Week 3 Section 3.2, 3,3 Rates of Change, The Derivative

Section 3.2 Rates of Change

Average Rate of Change

The average rate of change of $y = f(x)$ with respect to x from a to b is the quotient

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

For a linear function, the average rate of change is the slope of line

For a nonlinear function, the average rate of change is the slope of the secant line from $(a, f(a))$ to $(b, f(b))$

Then we let $(b, f(b))$ approach $(a, f(a))$ along the curve by letting b approach a.

Definition. Instantaneous rate of change

Given a function $y = f(x)$, the instantaneous rate of change of y with respect to x at $x = a$ is given by

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

or

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if these limits exist.

Definition. Average and Instantaneous Velocity

Suppose $s = s(t)$ describes the position of an object at time t. The **average velocity** from a to $a + h$ is

$$\text{average velocity} = \cancel{\frac{s(a+h) - s(a)}{h}} \qquad \frac{s(b)-s(a)}{b-a}$$

The **instantaneous velocity** (or simply velocity) $v(a)$ at time a is

$$v(a) = \lim_{h\to 0} (\text{average velocity}) = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h}$$

if this limit exists.

Ex.1) Find the slope of the tangent line to the graph of $f(x) = x^2 + 2x$ at the point $(1, 3)$.

Ex.2) If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $\underset{s(t)=}{y} = 10t - 1.86t^2$.

a) Find the average velocity over the given time intervals:

i) $[1, 1.1]$ $(1, s(1)) = (1, 8.14)$

$(1.1, s(1.1)) = (1.1, 8.7494)$

$$\text{ave. vel.} = \frac{\Delta s}{\Delta t} = \frac{8.7494 - 8.14}{1.1 - 1} = 6.094 \text{ m/s}$$

ii) $[1, 1.01]$ $(1, s(1)) = (1, 8.14)$

$(1.01, s(1.01)) = (1.01, 8.2026)$

$$\text{ave. vel.} = \frac{\Delta s}{\Delta t} = \frac{8.2026 - 8.14}{1.01 - 1} = 6.2614 \text{ m/s}$$

iii) $[1, 1.001]$ $(1, s(1)) = (1, 8.14)$

$(1.001, s(1.001)) = (1.001, 8.1463)$

$$\text{ave. vel.} = \frac{\Delta s}{\Delta t} = \frac{8.1463 - 8.14}{1.001 - 1} = 6.27814 \text{ m/s}$$

b) Use part a) to estimate the instantaneous velocity when $t = 1$.

6.27814 m/s

c) Find the exact value of the instantaneous velocity when $t = 1$.

$$\text{Instantaneous velocity} = \lim_{h\to 0} \frac{s(1+h) - s(1)}{h}$$

$$= \lim_{h\to 0} \frac{\left(10(1+h) - 1.86\underbrace{(1+h)^2}_{=1+2h+h^2}\right) - \underbrace{\left(10(1) - 1.86(1)^2\right)}_{=8.14}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{10} + 10h - \cancel{1.86} - 3.72h - 1.86h^2 - \cancel{8.14}}{h}$$

$$= \lim_{h\to 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(6.28 - 1.86h)}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{6.28 - 1.86h}{1} = 6.28 - 1.86(0) = 6.28 \text{ m/s}$$

Ex.3) Estimate the instantaneous rate of change of $f(x) = x^2$ at $x = 1$, and find exactly.

$[1, 1.1]$

$(1, f(1)) = (1, 1)$

$(1.1, f(1.1)) = (1, 1.21)$

ave. rate of change

$= \frac{\Delta y}{\Delta x} = \frac{1.21 - 1}{1.1 - 1} = 2.1$

$[1, 1.01]$

$(1, f(1)) = (1, 1)$

$(1.01, f(1.01)) = (1.01, 1.0201)$

ave. rate of change

$= \frac{\Delta y}{\Delta x} = \frac{1.0201 - 1}{1.01 - 1} = 2.01$

Instantaneous rate of change

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 \cancel{- 1}}{h}$$

$$= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{2+h}{1} = 2 + (0) = 2$$

Equation of tangent line

① point: (x_1, y_1)

② slope of tangent line $= m$

$$y - y_1 = m(x - x_1)$$

Ex.4) Find the equation of the tangent line of $f(x) = \sqrt{5+x}$ at $x = 20$.

① point: $\left(20, f(20)\right) = \left(20, \sqrt{5+20}\right) = (20, 5)$

② slope of tangent line

$$= \lim_{h\to 0} \frac{f(20+h) - f(20)}{h} = \lim_{h\to 0} \frac{\sqrt{5+(20+h)} - \sqrt{5+(20)}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{25+h} - 5\right)}{h} \frac{\left(\sqrt{25+h} + 5\right)}{\left(\sqrt{25+h} + 5\right)} \leftarrow (a-b)(a+b) = a^2 - b^2$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{25+h}\right)^2 - (5)^2}{h\left(\sqrt{25+h} + 5\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{25} + h \cancel{-25}}{h\left(\sqrt{25+h} + 5\right)} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{25+h} + 5\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{25+h} + 5} = \frac{1}{\sqrt{25+0} + 5} = \frac{1}{\sqrt{25} + 5} = \frac{1}{10} = m$$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 5 = \frac{1}{10}(x - 20)$$

$$\Rightarrow y - 5 = \frac{1}{10}x - 2$$

$$\boxed{\therefore y = \frac{1}{10}x + 3}$$

Ex.5) For the function shown below, at what labeled points is the instantaneous rate of change

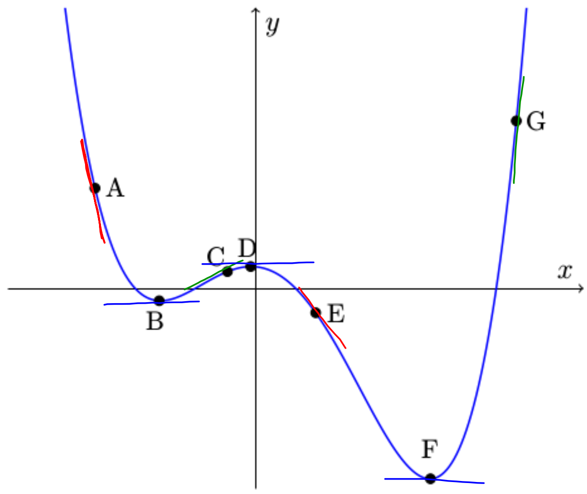
a) positive : C, G

e) most positive : G

b) negative : A, E

c) zero : B, D, F.

d) most negative A



Section 3.3 The Derivative

Definition. Derivative

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

Interpretations of the Derivative

The derivative has various applications and interpretations, including the following;

(a) **Slope of the tangent line:** $f'(a)$ is the slope of the line tangent to the graph of f at the point $(a, f(a))$.

(b) **Instantaneous rate of change:** $f'(a)$ is the instantaneous rate of change of $y = f(x)$ at $x = a$.

(c) **Velocity:** If $f(x)$ is the position of a moving object at time x, then $v = f'(a)$ is the velocity of the object at time $x = a$.

Ex.6) Given $f(x) = x^2 - 8x + 9$, find the derivative of f at $x = -2$ (use the definition) and the equation of tangent line at $x = -2$.

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\left(\underbrace{(-2+h)^2}_{=4-4h+h^2} - 8(-2+h) + 9\right) - \left((-2)^2 - 8(-2) + 9\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4} - 4h + h^2 \cancel{+16} - 8h \cancel{+9} \cancel{-29}}{h}$$

$$= \lim_{h\to 0} \frac{-12h + h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(-12+h)}{\cancel{h}} = \lim_{h\to 0} \frac{(-12+h)}{1}$$

$$= -12 + (0) = -12$$

. tangent line

① point : $(-2, f(-2)) = (-2, 29)$

② slope of tangent line $= -12$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 29 = -12(x - (-2))$$

$$\Rightarrow y - 29 = -12x - 24$$

$$\boxed{\therefore y = -12x + 5}$$

Definition. Derivative of function

The derivative of f is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Other Notations for the derivative of $y = f(x)$: $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x)$

: Lebniz form

Ex.7) Find $f'(x)$ if $f(x) = \frac{1-x}{2+x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left(\frac{1-(x+h)}{2+(x+h)}\right) - \left(\frac{1-x}{2+x}\right)}{h}$$

$$= \lim_{h\to 0} \left(\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}\right) \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{(1-x-h)}{(2+x+h)} \cdot \frac{(2+x)}{(2+x)} - \frac{(1-x)}{(2+x)} \cdot \frac{(2+x+h)}{(2+x+h)}\right) \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{2+x-2x-x^2-2h-xh}{(2+x+h)(2+x)} - \frac{(2+x+h-2x-x^2-xh)}{(2+x+h)(2+x)}\right) \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{\cancel{2}+\cancel{x}-\cancel{2x}-\cancel{x^2}-2h-\cancel{xh}-\cancel{2}-\cancel{x}-h+\cancel{2x}+\cancel{x^2}+\cancel{xh}}{(2+x+h)(2+x)}\right) \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{(2+x+h)(2+x)} \cdot \frac{1}{\cancel{h}} = \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)}$$

$$= \frac{-3}{(2+x+(0))(2+x)} = \frac{-3}{(2+x)(2+x)}$$

$$= \frac{-3}{(2+x)^2}$$

$$\therefore f'(x) = \frac{-3}{(2+x)^2}$$

Definition. Differentiable

A function f is differentiable at a if $f'(a)$ exists.

Theorem

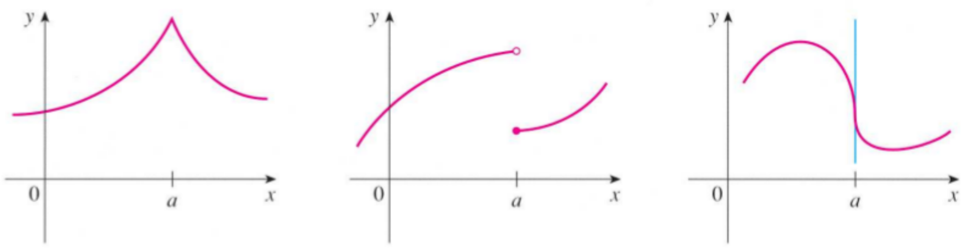
If f is differentiable at a, then f is continuous at a.

How can a function fail to be differentiable?

(a) If the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at that point and f is not differentiable there.

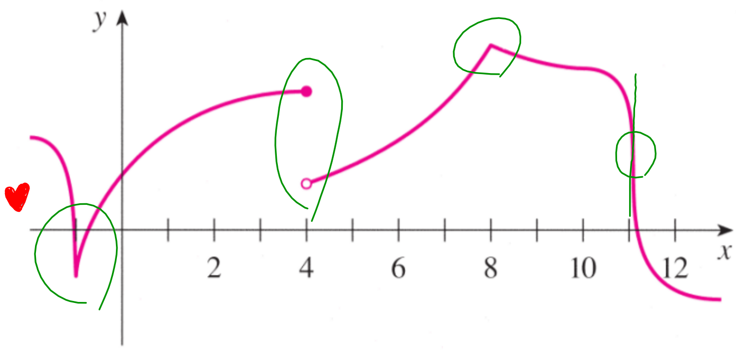
(b) If f is not continuous at a, then f is not differentiable at a.

(c) The curve has a vertical tangent line when $x = a$, f is not differentiable at a.

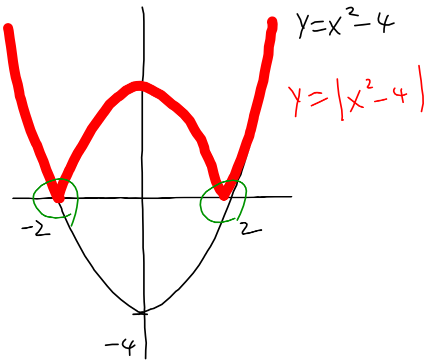


(a) A corner (b) A discontinuity (c) A vertical tangent

Ex.8) The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



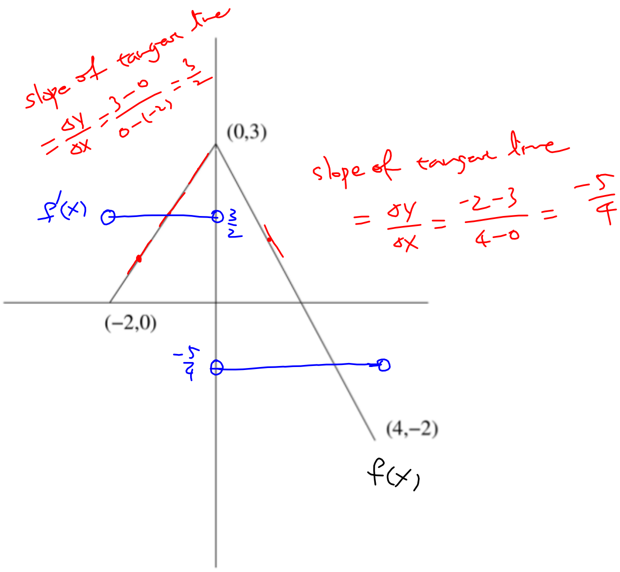
Ex.9) Where is $f(x) = |x^2 - 4|$ not differentiable.



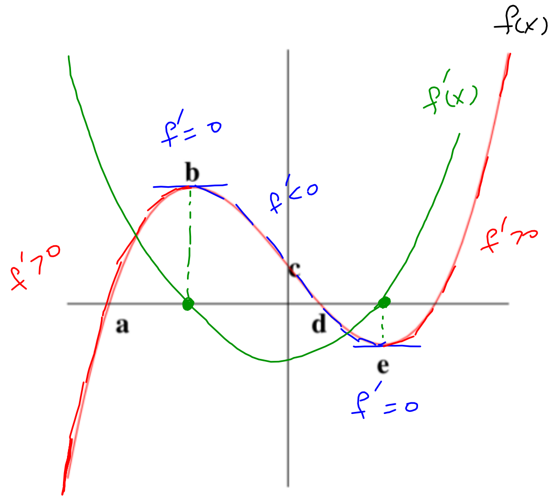
$\therefore$ f is NOT differentiable at $x = -2, 2$

(1) When $f(x)$ is decreasing, $f'(x) < 0$. (2) When $f(x)$ is increasing, $f'(x) > 0$. (3) When $f(x)$ is constant, $f'(x) = 0$.

Ex.10) Given the graph of $f(x)$ below, sketch the graph of the derivative.



a)



b)