Week 2 Section 3.1 Limits, Continuity

Section 3.1 Limits

Definitions

We write

$$\lim_{x \to a} f(x) = L$$

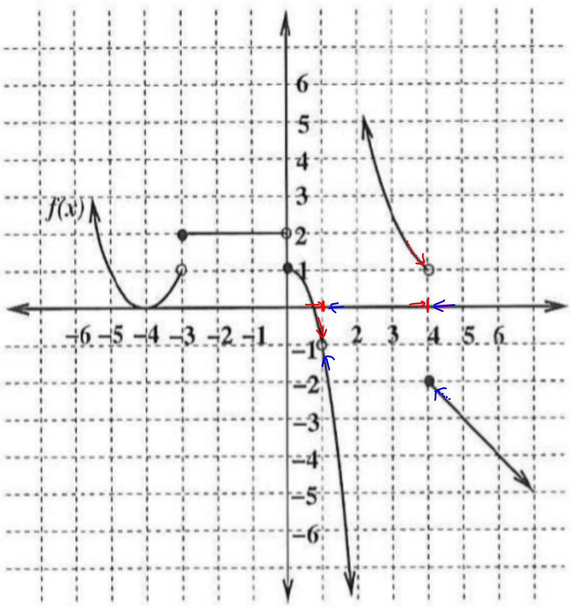
and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a but not equal to a.

This says that the value of $f(x)$ get closer and closer to the number L as x gets closer and closer to the number a (from either side of a), but $x \neq a$.

Left-Hand Limit: We write $\lim_{x \to a^-} f(x) = L$ if $f(x)$ is close to L whenever x is close to, but to the left of a.

Right-Hand Limit: We write $\lim_{x \to a^+} f(x) = L$ if $f(x)$ is close to L whenever x is close to, but to the right of a.

Ex.1) Use the graph below to find the following limits:



a) $\lim_{x \to -3^-} f(x) =$ b) $\lim_{x \to -3^+} f(x) =$ c) $\lim_{x \to -3} f(x) =$

d) $\lim_{x \to 1^-} f(x) =$ e) $\lim_{x \to 1^+} f(x) =$ f) $\lim_{x \to 1} f(x) =$

g) $\lim_{x \to 4^-} f(x) =$ h) $\lim_{x \to 4^+} f(x) =$ i) $\lim_{x \to 4} f(x) =$

j) $\lim_{x \to 2^-} f(x) =$ k) $\lim_{x \to 2^+} f(x) =$ l) $\lim_{x \to} f(x) =$

Ex.2) Use a table of values to estimate $\lim_{x\to 1} \frac{x-1}{x^2-1}$. Confirm your result graphically.

$f(x) = \frac{x-1}{x^2-1}$: undefined at $x = 1, -1$

Left limit

$x<1$	$f(x)$	$x>1$	$f(x)$
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975
⋮	⋮	⋮	⋮

$\lim_{x\to 1^-} f(x) = 0.5 = \lim_{x\to 1^+} f(x) = 0.5$

$\therefore \lim_{x\to 1} f(x) = 0.5$

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = A \quad \text{and} \quad \lim_{x\to a} g(x) = B$$

exist. Then

(a)
$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A+B$$

(b)
$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = A-B$$

(c)
$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x) = c\cdot A$$

(d)
$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A\cdot B$$

(e)
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0$$
$$= \frac{A}{B}$$

(f)
$$\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n \quad = A^n$$

(g)
$$\lim_{x \to a} c = c$$

(h)
$$\lim_{x \to a} x = a$$

(i)
$$\lim_{x \to a} x^n = a^n$$

(j)
$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

(k)
$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}$$

where n is a positive integer. (If n is even, we assume that $\lim_{x \to a} f(x) > 0$).

Limits of Polynomial and Rational Functions

(a) $\lim_{x \to a} f(x) = f(a)$, f any polynomial function

(b) $\lim_{x \to a} r(x) = r(a)$, r any rational function with nonzero denominator at $x = a$

Ex.3) Determine the follwing limits:

a) $\lim_{x \to 2} 10 = 10$

b) $\lim_{x \to 3} x^2 + 3x - 2 = (3)^2 + 3(3) - 2$

$= 16$

c) $\lim_{x \to 3} \frac{3x + 2}{x - 4} = \frac{3(3) + 2}{(3) - 4} = \frac{11}{-1} = -11$

Ex.4) Find each limit.

a) $\lim_{x \to 3} x^3 - 4x^2 + 2 = (3)^3 - 4(3)^2 + 2 = -7$

b) $\lim_{x \to 2} \frac{x^2 - 1}{x + 2} = \frac{(2)^2 - 1}{(2) + 2} = \frac{3}{4}$

c) If $f(x) = \begin{cases} x^2+3 & \text{if } x \le 3 \\ x-4 & \text{if } x > 3 \end{cases}$, find

i) $\lim_{x\to 0} f(x) = (0)^2 + 3 = 3$

ii) $\lim_{x\to 3^-} f(x) = (3)^2 + 3 = 12$

$x < 3$

$\neq$

iii) $\lim_{x\to 3^+} f(x) = (3) - 4 = -1$

$x > 3$

iv) $\lim_{x\to 3} f(x) = DNE$

d) $\lim_{x\to -2} \frac{x^2-4}{x+2} = \frac{(-2)^2-4}{(-2)+2} = \frac{0}{0}$: indeterminate form

$a^2 - b^2 = (a-b)(a+b)$

$= \lim_{x\to -2} \frac{x^2-4}{x+2}$

$= \lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)}{\cancel{(x+2)}} = \lim_{x\to -2} \frac{x-2}{1} = \frac{(-2)-2}{1} = \frac{-4}{1} = -4$

How to find limits algebraically $\lim_{x\to a} f(x)$

Try plugging a into the function:

(a) If you get a real number, that is your answer (unless you are dealing with a piecewise function) continuous

(b) If you get $\frac{0}{0}$ (indeterminate form), algebraically manipulate (usually factor), cancel, and plug in a again. : hole

(c) If you get $\frac{\text{none zero number}}{0}$ then the limit does not exist. $= \begin{cases} +\infty \\ -\infty \end{cases}$

: vertical asymptote

Ex.5) Evaluate each limit:

a) $\lim_{x\to 6} |x+3| = |(6)+3| = |9| = 9$

b) $\lim_{x\to 2} \frac{2x^2-x-6}{x-2}$ $\frac{-2}{3} = \frac{0}{0}$ $= \lim_{x\to 2} \frac{\cancel{(x-2)}(2x+3)}{\cancel{(x-2)}}$

$= \lim_{x\to 2} \frac{2x+3}{1} = \frac{2(2)+3}{1} = \frac{7}{1} = 7$

d) $\lim_{x\to 4^-} \frac{1}{x-4} = \frac{1}{0}\ \frac{+}{-} = -\infty$

$x<4$

$\neq$

e) $\lim_{x\to 4^+} \frac{1}{x-4} = \frac{1}{0}\ \frac{+}{+} = +\infty$

$x>4$

f) $\lim_{x\to 4} \frac{1}{x-4} =$ DNE

g) $\lim_{x\to 2} \frac{x-4}{(x-2)^2} = \frac{(2)-4}{(2-2)^2} = \frac{-2}{0}$

Left: $\lim_{x\to 2^-} \frac{x-4}{(x-2)^2} = \frac{-2}{(0)^2}\ \frac{-}{+} = -\infty$

$x<2$

$\parallel$

Right: $\lim_{x\to 2^+} \frac{x-4}{(x-2)^2} = \frac{-2}{(0)^2}\ \frac{-}{+} = -\infty$

$x>2$

$\therefore \lim_{x\to 2} \frac{x-4}{(x-2)^2} = -\infty$

NOTE

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

c) $\lim_{x\to 5} \frac{|x-5|}{x-5}$

Left: $\lim_{\substack{x\to 5^- \\ x<5}} \frac{|x-5|}{x-5}$, $|x-5| = -(x-5)$

$= \lim_{x\to 5^-} \frac{-\cancel{(x-5)}}{\cancel{x-5}} = \lim_{x\to 5^-} \frac{-1}{1} = -1$

$\neq$

Right: $\lim_{\substack{x\to 5^+ \\ x>5}} \frac{|x-5|}{x-5}$, $|x-5| = (x-5)$

$= \lim_{x\to 5^+} \frac{\cancel{x-5}}{\cancel{x-5}} = \lim_{x\to 5^+} \frac{1}{1} = 1$

$\therefore \lim_{x\to 5} \frac{|x-5|}{x-5} = \text{DNE}$

Ex.10) Find each limit.

a) $\lim\limits_{x \to -3} \dfrac{x^2}{x+3}$

Left: $\lim\limits_{\substack{x \to -3^- \\ x<-3}} \dfrac{x^2}{x+3} = \dfrac{9}{0} \begin{matrix} + \\ - \end{matrix} = -\infty$

$\neq$

Right: $\lim\limits_{\substack{x \to -3^+ \\ x>-3}} \dfrac{x^2}{x+3} = \dfrac{9}{0} \begin{matrix} + \\ + \end{matrix} = +\infty$

$\therefore \lim\limits_{x \to -3} \dfrac{x^2}{x+3} = DNE$

b) $\lim\limits_{x \to -2} \dfrac{x^2+x-2}{x+2} = \dfrac{0}{0}$

$= \lim\limits_{x \to -2} \dfrac{\cancel{(x+2)}(x-1)}{\cancel{x+2}} = \lim\limits_{x \to -2} \dfrac{x-1}{1} = \dfrac{(-2)-1}{1} = \dfrac{-3}{1} = -3$

Ex.6) Evaluate $\lim_{x \to -4} \dfrac{\sqrt{x^2+9}-5}{x+4}$. $= \dfrac{\sqrt{(-4)^2+9}-5}{(-4)+4} = \dfrac{0}{0}$

Multiply the conjugate

$$\lim_{x \to -4} \frac{\left(\sqrt{x^2+9}-5\right)}{(x+4)} \cdot \frac{\left(\sqrt{x^2+9}+5\right)}{\left(\sqrt{x^2+9}+5\right)}$$

$(a+b)(a-b) = a^2-b^2$

$$= \lim_{x \to -4} \frac{\left(\sqrt{x^2+9}\right)^2-(5)^2}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$= \lim_{x \to -4} \frac{x^2+9-25}{(x+4)\left(\sqrt{x^2+9}+5\right)} = \lim_{x \to -4} \frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$x^2-16 = (x+4)(x-4)$

$$= \lim_{x \to -4} \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}\left(\sqrt{x^2+9}+5\right)} = \lim_{x \to -4} \frac{x-4}{\sqrt{x^2+9}+5}$$

$$= \frac{(-4)-4}{\sqrt{(-4)^2+9}+5} = \frac{-8}{10} = -\frac{4}{5}$$

Ex.7) If $f(x) = 3x^2 + 4x - 7$ find $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

$$f(x+h) = 3(x+h)^2 + 4(x+h) - 7$$

$$x^2 + 2xh + h^2$$

$$= 3x^2 + 6xh + 3h^2 + 4x + 4h - 7$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(3x^2 + 6xh + 3h^2 + 4x + 4h - 7) - (3x^2 + 4x - 7)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{+4x} + 4h \cancel{-7} \cancel{-3x^2} \cancel{-4x} \cancel{+7}}{h}$$ Passover

$$= \lim_{h\to 0} \frac{6xh + 3h^2 + 4h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(6x + 3h + 4)}{\cancel{h}}$$ sacrifice

$$= \lim_{h\to 0} \frac{6x + 3h + 4}{1} = 6x + 3(0) + 4$$

$$= 6x + 4$$