Section 4.2 Derivatives of Products and Quotients

Product Rule

If $h(x) = f(x) \cdot g(x)$ and if $f'(x)$ and $g'(x)$ exist, then

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

ex) $\left((x^3)(x^2+4x)\right)' = 3x^2 \cdot (x^2+4x) + (x^3) \cdot (2x+4)$

Quotient Rule

If $h(x) = \dfrac{f(x)}{g(x)}$ and if $f'(x)$ and $g'(x)$ exist, then

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

ex) $\left(\dfrac{x^4}{x^2+5}\right)' = \dfrac{(4x^3)(x^2+5) - (x^4) \cdot (2x)}{(x^2+5)^2}$

Ex.8) Find the derivative of the following functions:

a) $h(x) = \underset{u}{x^2}\underset{v}{(x^2+4x)}$

$u = x^2$ $\quad v = x^2+4x$

$u' = 2x$ $\quad v' = 2x+4$

$h'(x) = u' \cdot v + u \cdot v'$

$= (2x) \cdot (x^2+4x) + (x^2) \cdot (2x+4)$

b) $h(x) = \underset{u}{(x^2+3)}\underset{v}{(\sqrt[4]{x} + \sqrt[8]{x^3})}$

$u = x^2+3$ $\quad v = x^{\frac{1}{4}} + x^{\frac{3}{8}}$

$u' = 2x$ $\quad v' = \frac{1}{4}x^{\frac{1}{4}-1} + \frac{3}{8}x^{\frac{3}{8}-1}$

$= \frac{1}{4}x^{-\frac{3}{4}} + \frac{3}{8}x^{-\frac{5}{8}}$

$h'(x) = u' \cdot v + u \cdot v'$

$= (2x) \cdot (x^{\frac{1}{4}} + x^{\frac{3}{8}}) + (x^2+3) \cdot \left(\frac{1}{4}x^{-\frac{3}{4}} + \frac{3}{8}x^{-\frac{5}{8}}\right)$

c) $h(x) = \dfrac{x^2+5}{3x}$ $=u$, $=v$

$u = x^2+5$ $\quad v = 3x$

$u' = 2x$ $\quad v' = 3$

$$h'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \frac{(2x)\cdot(3x) - (x^2+5)\cdot(3)}{(3x)^2}$$

d) $h(x) = \dfrac{3\sqrt{x}+7x}{x^2-4x+\frac{1}{x}}$ $=u$, $=v$

$u = 3x^{\frac{1}{2}} + 7x$ $\quad v = x^2 - 4x + x^{-1}$

$u' = \frac{3}{2}x^{-\frac{1}{2}} + 7$ $\quad v' = 2x - 4 - x^{-1-1}$

$= 2x - 4 - x^{-2}$

$$h'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \frac{(\frac{3}{2}x^{-\frac{1}{2}}+7)\cdot(x^2-4x+x^{-1}) - (3x^{\frac{1}{2}}+7x)\cdot(2x-4-x^{-2})}{(x^2-4x+x^{-1})^2}$$

e) $f(x) = 5x^4 e^x$ (u, v)

$u = 5x^4$ $\quad v = e^x$

$u' = 20x^3$ $\quad v' = e^x$

$$f'(x) = u' \cdot v + u \cdot v'$$

$$= (20x^3)\cdot e^x + 5x^4 \cdot e^x$$

f) $g(x) = \dfrac{x^2e^x + 5}{7 - e^x}$ $\quad u = x^2e^x+5$, $\quad v = 7-e^x$

$u = x^2 \cdot e^x + 5$

$u' = 2x \cdot e^x + x^2 \cdot e^x + 0$

$= 2x \cdot e^x + x^2 e^x$

$v = 7 - e^x$

$v' = 0 - e^x$

$= -e^x$

$$g'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \frac{(2x \cdot e^x + x^2 \cdot e^x) \cdot (7 - e^x) - (x^2 \cdot e^x + 5) \cdot (-e^x)}{(7 - e^x)^2}$$

$(e^x)' = e^x$ $\qquad (\ln x)' = \frac{1}{x}$

$(a^x)' = a^x \cdot \ln a$ $\qquad (\log_a x)' = \frac{1}{x \cdot \ln a}$

g) $h(x) = \dfrac{2^x + 5x^2}{\log_2 x^4 - \ln x}$ $\quad u = 2^x+5x^2$, $\quad v = \log_2 x^4 - \ln x$

$u = 2^x + 5x^2$

$u' = 2^x \cdot \ln 2 + 10x$

$v = \log_2 x^4 - \ln x$

$= 4 \log_2 x - \ln x$

$v' = 4 \frac{1}{x \cdot \ln 2} - \frac{1}{x}$

$= \frac{4}{x \cdot \ln 2} - \frac{1}{x}$

$$h'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \frac{(2^x \cdot \ln 2 + 10x) \cdot (\log_2 x^4 - \ln x) - (2^x + 5x^2) \cdot \left(\frac{4}{x \ln 2} - \frac{1}{x}\right)}{(\log_2 x^4 - \ln x)^2}$$

Ex.9) Find the equation of the tangent line to $f(x) = \frac{\underbrace{x^2+1}_{=u}}{\underbrace{3x^3-4x^2+2}_{=v}}$ at $x = 2$.

① point: $(2, f(2)) = (2, \frac{1}{2})$

$u = x^2+1$ $\quad v = 3x^3 - 4x^2 + 2$

$u' = 2x$ $\quad v' = 9x^2 - 8x$

② slope of tangent line $= f'(2)$

$$f'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \frac{(2x)\cdot(3x^3-4x^2+2) - (x^2+1)\cdot(9x^2-8x)}{(3x^3-4x^2+2)^2}$$

$$\Rightarrow f'(2) = \frac{(4)\cdot(10) - (5)(20)}{(10)^2} = \frac{-60}{100} = -\frac{3}{5}$$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - \frac{1}{2} = -\frac{3}{5}(x-2)$$

$$\Rightarrow y - \frac{1}{2} = -\frac{3}{5}x + \frac{6}{5}$$

$$\boxed{\therefore y = -\frac{3}{5}x + \frac{17}{10}}$$

Ex.10) Suppose that $f(2) = -1$, $g(2) = 3$, $f'(2) = -4$, and $g'(2) = 6$. Find $h'(2)$ for each of the following:

a) $h(x) = 2f(x) - 3g(x)$

$$h'(x) = 2f'(x) - 3g'(x)$$

$$\Rightarrow h'(2) = 2\underbrace{f'(2)}_{=-4} - 3\underbrace{g'(2)}_{=6} = 2(-4) - 3\cdot(6)$$

$$= -26$$

b) $h(x) = f(x)g(x)$

$$h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$\Rightarrow h'(2) = f'(2)\cdot g(2) + f(2)\cdot g'(2)$$

$$= (-4)\cdot(3) + (-1)\cdot(6) = -18$$

c) $h(x) = \dfrac{f(x)}{g(x)}$

$$h'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

$$\Rightarrow h'(2) = \frac{f'(2)\cdot g(2) - f(2)\cdot g'(2)}{(g(2))^2} = \frac{(-4)\cdot(3) - (-1)\cdot(6)}{(3)^2} = \frac{-12+6}{9} = \frac{-6}{9}$$

$$= -\frac{2}{3}$$

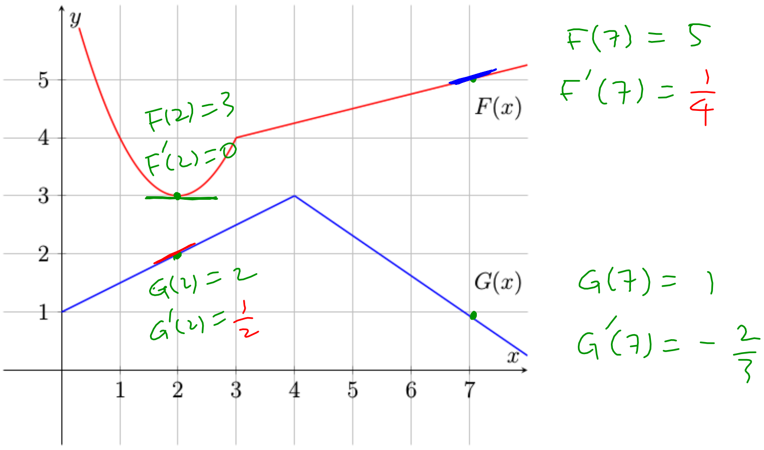
d) $h(x) = \dfrac{f(x)}{1+g(x)}$

$$h'(x) = \frac{f'(x)\cdot(1+g(x)) - f(x)\cdot g'(x)}{(1+g(x))^2}$$

$$\Rightarrow h'(2) = \frac{f'(2)\cdot(1+g(2)) - f(2)\cdot g'(2)}{(1+g(2))^2} = \frac{(-4)(1+3) - (-1)\cdot(6)}{(1+3)^2}$$

$$= \frac{-16+6}{16} = \frac{-10}{16} = \frac{-5}{8}$$

Ex.11) Let $P(x) = F(x)G(x)$ and $Q(x) = \dfrac{F(x)}{G(x)}$, where F and G are the functions whose graphs are shown below.



a) Find $P'(2)$

$$P'(x) = F'(x)\cdot G(x) + F(x)\cdot G'(x)$$

$$\Rightarrow P'(2) = F'(2)\cdot G(2) + F(2)\cdot G'(2)$$

$$= (0)\cdot(2) + (3)\cdot\left(\tfrac{1}{2}\right) = \frac{3}{2}$$

b) Find $Q'(7)$

$$Q'(x) = \frac{F'(x)\cdot G(x) - F(x)\cdot G'(x)}{(G(x))^2}$$

$$\Rightarrow Q'(7) = \frac{F'(7)\cdot G(7) - F(7)\cdot G'(7)}{(G(7))^2} = \frac{\left(\frac{1}{4}\right)(1) - (5)\cdot\left(-\frac{2}{3}\right)}{(1)^2}$$

$$= \frac{1}{4} + \frac{10}{3} = \frac{43}{12}$$