MATH 142 Business Calculus

Fall 2019, WEEK 2

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Week 2 Section 3.1 Limits, Continuity

Section 3.1 Limits

Definitions

We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a but not equal to a.

This says that the value of $f(x)$ get closer and closer to the number L as x gets closer and closer to the number a (from either side of a), but $x \neq a$.

Left-Hand Limit: We write $\lim_{x \to a^-} f(x) = L$ if $f(x)$ is close to L whenever x is close to, but to the left of a.

Right-Hand Limit: We write $\lim_{x \to a^+} f(x) = L$ if $f(x)$ is close to L whenever x is close to, but to the right of a.

Existence of a Limit

If the function $f(x)$ is defined near $x = a$ but not necessarily at $x = a$, then

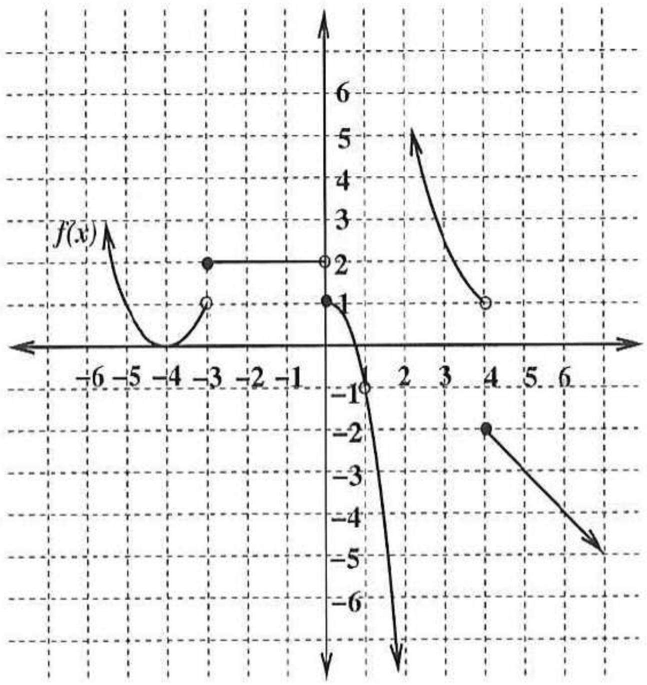
$$\lim_{x \to a} f(x) = L$$

if and only if both the limits

$$\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x)$$

exist and are equal to the same number L.

Ex.1) Use the graph below to find the following limits:



a) $\lim_{x \to -3^-} f(x) =$ b) $\lim_{x \to -3^+} f(x) =$ c) $\lim_{x \to -3} f(x) =$

d) $\lim_{x \to 1^-} f(x) =$ e) $\lim_{x \to 1^+} f(x) =$ f) $\lim_{x \to 1} f(x) =$

g) $\lim_{x \to 4^-} f(x) =$ h) $\lim_{x \to 4^+} f(x) =$ i) $\lim_{x \to 4} f(x) =$

j) $\lim_{x \to 2^-} f(x) =$ k) $\lim_{x \to 2^+} f(x) =$ l) $\lim_{x \to} f(x) =$

Ex.2) Use a table of values to estimate $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$. Confirm your result graphically.

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

(a)
$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

(b)
$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

(c)
$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$$

(d)
$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

(e)
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0$$

(f)
$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

(g)
$$\lim_{x\to a} c = c$$

(h)
$$\lim_{x\to a} x = a$$

(i)
$$\lim_{x\to a} x^n = a^n$$

(j)
$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$$

(k)
$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}$$

where n is a positive integer. (If n is even, we assume that $\lim_{x\to a} f(x) > 0$).

Ex.3) Determine the follwing limits:

a) $\lim_{x\to 2} 10$

b) $\lim_{x\to 3} x^2 + 3x - 2$

c) $\lim_{x\to 3} \dfrac{3x+2}{x-4}$

Limits of Polynomial and Rational Functions

(a) $\lim_{x\to a} f(x) = f(a)$, f any polynomial function

(b) $\lim_{x\to a} r(x) = r(a)$, r any rational function with nonzero denominator at $x = a$

Ex.4) Find each limit.

a) $\lim_{x\to 3} x^3 - 4x^2 + 2$

b) $\lim_{x\to 2} \dfrac{x^2-1}{x+2}$

c) If $f(x) = \begin{cases} x^2+3 & \text{if} \quad x \leq 3 \\ x-4 & \text{if} \quad x > 3 \end{cases}$, find

i) $\lim_{x\to 0} f(x)$

ii) $\lim_{x\to 3^-} f(x)$

iii) $\lim_{x\to 3^+} f(x)$

iv) $\lim_{x\to 3} f(x)$

d) $\lim_{x\to -2} \dfrac{x^2-4}{x+2}$

How to find limits algebraically $\lim_{x \to a} f(x)$

Try plugging a into the function:

(a) If you get a real number, that is your answer (unless you are dealing with a piecewise function)

(b) If you get $\frac{0}{0}$ (indeterminate form), algebraically manipulate (usually factor), cancel, and plug in a again.

(c) If you get $\frac{\text{none zero number}}{0}$ then the limit does not exist.

Ex.5) Evaluate each limit:

a) $\lim_{x \to 6} |x + 3|$

b) $\lim_{x \to 2} \frac{2x^2 - x - 6}{x - 2}$

c) $\lim_{x \to 5} \frac{|x - 5|}{x - 5}$

Ex.6) Find each limit.

a) $\lim_{x \to -3} \frac{x^2}{x+3}$

b) $\lim_{x \to -2} \frac{x^2+x-2}{x+2}$

Ex.7) Evaluate $\lim_{x \to -4} \frac{\sqrt{x^2+9}-5}{x+4}$.

Ex.8) If $f(x) = 3x^2 + 4x - 7$ find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

Ex.9) Evaluate and examine the behavior of $f(x) = \dfrac{7-x}{x^2}$ at $x = 0$.

Definition. The vertical line $x = a$ is a **vertical asymptote** for the graph of $y = f(x)$ if

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

Locating Vertical Asymptotes of Rational Functions

If $f(x) = \dfrac{n(x)}{d(x)}$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

Ex.10) Find vertical asymptote(s) of $f(x) = \dfrac{x-3}{x^2-4x+3}$.

Continuity

A function f is **continuous** at the point $x = a$ if all of the following are true:

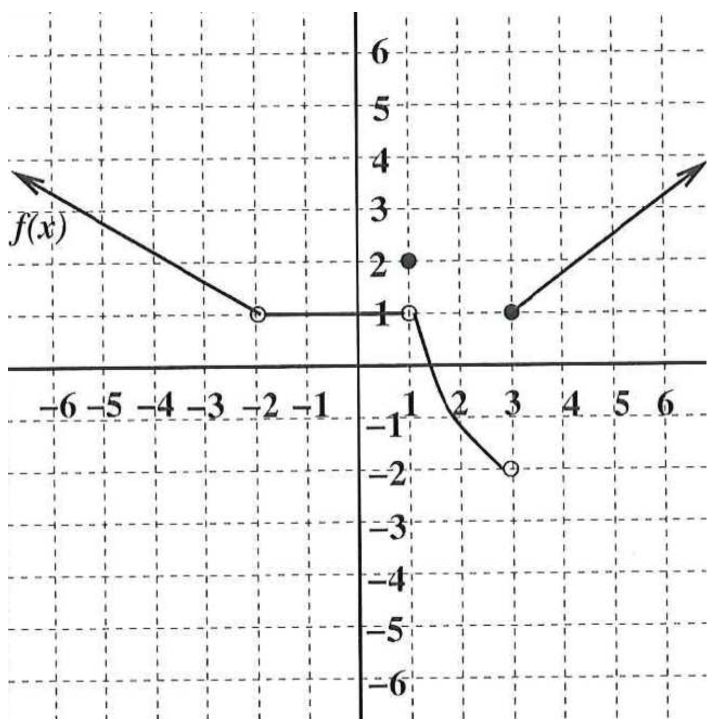
(a) $f(a)$ is defined

(b) $\lim_{x \to a} f(x)$ exists.

(c) $\lim_{x \to a} f(x) = f(a)$

If one or more of the conditions are not met, we say f is discontinuous at $x = a$.

Ex.11) For what values of x is the function discontinous? Explain.



> **Definition.** A function is **continous on an interval** if it is continuous at each point on the interval.

Ex.12) Find the intervals on which the following functions are continuous.

a) $f(x) = 3x^9 + 4x^5 - 7$

b) $g(x) = 2^{3x} + 4$

c) $h(x) = \log_8(x-3) - 2$

d) $k(x) = \dfrac{x+2}{x^2 - 3x - 10}$

e) $m(x) = \begin{cases} \dfrac{x-5}{x+2} & \text{if } x \le -3 \\ 2x^2 & \text{if } -3 < x \le 0 \\ \dfrac{x}{x-4} & \text{if } x > 0 \end{cases}$

Ex.13) Find the value(s) of k that make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} 2x - 5 & \text{if } x \leq 1 \\ x^2 + k & \text{if } x > 1 \end{cases}$$

Ex.14) Find the value(s) of A that make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ Ax^2 - 6x + 10 & \text{if } x \geq 2 \end{cases}$$