Ex.8) Evaluate and examine the behavior of $f(x) = \dfrac{7-x}{x^2}$ at $x = 0$.

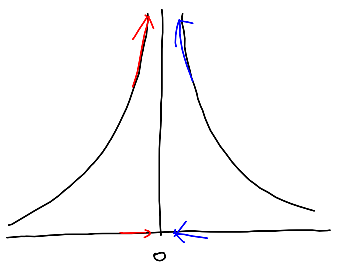
Left: $\lim\limits_{x \to 0^-} \dfrac{7-x}{x^2} = \dfrac{7}{0^2} \quad \dfrac{+}{+} = +\infty$

$x < 0$

Right: $\lim\limits_{x \to 0^+} \dfrac{7-x}{x^2} = \dfrac{7}{0^2} \quad \dfrac{+}{+} = +\infty$

$x > 0$

$\therefore \lim\limits_{x \to 0} \dfrac{7-x}{x^2} = +\infty$



$y = \dfrac{7-x}{x^2}$

Vertical Asymptote

Definition. The vertical line $x = a$ is a **vertical asymptote** for the graph of $y = f(x)$ if

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

Locating Vertical Asymptotes of Rational Functions

If $f(x) = \frac{n(x)}{d(x)}$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

$$\frac{\text{none zero}}{0}$$

Ex.9) Find vertical asymptote(s) of $f(x) = \dfrac{x-3}{x^2-4x+3}$.

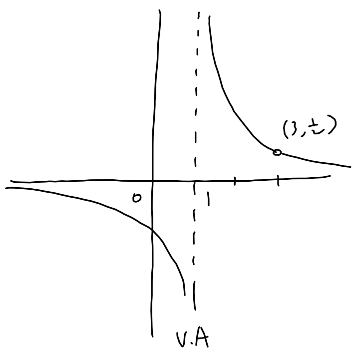
$f(x) = \dfrac{x-3}{(x-3)(x-1)}$: Domain: $\mathbb{R}$ except $x=1, 3$

· V.A: $x = 1$: $\dfrac{\text{none zero}}{\text{zero}}$

· Hole at $x = 3$: $\dfrac{0}{0}$

$$\lim_{x\to 3} \frac{\cancel{x-3}}{\cancel{(x-3)}(x-1)} = \lim_{x\to 3} \frac{1}{x-1} = \frac{1}{(3)-1} = \frac{1}{2}$$

$\therefore$ Hole at $(3, \frac{1}{2})$



Continuity

A function f is **continuous** at the point $x = a$ if all of the following are true:

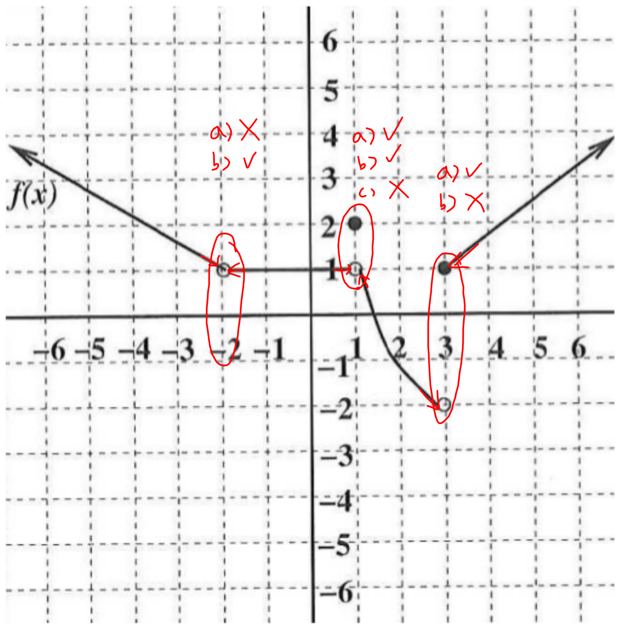
(a) $f(a)$ is defined

(b) $\lim_{x \to a} f(x)$ exists.

(c) $\lim_{x \to a} f(x) = f(a)$

If one or more of the conditions are not met, we say f is discontinuous at $x = a$.

Ex.11) For what values of x is the function discontinous? Explain.



Definition. A function is **continous on an interval** if it is continuous at each point on the interval.

Ex.12) Find the intervals on which the following functions are continuous.

a) $f(x) = 3x^9 + 4x^5 - 7$

$(-\infty, \infty)$

b) $g(x) = 2^{3x} + 4$

$(-\infty, \infty)$

c) $h(x) = \log_8 (\overset{>0}{x-3}) - 2$

$x - 3 > 0$

$\therefore x > 3$

$\therefore (3, \infty)$

d) $k(x) = \dfrac{x+2}{x^2 - 3x - 10}$

$x^2 - 3x - 10 = 0$

$\Rightarrow (x-5)(x+2) = 0$

$\therefore x = 5, -2$

$\therefore (-\infty, -2), (-2, 5), (5, \infty)$

$$\text{e) } m(x) = \begin{cases} \dfrac{x-5}{x+2} & \text{if } x \leq -3 \\ 2x^2 & \text{if } -3 < x \leq 0 \\ \dfrac{x}{x-4} & \text{if } x > 0 \end{cases}$$

i) $x < -3$

$m(x) = \dfrac{x-5}{x+2}$

since $x = -2$ is not in $x < 3$,

$\therefore$ $m(x)$ is conti. in $x < -3$

ii) At $x = -3$

① $m(-3) = \dfrac{(-3)-5}{(-3)+2} = \dfrac{-8}{-1} = 8$

② Left: $\lim\limits_{\substack{x \to -3^- \\ x < -3}} m(x) = \lim\limits_{x \to -3^-} \dfrac{x-5}{x+2} = \dfrac{(-3)-5}{(-3)+2} = \dfrac{-8}{-1} = 8$

$\neq$

Right: $\lim\limits_{\substack{x \to -3^+ \\ x > -3}} m(x) = \lim\limits_{x \to -3^+} 2x^2 = 2(-3)^2 = 18$

$\therefore$ $m(x)$ is NOT conti. at $x = -3$

iii) $-3 < x < 0$

$m(x) = 2x^2$

$\therefore$ m is conti. on $-3 < x < 0$

iv) At $x = 0$

① $m(0) = 2(0)^2 = 0$

② Left: $\lim\limits_{\substack{x \to 0^- \\ x < 0}} m(x) = \lim\limits_{x \to 0^-} 2x^2 = 2(0)^2 = 0$

$\parallel$

Right: $\lim\limits_{\substack{x \to 0^+ \\ x > 0}} m(x) = \lim\limits_{x \to 0^+} \dfrac{x}{x-4} = \dfrac{0}{0-4} = \dfrac{0}{-4} = 0$

$\therefore \lim\limits_{x \to 0} m(x) = 0$

③ $m(0) = 0 = \lim\limits_{x \to 0} m(x)$

$\therefore$ $m(x)$ is conti. at $x = 0$

v) $x > 0$

$m(x) = \dfrac{x}{x-4}$

$\therefore$ $m(x)$ is Not conti. at $x = 4$

$\therefore$ $m(x)$ is conti. on $(0, 4), (4, \infty)$

$\therefore$ $m(x)$ is NOT conti. at $x = -3, 4$

or $(-\infty, -3), (-3, 4), (4, \infty)$

Ex.13) Find the value(s) of k that make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} 2x-5 & \text{if } x \leq 1 \\ x^2+k & \text{if } x > 1 \end{cases}$$

At $x=1$

① $f(1) = 2(1) - 5 = -3$

② Left : $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x-5 = 2(1) - 5 = -3$

$x<1$

Right: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2+k = (1)^2 + k = 1+k$

$x>1$

$\Rightarrow -3 = 1+k$

$\therefore k = -4$

$\therefore \lim_{x \to 1} f(x) = -3$

③ $f(1) = -3 = \lim_{x \to 1} f(x)$

$\therefore f(x)$ is continuous at $x=1$ (everywhere) if $k=-4$.

Ex.14) Find the value(s) of A that make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ Ax^2 - 6x + 10 & \text{if } x \geq 2 \end{cases}$$

At $x=2$

① $f(2) = A(2)^2 - 6(2) + 10 = 4A - 12 + 10 = 4A - 2$

② Left: $\lim_{x \to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{x^2-4}{x-2} \; \left(= \tfrac{0}{0}\right) = \lim_{x\to 2^-} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x\to 2^-} \dfrac{x+2}{1} = (2)+2 = 4$

$x<2$

Right: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} Ax^2 - 6x + 10 = A(2)^2 - 6(2) + 10 = 4A - 2$

$x>2$

$\Rightarrow \; 4 = 4A - 2$

$\Rightarrow \; 6 = 4A$

$\therefore \; A = \dfrac{6}{4} = \dfrac{3}{2}$

$\therefore \lim_{x\to 2} f(x) = 4$ if $A = \frac{3}{2}$

③ $f(2) = 4\left(\frac{3}{2}\right) - 2 = 4 = \lim_{x\to 2} f(x)$ ✓

$\therefore$ $f(x)$ is conti. at $x=2$ (everywhere) if $A = \frac{3}{2}$

Week 3 Section 3.2, 3,3 Rates of Change, The Derivative

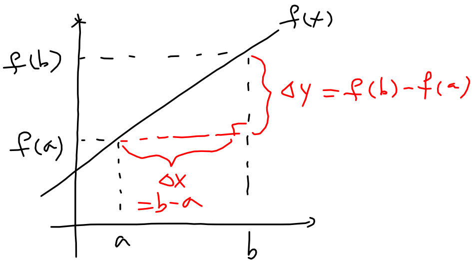
Section 3.2 Rates of Change

Average Rate of Change

The average rate of change of $y = f(x)$ with respect to x from a to b is the quotient

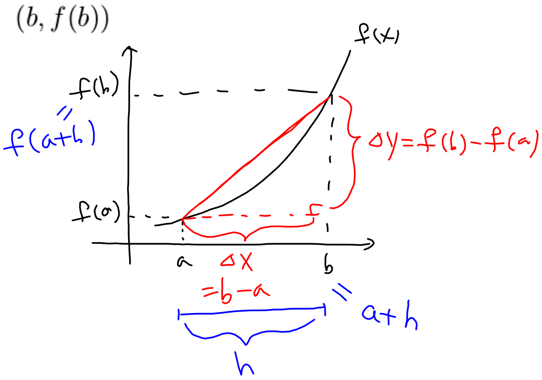
$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

For a linear function, the average rate of change is the slope of line



$$\text{slope of line} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

For a nonlinear function, the average rate of change is the slope of the secant line from $(a, f(a))$ to $(b, f(b))$



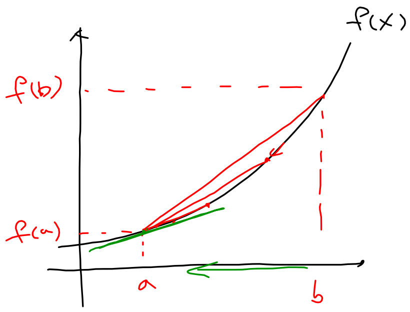
slope of secant line

$$= \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$= \frac{f(a+h) - f(a)}{h}$$: Difference Quotient.

Then we let $(b, f(b))$ approach $(a, f(a))$ along the curve by letting b approach a.



Definition. Instantaneous rate of change

Given a function $y = f(x)$, the instantaneous rate of change of y with respect to x at $x = a$ is given by

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

or

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$: slope of tangent line

$= f'(a)$: derivative.

if these limits exist.

Definition. Average and Instantaneous Velocity

Suppose $s = s(t)$ describes the position of an object at time t. The **average velocity** from a to $a+h$ is

$$\text{average velocity} = \cancel{\frac{s(a+h)-s(a)}{h}}$$

The **instantaneous velocity** (or simply velocity) $v(a)$ at time a is

$$v(a) = \lim_{h\to 0}(\text{average velocity}) = \lim_{h\to 0}\frac{s(a+h)-s(a)}{h}$$

if this limit exists.

Ex.1) Find the slope of the tangent line to the graph of $f(x) = x^2 + 2x$ at the point $(1, 3)$.

$a = 1$

· slope of tangent: $\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h\to 0} \frac{\left((1+h)^2 + 2(1+h)\right) - \left((1)^2 + 2(1)\right)}{h}$ $\quad (1+h)^2 = 1+2h+h^2$

$= \lim_{h\to 0} \frac{\cancel{1} + 2h + h^2 \cancel{+2} + 2h \cancel{-3}}{h}$

$= \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$

$= \lim_{h\to 0} \frac{4+h}{1} = 4 + (0) = 4$: slope of tangent line

point: $(1, 3)$

$y - y_1 = m(x - x_1)$

$\Rightarrow y - 3 = 4(x-1)$

$\Rightarrow y - 3 = 4x - 4$

$\boxed{\therefore\ y = 4x - 1}$