Section 4.3 The Chain Rule

The Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F(x) = f(g(x))$ is differentiable at x and is given by

$$F'(x) = f'(g(x)) \cdot g'(x)$$

ex) $\left((x^3+4)^4\right)' = 4(x^3+4)^3 \cdot (3x^2)$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

General Derivative Rules

- If $y = [f(x)]^n$ then $y' = n[f(x)]^{n-1} \cdot f'(x)$
- If $y = e^{f(x)}$ then $y' = e^{f(x)} \cdot f'(x)$

$(e^x)' = e^x$

ex) $\left(e^{(x^2+1)}\right)' = e^{(x^2+1)} \cdot (2x)$

- If $y = \ln(f(x))$ then $y' = \frac{1}{f(x)} \cdot f'(x)$

$(\ln x)' = \frac{1}{x}$

ex) $\left(\ln(x^3+2x)\right)' = \frac{1}{(x^3+2x)} \cdot (3x^2+2)$

$(a^x)' = a^x \cdot \ln a$

- If $y = b^{f(x)}$ then $y' = b^{f(x)} \cdot \ln b \cdot f'(x)$

ex) $\left(2^{(x^3)}\right)' = 2^{(x^3)} \cdot \ln 2 \cdot 3x^2$

- If $y = \log_b f(x)$ then $y' = \frac{1}{f(x) \cdot \ln b} \cdot f'(x)$

$(\log_b x)' = \frac{1}{x \cdot \ln b}$

ex) $\left(\log_7(2x^4+x^3)\right)'$

$= \frac{1}{(2x^4+x^3) \cdot \ln 7} \cdot (8x^3+3x^2)$

Ex.1) Differentiate the following:

a) $f(x) = (4x^2 + 7x)^5$

$$\frac{df}{dx} = f'(x) = 5(4x^2+7x)^4 \cdot (8x+7)$$

b) $g(x) = 6(x^{\frac{1}{2}} - 3x)^4$

$$\frac{dg}{dx} = g'(x) = 24(x^{\frac{1}{2}}-3x)^3 \cdot \left(\frac{1}{2}x^{\frac{1}{2}-1}-3\right)$$

$$= 24(x^{\frac{1}{2}}-3x)^3\left(\frac{1}{2}x^{-\frac{1}{2}}-3\right)$$

c) $y = \dfrac{3}{(t^2 + 3t + 4)^4} = 3(t^2+3t+4)^{-4}$

$$\frac{dy}{dt} = y' = -12(t^2+3t+4)^{-5} \cdot (2t+3)$$

d) $F(x) = e^{(x^2)}$

$$\frac{dF}{dx} = F'(x) = e^{(x^2)} \cdot (2x)$$

$u = 3x^5 \qquad v = e^{(x^4)}$

$u' = 15x^4 \qquad v' = e^{(x^4)} \cdot (4x^3)$

$= 4 \cdot x^3 \cdot e^{x^4}$

e) $H(x) = \underbrace{3x^5}_{u=}\underbrace{e^{x^4}}_{=v}$

$$\frac{dH}{dx} = H'(x) = u' \cdot v + u \cdot v'$$

$$= (15x^4) \cdot e^{x^4} + 3x^5 \cdot 4x^3 \cdot e^{x^4} = 15x^4 \cdot e^{x^4} + 12x^8 \cdot e^{x^4}$$

f) $h(x) = \dfrac{3}{\sqrt[4]{(3x+2)^5}} = 3(3x+2)^{-\frac{5}{4}}$

$$\frac{dh}{dx} = h'(x) = 3 \cdot \left(-\frac{5}{4}\right)(3x+2)^{-\frac{5}{4}-1} \cdot (3)$$

$$= -\frac{45}{4}(3x+2)^{-\frac{9}{4}}$$

g) $f(x) = \underbrace{(4x^2+5)^6}_{u=}\underbrace{(\sqrt[3]{(3x^4-5x+7)^4})}_{=v}$

$u = ((4x^2+5))^6$

$u' = 6(4x^2+5)^5 \cdot (8x)$

$= 48x(4x^2+5)^5$

$v = ((3x^4-5x+7))^{\frac{4}{3}}$

$v' = \frac{4}{3}(3x^4-5x+7)^{\frac{4}{3}-1} \cdot (12x^3-5)$

$= \frac{4}{3}(3x^4-5x+7)^{\frac{1}{3}} \cdot (12x^3-5)$

$\frac{df}{dx} = f'(x) = u' \cdot v + u \cdot v'$

$= 48x \cdot (4x^2+5)^5 \cdot (3x^4-5x+7)^{\frac{4}{3}} + (4x^2+5)^6 \cdot \frac{4}{3}(3x^4-5x+7)^{\frac{1}{3}} \cdot (12x^3-5)$

h) $y = \log_8(\underbrace{\sqrt{1+x^2}+10x}_{=u})$

$\frac{dy}{dx} = y' = \frac{1}{(\quad u \quad)\cdot \ln 8} \cdot \frac{du}{dx}$

$u = (1+x^2)^{\frac{1}{2}} + 10x$

$u' = \frac{1}{\cancel{2}}(1+x^2)^{\frac{1}{2}-1} \cdot (\cancel{2}x) + 10$

$= x(1+x^2)^{-\frac{1}{2}} + 10$

$= \frac{1}{(\sqrt{1+x^2}+10x)\cdot \ln 8} \cdot \left(x(1+x^2)^{-\frac{1}{2}} + 10\right)$

i) $y = \log_6\left(\frac{x-1}{x+2}\right)$

$y' = \frac{1}{\left(\frac{x-1}{x+2}\right)\cdot \ln 6} \cdot \left(\frac{x-1}{x+2}\right)'$

$\frac{(1)\cdot(x+2) - (x-1)\cdot(1)}{(x+2)^2}$

$= \frac{x+2-x+1}{(x+2)^2} = \frac{3}{(x+2)^2}$

$= \frac{1}{\left(\frac{x-1}{x+2}\right)\cdot \ln 6} \cdot \left(\frac{3}{(x+2)^2}\right)$

Or using the properties of Log. $\left(\log_a \frac{x}{y} = \log_a x - \log_a y\right)$

$y = \log_6\left(\frac{x-1}{x+2}\right) = \log_6(x-1) - \log_6(x+2)$

$\Rightarrow \frac{dy}{dx} = y' = \frac{1}{(x-1)\cdot \ln 6} \cdot (1) - \frac{1}{(x+2)\cdot \ln 6} \cdot (1)$

$= \frac{1}{(x-1)\ln 6} - \frac{1}{(x+2)\cdot \ln 6}$

$u = 5x^{-3}$ $\quad v = \ln((x^2+x)^5)$

$u' = -15x^{-4}$ $\quad = 5\ln((x^2+x))$

$v' = 5 \cdot \frac{1}{(x^2+x)} \cdot (2x+1)$

$= \frac{10x+5}{x^2+x}$

j) $y = \frac{5\ln((x^2+x)^5)}{x^3} = \underbrace{5 \cdot x^{-3}}_{=u} \cdot \underbrace{\ln((x^2+x)^5)}_{=v}$

$\frac{dy}{dx} = y' = u' \cdot v + u \cdot v'$

$= \left(-15x^{-4}\right) \cdot \left(5\ln(x^2+x)\right) + 5 \cdot x^{-3} \cdot \left(\frac{10x+5}{x^2+x}\right)$

k) $y = \ln\left(\frac{x^4 \cdot (x^2+2)^3}{(2x+4)^2}\right) = \ln(x^4) + \ln(x^2+2)^3 - \ln(2x+4)^2$

$= 4\ln x + 3\ln(x^2+2) - 2\ln(2x+4)$

$\frac{dy}{dx} = y' = 4 \cdot \frac{1}{x} + 3 \cdot \frac{1}{(x^2+2)} \cdot (2x) - 2 \cdot \frac{1}{(2x+4)} \cdot (2) = \frac{4}{x} + \frac{6x}{x^2+2} - \frac{4}{2x+4}$

l) $y = \underbrace{x^3}_{u=} \cdot \underbrace{5^{x^4+2}}_{=v}$

$u = x^3$ $\quad v = 5^{(x^4+2)}$

$u' = 3x^2$ $\quad v' = 5^{(x^4+2)} \cdot (\ln 5) \cdot (4x^3)$

$\frac{dy}{dx} = y' = u' \cdot v + u \cdot v'$

$= (3x^2) \cdot 5^{(x^4+2)} + x^3 \cdot \left(5^{(x^4+2)} \cdot (\ln 5) \cdot 4x^3\right)$

Ex.2) Find the value(s) of x where the tangent line is horizontal for $f(x) = \dfrac{\underbrace{x^2}_{=u}}{\underbrace{(2-3x)^3}_{=v}}$

$f' = 0$

$u = x^2$ $\qquad$ $v = ((2-3x))^3$

$u' = 2x$ $\qquad$ $v' = 3(2-3x)^2 \cdot (-3)$

$\qquad\qquad\qquad = -9(2-3x)^2$

$$f'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \frac{(2x)\cdot(2-3x)^3 - x^2 \cdot (-9(2-3x)^2)}{((2-3x)^3)^2}$$

$$= \frac{2x \cdot (2-3x)^{\cancel{3}\,1} + 9x^2 \cdot \cancel{(2-3x)^2}}{(2-3x)^{\cancel{6}\,4}} = 0$$

$$\Rightarrow \frac{2x \cdot (2-3x) + 9x^2}{(2-3x)^4} = 0$$

$$\Rightarrow 2x(2-3x) + 9x^2 = 0$$

$$\Rightarrow 4x - 6x^2 + 9x^2 = 0$$

$$\Rightarrow 4x + 3x^2 = 0$$

$$\Rightarrow x(4+3x) = 0$$

$$\therefore x = 0, -\frac{4}{3}$$

Ex.3) Suppose $w(x) = u(v(x))$ and $u(0) = 1$, $v(0) = 2$, $u'(0) = 3$, $u'(2) = 4$, $v'(0) = 5$, and $v'(2) = 6$. Find $w'(0)$.

$w'(x) = u'(v(x)) \cdot v'(x)$

$\Rightarrow w'(0) = u'(v(0)) \cdot v'(0)$

$= u'(2) \cdot (5)$

$= (4) \cdot (5) = 20$

Ex.4) Let $y = \ln u$ and $u = 5x^4 + x^6$. Find $\dfrac{dy}{dx}$.