b) $g(x) = \dfrac{2}{x^2 - 16}$

c) $h(x) = (x+2)e^x$ Domain: $\mathbb{R}$

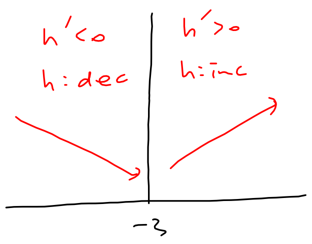
Step 1) C.N

$$h'(x) = (1)\cdot e^x + (x+2)\cdot e^x$$
$$= e^x(1 + x + 2)$$
$$= e^x(x+3) = 0$$

$\neq 0$ $\quad$ 0

$\therefore x = -3$: C.N

Step 2) Sign chart for h'



$\therefore$ dec. $(-\infty, -3)$

inc. $(-3, \infty)$

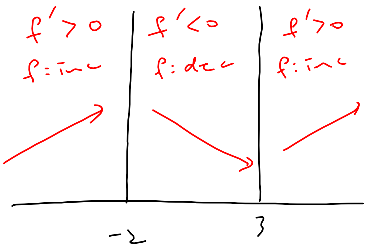
local min: $(-3, h(-3))$

$= \left(-3, -\dfrac{1}{e^3}\right)$

d) $f'(x) = (x+2)^5(x-3)^7 \underbrace{e^{(x-6)}}_{\neq 0,\ >0} = 0$

$\therefore\ x = -2,\ 3$: C.N's

sign chart for f'



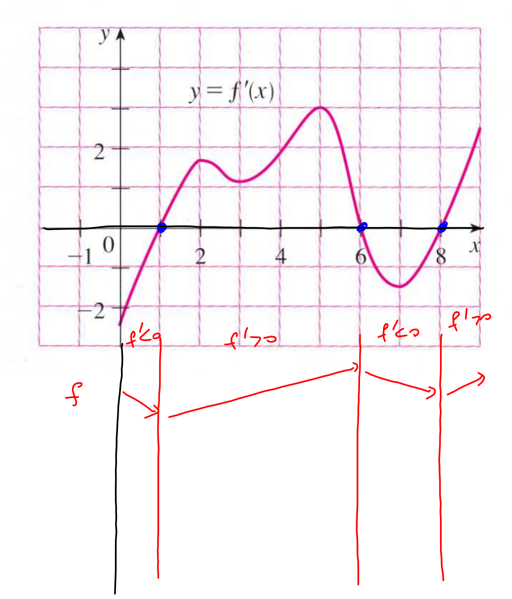
$\therefore$ inc.: $(-\infty, -2)$, $(3, \infty)$

dec.: $(-2, 3)$

local Max. at $x = -2$

local min. at $x = 3$

Ex.8) The graph of the derivative $f'(x)$ is shown below.



dec.: $(0, 1), (6, 8)$

inc.: $(1, 6), (8, 9)$

local min. at $x = 1, 8$

local Max. at $x = 6$

Section 5.2 The Second Derivative

Definition. Second Derivative: Given a function $y = f(x)$, the **second derivative**, denoted by $f''(x)$, is defined to be the derivative of the first derivative. Thus,

$$(f'(x))' = f''(x) = \frac{d}{dx}(f'(x))$$

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2}$$

ex) $f(x) = x^4$

$f' = 4x^3$

$f'' = (4x^3)' = 12x^2$

Ex.1) Find the first and second derivative of the following functions:

a) $f(x) = 2x^3 - 14x^2 + 3x - 16$

$f'(x) = 6x^2 - 28x + 3$

$f''(x) = 12x - 28$

b) $y = \underline{3x^2}\,\underline{\ln x}$

$y' = 6x \cdot \ln x + 3x^2 \cdot \frac{1}{x}$

$= \underline{6x} \cdot \underline{\ln x} + 3x$

$y'' = \overbrace{6 \cdot \ln x + 6x \cdot \frac{1}{x}} + 3$

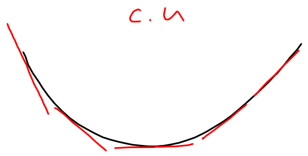
$= 6 \cdot \ln x + 6 + 3$

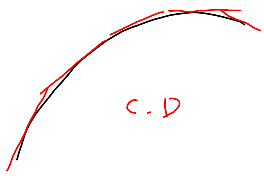
$\therefore\ y'' = 6 \cdot \ln x + 9$

Definition. Concave Up and Down:

(a) We say that the graph of f is concave up ($\smile$) on (a, b) if $f'(x)$ is increasing on (a, b).

(b) We say that the graph of f is concave down ($\frown$) on (a, b) if $f'(x)$ is decreasing on (a, b).





Test for Concavity

(a) If $f''(x) > 0$ on (a, b), then the graph of f is concave up ($\smile$) on (a, b).

(b) If $f''(x) < 0$ on (a, b), then the graph of f is concave down ($\frown$) on (a, b).

Definition. Inflection Point: A point $(c, f(c))$ on the graph of f is an **inflection point** and c is and **inflection value** if $f(c)$ is defined and the concavity of the graph of f changes at $(c, f(c))$.

Locating Inflection Points:

(a) Determine the values of x where the second derivative is zero or where the second derivative is undefined. $f''=0$, or f'': DNE

(b) Place these values on a number line and create a sign chart for the second derivative.

(c) The point is an inflection point if the second derivative changes sign and if the x-value is in the domain of $f(x)$.

Ex.2) Given $f(x) = -x^3 + 2x^2 - 3x + 9$ determine the intervals where $f'(x)$ is increasing and decreasing.

$f'' > 0$ $\quad\therefore f$: C.U

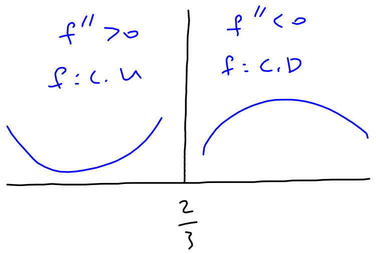
$f'' < 0$ $\quad\therefore f$: C.D

$f' = -3x^2 + 4x - 3$

$f'' = -6x + 4 = 0$

$\therefore x = \frac{4}{6} = \frac{2}{3}$

Sign chart for f''



$\therefore$ C.U: $(-\infty, \frac{2}{3})$

C.D: $(\frac{2}{3}, \infty)$

Inf: $(\frac{2}{3}, f(\frac{2}{3})) = (\frac{2}{3}, \frac{205}{27})$

Ex.3) Determine the intervals where the functions below are concave up and concave down and locate any inflection points.

a) $f(x) = x^4 - 6x^2$

$f' = 4x^3 - 12x$

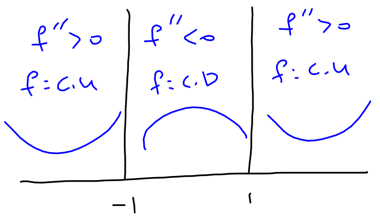
$f'' = 12x^2 - 12$

$= 12(x^2 - 1)$

$= 12(x-1)(x+1) = 0$

$\therefore x = 1, -1$

sign chart for f''



$\therefore$ C.U: $(-\infty, -1), (1, \infty)$

C.D: $(-1, 1)$

Inf.: $(-1, f(-1)) = (-1, -5)$

$(1, f(1)) = (1, -5)$

b) $f(x) = \ln(x^2 + 6x + 13)$ Domain: $\mathbb{R}$

$$f'(x) = \frac{1}{(x^2+6x+13)} \cdot (2x+6)$$

$$= \frac{2x+6}{x^2+6x+13}$$

$$f''(x) = \frac{(2)\cdot(x^2+6x+13) - (2x+6)\cdot(2x+6)}{(x^2+6x+13)^2}$$

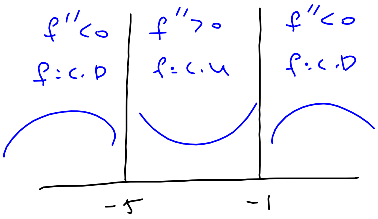
$$= \frac{2x^2+12x+26-4x^2-24x-36}{(x^2+6x+13)^2}$$

$$= \frac{-2x^2-12x-10}{(x^2+6x+13)^2} = \frac{-2(x^2+6x+5)}{(x^2+6x+13)^2}$$

$$= \frac{-2(x+1)(x+5)}{(x^2+6x+13)^2} = 0$$

$\therefore\ x = -1, -5$

sign chart for f''



$\therefore$ C.D: $(-\infty, -5)$, $(-1, \infty)$

C.U: $(-5, -1)$

Inf: $(-5, f(-5)) = (-5, \ln 8)$

$(-1, f(-1)) = (-1, \ln 8)$

Ex.4) Suppose the function $f(x)$ has a domain of all real numbers except $x = 1$. The second derivative of $f(x)$ is shown below,

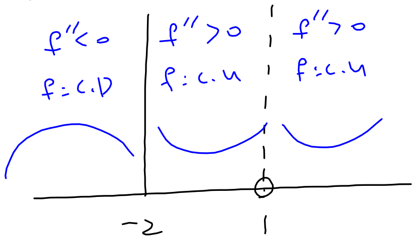
$$f''(x) = \frac{2x+4}{(x-1)^4}.$$

Determine the intervals where the function is concave up and concave down and locate any inflection points.

$$f'' = \frac{2(x+2)}{(x-1)^4} = 0$$

$\therefore\ x = -2$, P.N.s: $x = -2,\ 1$

sign chart for f''

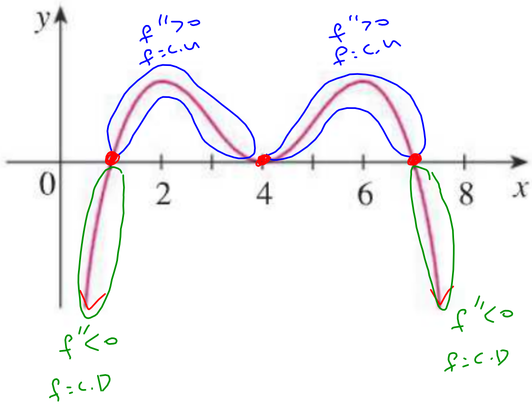


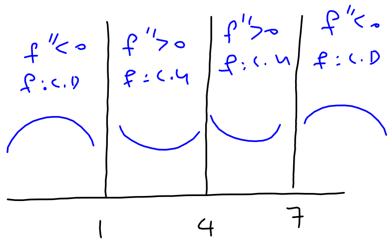
$\therefore$ C.D: $(-\infty, -2)$

C.U: $(-2, 1)$, $(1, \infty)$

Inf. at $x = -2$

Ex.5) The graph of the second derivative f'' of a function f is shown below.





(a) On what intervals is f concave upward?

$(1, 4), (4, 7)$

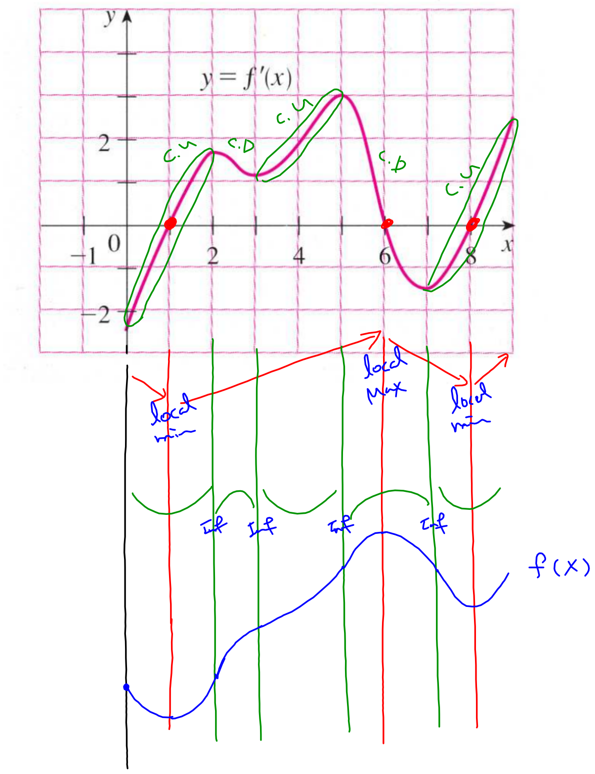
(b) On what intervals is f concave downward?

$(-\infty, 1), (7, \infty)$

(c) Stat the x-coordinates of the points of inflection.

$x = 1, 7$

Ex.6) The graph of the derivative f' of a function f is shown below



<u>Second Derivative Test</u>

Suppose that f is defined on (a,b), $f'(c)=0$, and $c \in (a,b)$.

(a) If $f''(c) > 0$, then $f(c)$ is a relative (local) minimum.

(b) If $f''(c) < 0$, then $f(c)$ is a relative (local) maximum.

Ex.7) Determine where the local extrema of $f(x) = -x^3 + 12x - 5$ occur (and classify) using the Second Derivative Test for Local Extrema.

Step 1) C.N

$f' = -3x^2 + 12$

$= -3(x^2 - 4)$

$= -3(x-2)(x+2) = 0$

$\therefore x = 2, -2$: C.Ns

Step 2) Use the Second Derivative Test

$f'' = -6x$

Then

$f''(-2) = -6(-2) = 12 > 0$:

$\therefore$ f has a local min at $x=-2$

$\therefore$ local min: $(-2, f(-2)) = (-2, -21)$

$f''(2) = -6(2) = -12 < 0$:

$\therefore$ f has a local Max at $x=2$

$\therefore$ local Max: $(2, f(2)) = (2, 11)$