MATH 142 Business Calculus

Fall 2019, WEEK 3

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Week 3 Section 3.2, 3,3 Rates of Change, The Derivative

Section 3.2 Rates of Change

Average Rate of Change

The average rate of change of $y = f(x)$ with respect to x from a to b is the quotient

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

For a linear function, the average rate of change is the slope of line

For a nonlinear function, the average rate of change is the slope of the secant line from $(a, f(a))$ to $(b, f(b))$

Then we let $(b, f(b))$ approach $(a, f(a))$ along the curve by letting b approach a.

Definition. Instantaneous rate of change

Given a function $y = f(x)$, the instantaneous rate of change of y with respect to x at $x = a$ is given by

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

or

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if these limits exist.

Ex.1) Find the slope of the tangent line to the graph of $f(x) = x^2 + 2x$ at the point $(1, 3)$.

Definition. Average and Instantaneous Velocity

Suppose $s = s(t)$ describes the position of an object at time t. The **average velocity** from a to $a + h$ is

$$\text{average velocity} = \frac{s(b) - s(a)}{b - a}$$

The **instantaneous velocity** (or simply velocity) $v(a)$ at time a is

$$v(a) = \lim_{h \to 0} (\text{average velocity}) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

if this limit exists.

Ex.2) If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.

a) Find the average velocity over the given time intervals:

i) $[1, 1.1]$

ii) $[1, 1.01]$

iii) $[1, 1.001]$

b) Use part a) to estimate the instantaneous velocity when $t = 1$.

c) Find the exact value of the instantaneous velocity when $t = 1$.

Ex.3) Estimate the instantaneous rate of change of $f(x) = x^2$ at $x = 1$, and find exactly.

Ex.4) Find the equation of the tangent line of $f(x) = \sqrt{5+x}$ at $x = 20$.

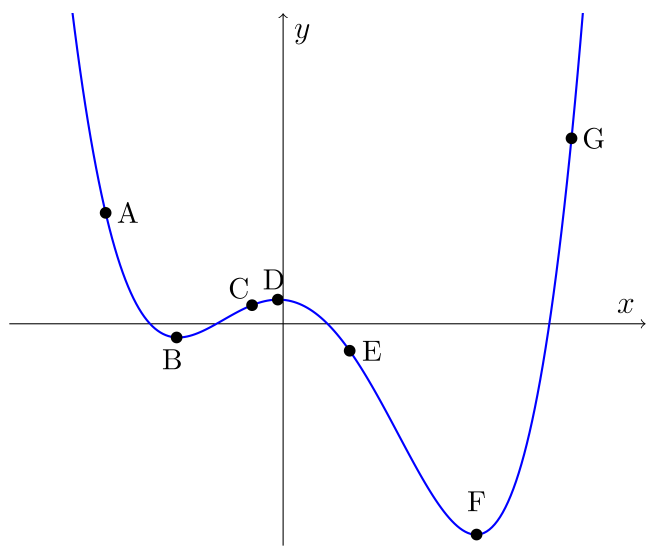
Ex.5) For the function shown below, at what labeled points is the instantaneous rate of change

a) positive

b) negative

c) zero

d) most negative



Section 3.3 The Derivative

Definition. Derivative

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

Interpretations of the Derivative

The derivative has various applications and interpretations, including the following;

(a) **Slope of the tangent line:** $f'(a)$ is the slope of the line tangent to the graph of f at the point $(a, f(a))$.

(b) **Instantaneous rate of change:** $f'(a)$ is the instantaneous rate of change of $y = f(x)$ at $x = a$.

(c) **Velocity:** If $f(x)$ is the position of a moving object at time x, then $v = f'(a)$ is the velocity of the object at time $x = a$.

Ex.6) Given $f(x) = x^2 - 8x + 9$, find the derivative of f at $x = -2$ (use the definition) and the equation of tangent line at $x = -2$.

Definition. Derivative of function

The derivative of f is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Other Notations for the derivative of $y = f(x)$: $f'(x) = y' = \dfrac{dy}{dx} = \dfrac{df}{dx} = \dfrac{d}{dx}f(x)$

Ex.7) Find $f'(x)$ if $f(x) = \dfrac{1-x}{2+x}$

Definition. Differentiable

A function f is differentiable at a if $f'(a)$ exists.

Theorem

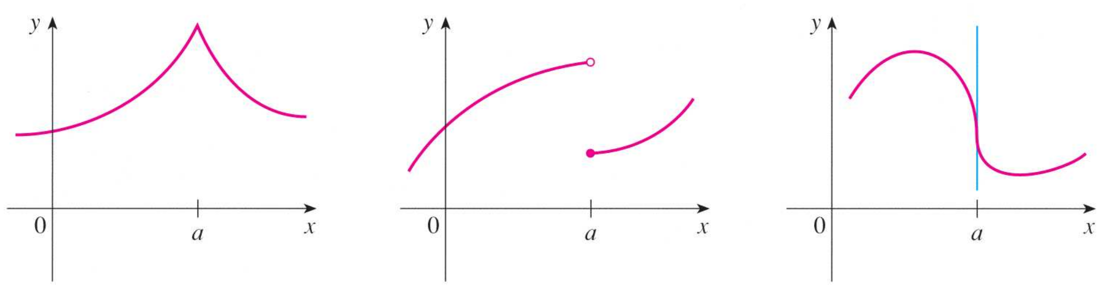
If f is differentiable at a, then f is continuous at a.

<u>How can a function fail to be differentiable?</u>

(a) If the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at that point and f is not differentiable there.

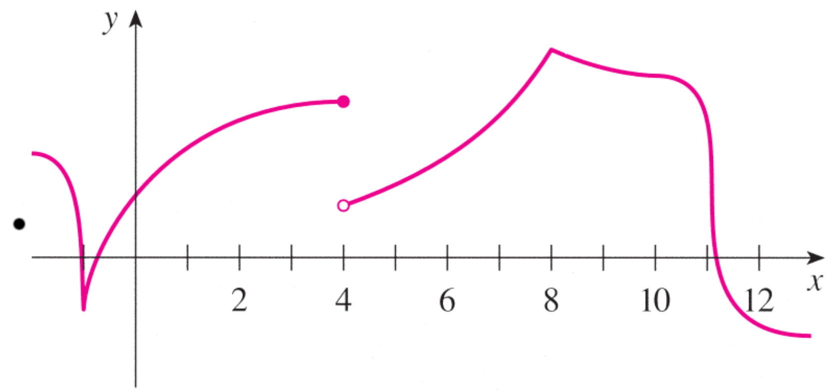
(b) If f is not continuous at a, then f is not differentiable at a.

(c) The curve has a vertical tangent line when $x = a$, f is not differentiable at a.



(a) A corner (b) A discontinuity (c) A vertical tangent

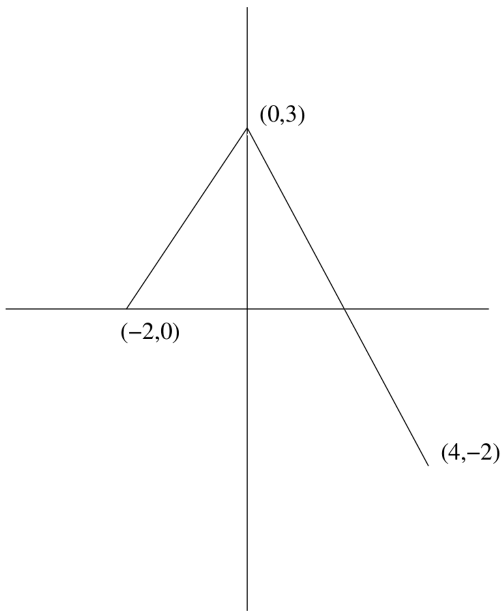
Ex.8) The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



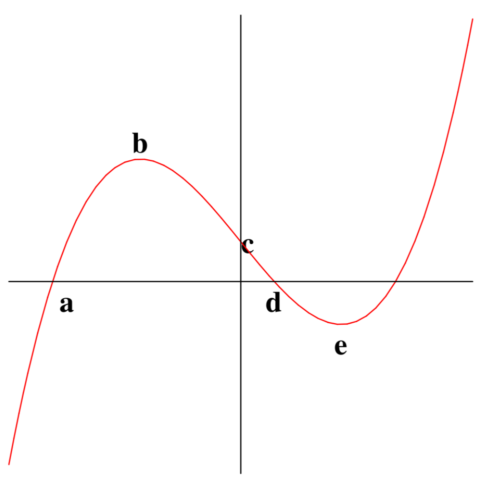
Ex.9) Where is $f(x) = |x^2 - 4|$ not differentiable.

(1) When $f(x)$ is decreasing, $f'(x) < 0$.
(2) When $f(x)$ is increasing, $f'(x) > 0$.
(3) When $f(x)$ is constant, $f'(x) = 0$.

Ex.10) Given the graph of $f(x)$ below, sketch the graph of the derivative.



a)



b)