^m)^n = a^{m \cdot n}$$

$$\Rightarrow 3^{2(x-1)} = 3^{1+x}$$

$$\Rightarrow 2(x-1) = 1+x$$

$$\Rightarrow 2x - 2 = 1 + x$$

$$-x \quad +2 \qquad -x \quad +2$$

$$\boxed{x = 3}$$

b) $x^2e^x - 5xe^x = 0$

$$\Rightarrow x \cdot e^x (x-5) = 0$$

$x = 0$ $\quad \neq 0$ $\quad x = 5$

$$\therefore x = 0,\ 5$$

Growth and Decay Applications

Function of the form

$$y = Ce^{kt}$$

where C is the initial amount, k is the relative growth(or decay) rate, and t is time. If k is positive, it is **exponential growth model**, if k is negative, it is **exponential decay model**.

Ex18) The population of a particular city grows continuously at a relative growth rate of 5.4%. If 30,000 people currently live in the city, what will be the population in eight years?

$k = 0.054$

$P(t) = P_0 \cdot e^{kt}$

$\Rightarrow \boxed{P(t) = 30000 \cdot e^{0.054t}}$

$\Rightarrow P(8) = 30000 \cdot e^{0.054 \cdot (8)} \approx 46{,}210$ people

Ex19) The population of an undesirable city is modeled by

$$p(t) = ce^{-0.09t}$$

where t represents the number of years since 1950, $p(t)$ represents the population in the t^{th}year, and c is a constant representing the population in 1950.

a) If the city had a population of 20,000 in 1990, what is the value of c?

b) Use the model to predict the population in 2008.

$$-\frac{5}{64}x^2 + 25x - 1000 = 0$$

$$x = \frac{-25 \pm \sqrt{(25)^2 - 4\cdot\left(-\frac{5}{64}\right)\cdot(-1000)}}{2\cdot\left(-\frac{5}{64}\right)}$$

Note

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Growth and Decay Applications

Function of the form

$$y = Ce^{kt}$$

where C is the initial amount, k is the relative growth(or decay) rate, and t is time. If k is positive, it is **exponential growth model**, if k is negative, it is **exponential decay model**.

Ex18) The population of a particular city grows continuously at a relative growth rate of 5.4%. If 30,000 people currently live in the city, what will be the population in eight years?

$$P(t) = P_0 \cdot e^{kt}$$

$$\Rightarrow P(t) = 30000 \cdot e^{0.054t}$$

$$P(8) = 30000 \cdot e^{0.054(8)} \approx 46,210 \text{ people}$$

Ex19) The population of an undesirable city is modeled by

$$p(t) = ce^{-0.09t}$$

where t represents the number of years since 1950, $p(t)$ represents the population in the t^{th}year, and c is a constant representing the population in 1950.

a) If the city had a population of 20,000 in 1990, what is the value of c?

$$\Rightarrow \frac{c \cdot e^{-0.09 \cdot (40)}}{e^{-0.09(40)}} = \frac{20000}{e^{-0.09(40)}}$$

$$\therefore\ c = \frac{20000}{e^{-0.09(40)}} \approx 731,965 \text{ people in 1950}$$

$\Rightarrow$ Complete the model:

$$\boxed{P(t) = 731965 \cdot e^{-0.09t}}$$

b) Use the model to predict the population in 2008.

$$P(58) = 731965 \cdot e^{-0.09(58)}$$

$$\approx 3958 \text{ people in 2008.}$$

$A = P(1+r)^t$

Compound Interest

If a principal P (present value) is invested at an annual rate r (expressed as a decimal) compounded m times a year, then the amount A (future value) in the account at the end of t years is given by

$$\boxed{A = P(1 + \frac{r}{m})^{mt}}$$

Quarterly : $m = 4$. montly : $m = 12$. daily : $m = 365$

Ex20) If \$5,000 is invested in an account paying 2.5% compounded monthly, how much will be in the account at the end of 10 years?

$m = 12$

$$A(t) = 5000 \cdot \left(1 + \frac{0.025}{12}\right)^{12t}$$

after 10 years

$$\Rightarrow A(10) = 5000\left(1 + \frac{0.025}{12}\right)^{12(10)} \approx \$\,6418.46$$

Continuously Compounded Interest

$$\boxed{A = Pe^{rt}}$$

where P is principal, r is the annual interest rate compounded continuoulsy (as a decimal), t is time, and A is the accumulated amount at the end of t years.

Ex21) What amount will an account have after five years if \$1,000 is invested at an annual rate of 3.25% compounded continuously?

$$A(t) = P \cdot e^{rt}$$

$$\Rightarrow A(t) = 1000 \cdot e^{0.0325t}$$

$$A(5) = 1000 \cdot e^{0.0325(5)} \approx \$\,1176.45$$

Effective Yield $A = P\left(1+\frac{r}{12}\right)^{12t}$

If \$1000 is invested at an annual rate of 9% compounded monthly, then at the end of a year there is

$$\boxed{F = \$1000\left(1+\frac{0.09}{12}\right)^{12}} = \$1093.81 = 1000(1+0.09381)$$

in the account. This is the same amount that is obtainable if the same principal of \$1000 is invested for one year at an annual rate of 9.381%. We call the rate 9.381% the **effective annual yield**. The 9% annual rate is often refered to as the **nominal rate**.

If we let r_{eff} be the effective annual yield for 1 year, then r_{eff} must satisfy

$$\cancel{P}\left(1+\frac{r}{m}\right)^m = \cancel{P}(1+r_{eff})$$

$$\Rightarrow \left(1+\frac{r}{m}\right)^m = 1 + r_{eff}$$

$$-1 \qquad -1$$

$$\therefore \left(1+\frac{r}{m}\right)^m - 1 = r_{eff}$$

thus

$$\boxed{r_{eff} = \left(1+\frac{r}{m}\right)^m - 1}$$

and

$$\boxed{r_{eff} = e^r - 1}$$

is the effective annual yield compounded continuously.

Ex22) You have been doing some research and have found that you can either invest your money at an annual interest rate of 3.55% compounded monthly or 3.50% compounded continuously. Which one would you choose?

monthly

$$r_{eff} = \left(1+\frac{0.0355}{12}\right)^{12} - 1 \simeq 0.036083$$

$\therefore$ effective annual yield is 3.6083 %

continuosly

$$r_{eff} = e^{0.035} - 1 \simeq 0.035620$$

$\therefore$ effective annual yield is 3.5620 %

Calculator Functions

TVM Solver: We can use the TVM Solver on our calculator to solve problems involving compound interest. To access the Finance Menu, you need to press APPS $->$ 1:Finance (Please note that if you have a plain TI-83, you need to press 2nd x^{-1} to access the Finance Menu). Below we define the inputs on the TVM Solver:

- N = the total number of compounding periods
- $I\%$ = interest rate (as a percentage)
- PV = present value (principal amount). Entered as a negative number if invested, a positive number if borrowed.
- PMT = payment amount (0 for this class)
- FV = future value (accummulated amount)
- $P/Y = C/Y$ = the number of compounding periods per year.

Move the cursor to the value you are solving for and hit ALPHA and the ENTER.

Effective Yield: We use the C:Eff(option on the Finance Menu to compute the effective rate of interest. The inputs are as follows:

EFF(annual interest rate as a percentage, the number of compounding periods per year)

Section 1.5 Logarithms

Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function and therefore has an inverse function. The inverse function f^{-1} is called the **Logarithmic function with base** a and is denoted $\log_a$.

Definition of the Logarithmic function

Let a be a positive number with $a \neq 1$. The **Logarithmic function with base** a, denoted by $\log_a$, is defined by

$$y = \log_a x \iff a^y = x$$

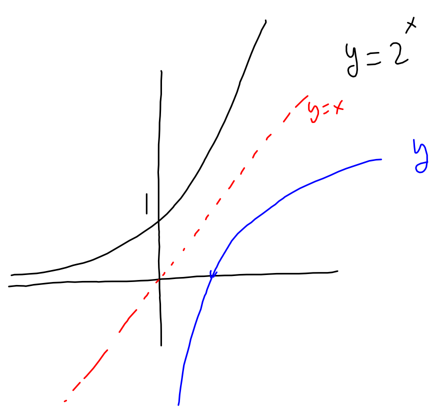
exp.

exp

$$a^y = x \iff \log_a x = y$$

base

base



$y = \log_2 x$

domain : $(0, \infty)$

range : $(-\infty, \infty)$

Common Logarithm

The logarithm with base 10 is called the **Common logarithm** and is denoted by omitting the base

$$\log x = \log_{10} x$$

Natural Logarithm

The logarithm with base e is called the **Natural logarithm** and is denoted by ln

$$\ln x = \log_e x$$

The natural logarithm function $y = \ln x$ is the inverse function of the exponential function $y = e^x$ which is

$$\ln x = y \iff e^y = x$$

Ex23) Solve for x, y, or b without a calculator

Note

$a^x = y \Leftrightarrow \log_a y = x$

a) $\log_3 x = 2$ exp, base $\Leftrightarrow 3^2 = x$

$\therefore x = 9$

b) $\log_b e^{-4} = -4$ exp, base $\Leftrightarrow b^{-4} = e^{-4}$

$\therefore b = e$

c) $\log_{49}(\frac{1}{7}) = y$ exp, base $\Leftrightarrow 49^y = \frac{1}{7}$

$\Rightarrow (7^2)^y = 7^{-1}$

$\Rightarrow 7^{2y} = 7^{-1}$

$\Rightarrow 2y = -1$

$\therefore y = -\frac{1}{2}$

Properties of Logarithm

1. $\log_a 1 = 0 \iff a^0 = 1$
2. $\log_a a = 1 \iff a^1 = a$ ex) $\log_3 3 = 1$, $\log_7 7 = 1$
3. $\log_a a^x = x$ ← $\log_a a^x = x \log_a a = x$, $\log_a a = 1$
4. $a^{\log_a x} = x$ ← $a^{\log_a x} = x^{\log_a a} = x^1 = x$, $\log_a a = 1$
5. $\log_a b^m = m \log_a b$
6. $\log_{a^n} b = \frac{1}{n}\log_a b$

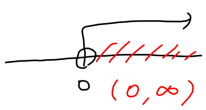
$$\boxed{\log_{a^m} b^n = \frac{n}{m}\log_a b}$$

7. $\log_a xy = \log_a x + \log_a y$ ex) $\log_a 2\cdot 3 = \log_a 2 + \log_a 3$
8. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ ex) $\log_7\left(\frac{2}{3}\right) = \log_7 2 - \log_7 3$
9. $\cancel{\log_a b} = \frac{\log_c b}{\log_c a}$ and $\log_a b = \frac{1}{\log_b a}$
10. $\log_a M = \log_a N \iff M = N$

Ex24) Find the domain of each funtion

a) $f(x) = 2 + \log_5 x$

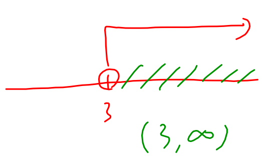
$x > 0$ $\therefore$ domain is $(0, \infty)$



b) $g(x) = \log(x-3)$

$x - 3 > 0$

$\Rightarrow x > 3$ $\therefore$ domain is $(3, \infty)$



Ex25) Solve the following for x:

a) $\log_b x = \frac{2}{3}\log_b 27 + 2\log_b 2 - \log_b 3$

Note

$m\log_a b = \log_a b^m$

$= \log_b (27)^{\frac{2}{3}} + \log_b 2^2 - \log_b 3$

$= \log_b\left(\frac{(27)^{\frac{2}{3}}\cdot 2^2}{3}\right) = \log_b\left(\frac{\cancel{9}^{3}\cdot 4}{\cancel{3}^{1}}\right) = \log_b 12$

$(3^3)^{\frac{2}{3}} = 3^{3\cdot\frac{2}{3}} = 3^2$

b) $\log x + \log(x-3) = 1$

$\Rightarrow \log x\cdot(x-3) = 1$

since base = 10

$\Rightarrow x(x-3) = 10$

$x^2 - 3x = 10$

$\Rightarrow x^2 - 3x - 10 = 0$

A = -3, M = -10 : 2, -5

$(x+2)(x-5) = 0$

$\cancel{x=-2}$ $x = 5$

$\boxed{\therefore\ x = 5}$

Ex26) Find the domain of $f(x) = \dfrac{\sqrt[7]{2x-9}}{\log_2(4x-1)}$. ← no restriction

$4x - 1 > 0$ and $\log_2(4x-1) \neq 0$

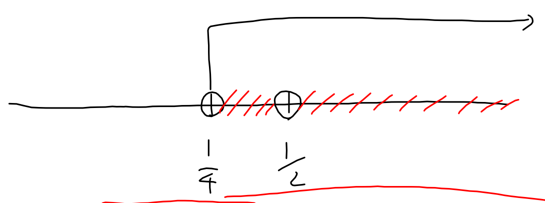
+1 +1

$\Rightarrow \frac{4x}{4} > \frac{1}{4}$ $\Rightarrow 4x - 1 \neq 1$

+1 +1

$\Rightarrow x > \frac{1}{4}$ $\Rightarrow \frac{4x}{4} \neq \frac{2}{4}$

and $\Rightarrow x \neq \frac{1}{2}$



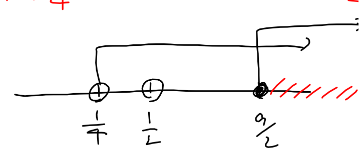
$\boxed{\therefore\ \left(\frac{1}{4}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)}$

ex) $\dfrac{\sqrt{2x-9}}{\log_2(4x-1)}$

$4x-1 > 0$ and $4x-1 \neq 1$

and $2x - 9 \geq 0$

$x > \frac{1}{4}$ and $x \neq \frac{1}{2}$ and $x \geq \frac{9}{2}$



$\therefore\ \left[\frac{9}{2}, \infty\right)$

Ex27) Evaluate $\log_8 17$. $= \dfrac{\ln 17}{\ln 8} \approx 1.3625$

$= \dfrac{\log 17}{\log 8}$

Note

$\log_a b = \dfrac{\ln b}{\ln a}$

Ex28) Solve $1.02^{4x} = 2$ for x rounded to four decimal places.

take ln on both sides

$\ln 1.02^{4x} = \ln 2$

$\Rightarrow \dfrac{4x \cdot \ln(1.02)}{\ln(1.02)} = \dfrac{\ln 2}{\ln(1.02)}$

$\frac{1}{4} \cdot 4x = \dfrac{\ln 2}{\ln 1.02} \cdot \frac{1}{4}$

$\therefore x = \dfrac{\ln 2}{4 \cdot \ln 1.02}$

≈ 8.7507

Ex29) Given $\log_b 3 = 0.6826$ and $\log_b 4 = 0.8614$, find the following

a) $\log_b 48$

since $48 = 3 \cdot 4^2$

$\Rightarrow \log_b 48 = \log_b 3 \cdot 4^2 = \log_b 3 + \log_b 4^2$

$= \log_b 3 + 2 \log_b 4 = 0.6826 + 2 \cdot 0.8614$

$= 2.4054$

b) $\log_b \left(\frac{16}{b^2}\right)$

$= \log_b 16 - \log_b b^2 = \log_b 4^2 - \log_b b^2$

$= 2 \log_b 4 - 2 \underbrace{\log_b b}_{=1}$

$= 2 \cdot 0.8614 - 2 = -0.2772$

Ex30) How long will it take for the amount in an account to double if the money is compounded continuously at an interest rate of 2.5% per year?

$$A = P \cdot e^{rt}$$

$$\Rightarrow A = P \cdot e^{0.025t}$$

double of initial amount

$$\Rightarrow 2\cancel{P} = \cancel{P} \cdot e^{0.025t}$$

take ln on both sides

$$\Rightarrow \ln 2 = \ln e^{0.025t}$$

$$\Rightarrow \ln 2 = 0.025t \cdot \underbrace{\ln e}_{=1}$$

$$\Rightarrow \frac{\ln 2}{0.025} = \frac{0.025\, t}{0.025}$$

$$\Rightarrow t = \frac{\ln 2}{0.025} \approx 27.7259 \text{ years}$$