Section 5.3 Limits at Infinity

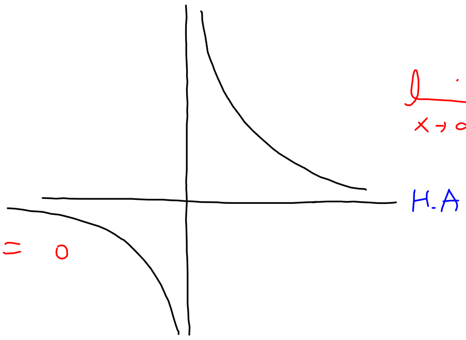
Definition. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

ex1 $f(x) = \frac{1}{x}$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$



$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Definition. Horizontal Asymptote: The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

$$\frac{1}{\infty} = 0$$

$$\frac{\infty}{1} = \infty$$

Ex.8) Find the limits:

a) $\lim_{x\to\infty} x = \infty$

b) $\lim_{x\to-\infty} x = -\infty$

c) $\lim_{x\to\infty} (\cancel{x} - x^2) = -\infty$

d) $\lim_{x\to-\infty} (\cancel{x} - x^3) = +\infty$

e) $\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$

f) $\lim_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$

g) $\lim_{x \to \infty} \frac{1}{x^4} = \frac{1}{\infty} = 0$

Higher Order Rule

h) $\lim_{x \to \infty} \frac{7x + 12}{x^2 + 10x + 5} = \lim_{x \to \infty} \frac{7\cancel{x}}{x^{\cancel{2}}} = \lim_{x \to \infty} \frac{7}{x} = \frac{7}{\infty} = 0$

H.O

i) $\lim_{x \to \infty} \frac{7x^3 + 4x}{2x^3 - x^2 + 3} = \lim_{x \to \infty} \frac{7\cancel{x^3}}{2\cancel{x^3}} = \frac{7}{2}$

j) $\lim\limits_{x\to-\infty} \frac{3x+4}{2x+1} \overset{H.O}{=} \lim\limits_{x\to-\infty} \frac{3\cancel{x}}{2\cancel{x}} = \frac{3}{2}$

k) $\lim\limits_{x\to-\infty} \frac{x^2+7x+12}{x+5} \overset{H.O}{=} \lim\limits_{x\to-\infty} \frac{x^{\cancel{2}}}{\cancel{x}} = \lim\limits_{x\to-\infty} \frac{x}{1} = -\infty$

l) $\lim\limits_{t\to\infty} \frac{t^4-t^2+1}{t^5+t^3-t} \overset{H.O}{=} \lim\limits_{t\to\infty} \frac{\cancel{t^4}}{t^{\cancel{5}}{}^{1}} = \lim\limits_{t\to\infty} \frac{1}{t} = \frac{1}{\infty} = 0$

m) $\lim_{x\to-\infty} \frac{x^4+2x+3}{x(x^2-1)} = \lim_{x\to-\infty} \frac{x^4+2x+3}{x^3-x} \overset{H.O}{=} \lim_{x\to-\infty} \frac{x^{\cancel{4}\,1}}{\cancel{x^3}} = \lim_{x\to-\infty} \frac{x}{1} = -\infty$

n) $\lim_{x\to\infty} \frac{\sqrt{1+4x^2}}{4+x} \overset{H.O}{=} \lim_{x\to\infty} \frac{\sqrt{4x^2}}{x}$ $= \sqrt{(2x)^2} = |2x|$ $= 2x$

$= \lim_{x\to\infty} \frac{2\cancel{x}}{\cancel{x}} = \frac{2}{1} = 2$

NOTE

$\sqrt{x^2} = |x|$

ex) $\sqrt{(-2)^2} = |-2| = 2$

o) $\lim_{x\to-\infty} \frac{\sqrt{1+4x^2}}{4+x} \overset{H.O}{=} \lim_{x\to-\infty} \frac{\sqrt{4x^2}}{x}$ $= \sqrt{(2x)^2} = |2x|$ $= -(2x)$

$= \lim_{x\to-\infty} \frac{-(2\cancel{x})}{\cancel{x}} = \frac{-2}{1} = -2$

p) $\lim_{x\to\infty} \frac{3}{1+e^{-5x}}$ $= \lim_{x\to\infty} \frac{3}{1+0} = \frac{3}{1} = 3$

$e^{-5x} = e^{-\infty} = 0$

q) $\lim_{x\to-\infty} \frac{3}{1+e^{-5x}}$ $= \frac{3}{\infty} = 0$

$e^{-5x} = e^{+\infty} = \infty$

r) $\lim_{x\to\infty} \frac{e^{3x}-2e^{-3x}}{e^{3x}+e^{-3x}}$ $= \lim_{x\to\infty} \frac{\cancel{e^{3x}}}{\cancel{e^{3x}}} = 1$

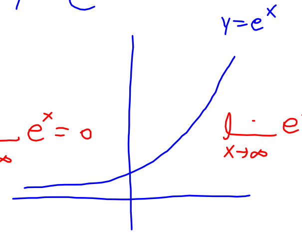
$e^{-3x} = e^{-\infty} = 0$

s) $\lim_{x\to-\infty} \frac{e^{3x}-2e^{-3x}}{e^{3x}+e^{-3x}}$ $= \lim_{x\to-\infty} \frac{-2\cancel{e^{-3x}}}{\cancel{e^{-3x}}} = \frac{-2}{1} = -2$

$e^{3x} = e^{-\infty} = 0$

NOTE

$y = e^x$



$\lim_{x\to-\infty} e^x = 0$ $\lim_{x\to\infty} e^x = \infty$

$$\boxed{\begin{aligned} \therefore\ e^{\infty} &= \infty \\ e^{-\infty} &= 0 \end{aligned}}$$

Finding limits at infinity for a rational function, $f(x)$: Look for the highest degree of x:

(a) If it is in the denominator, then $\lim\limits_{x\to\pm\infty} f(x) = 0$.

(b) If it is in the numerator, then $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$.

(c) If the degree of the polynomial in the numerator and denominator is the same then $\lim\limits_{x\to\pm\infty} f(x) =$ ratio of the leading coefficients.

Finding the Vertical Asymptote and Horizontal Asymptote.

(a) Vertical asymptote: undefined point but if it could be cancelled, it is not vertical asymptote but hole.

(b) Horizontal asymptote: use infinite limit $x \to \infty$ and $x \to -\infty$.

Ex.9) Find all horizontal and vertical asymptotes of the functions below.

a) $f(x) = \dfrac{3x+2}{x-4}$

· V.A : (deno = 0) : $x - 4 = 0$

$\boxed{\therefore\ x = 4 : \text{V.A}}$

· H.A : $\displaystyle\lim_{x \to \infty} \frac{3x+2}{x-4} \overset{\text{H.O}}{=} \lim_{x \to \infty} \frac{3\cancel{x}}{\cancel{x}} = \frac{3}{1} = 3$

$\displaystyle\lim_{x \to -\infty} \frac{3x+2}{x-4} \overset{\text{H.O}}{=} \lim_{x \to -\infty} \frac{3\cancel{x}}{\cancel{x}} = \frac{3}{1} = 3$

$\boxed{\therefore\ y = 3 : \text{H.A}}$

b) $g(x) = \dfrac{x^2+9}{x}$

$\boxed{\cdot \text{V.A}: x = 0}$

$\cdot$ H.A: $\lim\limits_{x\to\infty} \dfrac{x^2+9}{x} \overset{\text{H.O}}{=\!=} \lim\limits_{x\to\infty} \dfrac{x^{\cancel{2}}}{\cancel{x}} = \lim\limits_{x\to\infty} \dfrac{x}{1} = \infty$

$\lim\limits_{x\to-\infty} \dfrac{x^2+9}{x} \overset{\text{H.O}}{=\!=} \lim\limits_{x\to-\infty} \dfrac{x^{\cancel{2}}}{\cancel{x}} = \lim\limits_{x\to-\infty} \dfrac{x}{1} = -\infty$

$\boxed{\therefore \text{ No H.A}}$

c) $h(x) = \dfrac{x+3}{x^2+7x+12}$

$= \dfrac{x+3}{(x+3)(x+4)}$

$\cdot$ V.A: $x = \cancel{-3}, -4$

$\cdot$ hole: $\left(-3, \underline{\lim\limits_{x\to-3} f(x)}\right)$

$= \lim\limits_{x\to-3} \dfrac{\cancel{x+3}}{\cancel{(x+3)}(x+4)} = \lim\limits_{x\to-3} \dfrac{1}{x+4} = \dfrac{1}{(-3)+4} = \dfrac{1}{1} = 1$

$= (-3, 1)$

$\cdot$ H.A: $\lim\limits_{x\to\infty} \dfrac{x+3}{x^2+7x+12} \overset{\text{H.O}}{=\!=} \lim\limits_{x\to\infty} \dfrac{\cancel{x}^{\,1}}{x^{\cancel{2}}} = \lim\limits_{x\to\infty} \dfrac{1}{x} = \dfrac{1}{\infty} = 0^+$

$\lim\limits_{x\to-\infty} \dfrac{x+3}{x^2+7x+12} \overset{\text{H.O}}{=\!=} \lim\limits_{x\to-\infty} \dfrac{\cancel{x}}{x^{\cancel{2}\,1}} = \lim\limits_{x\to-\infty} \dfrac{1}{x} = \dfrac{1}{-\infty} = 0^-$

$\therefore\ y = 0$: H.A

Ex.10) Find all horizontal and vertical asymptotes of $f(x) = \dfrac{6e^x}{1-4e^x}$.

$\cdot$ V.A (deno $=0$)

$1-4e^x = 0$

$\Rightarrow -4e^x = -1$

$\Rightarrow e^x = \frac{1}{4}$

ln ln

$\Rightarrow \ln e^{x} = \ln\frac{1}{4}$

$\Rightarrow x \cdot \underbrace{\ln e}_{=1} = \ln\frac{1}{4}$

$\therefore x = \ln\frac{1}{4}$: V.A

$\cdot$ H.A

$$\lim_{x\to\infty} \frac{6e^x}{1-4e^x} \overset{H.O}{=} \lim_{x\to\infty} \frac{6\cancel{e^x}}{-4\cancel{e^x}} = -\frac{6}{4} = -\frac{3}{2}$$

$$\lim_{x\to-\infty} \frac{6\underbrace{e^x}_{=e^{-\infty}=0}}{1-4\underbrace{e^x}_{=e^{-\infty}=0}} = \frac{0}{1-0} = 0$$

$\therefore$ H.As: $y = -\frac{3}{2}$ (positive)

$y = 0$ (negative)

Section 5.4 Additional Curve Sketching

Graphing Strategy:

(a) Analyze $f(x)$
  i. Find the domain of f.
  ii. Find the intercepts.
  iii. Find asymptotes.

(b) Analyze $f'(x)$
  i. Find the critical values.
  ii. Use sign chart for $f'(x)$ and determine intervals of increasing/decreasing.
  iii. Find all relative(local) extrema.

(c) Analyze $f''(x)$
  i. Use sign chart for $f''(x)$ and find intervals where the graph of the function is cancave up and concave down.
  ii. Find all inflection values.

(d) Sketch the graph of f using all of the above information. Plot additional points as needed.

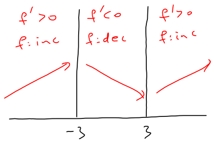
Ex.11) Use the graphing strategy to sketch a graph of $f(x) = x^3 - 27x$.

Step 1) $f(x) = x^3 - 27x$

- Domain: $\mathbb{R}$
- x-intcp $(y=0)$: $0 = x^3 - 27x$
  $\Rightarrow 0 = x(x^2-27)$
  $\therefore x = 0, -\sqrt{27}, \sqrt{27}$: x-intcp.
- y-intcp $(x=0)$: $y = (0)^3 - 27(0)$
  $= 0$
  $\therefore y = 0$: y-intcp
- V.A: none
- H.A: $\lim_{x\to\infty} x^3 - 27x = \infty$
  $\lim_{x\to-\infty} x^3 - 27x = -\infty$

Step 2) $f'(x) = 3x^2 - 27$
$= 3(x^2-9)$
$= 3(x+3)(x-3) = 0$
$\therefore x = -3, 3$: C.Ns

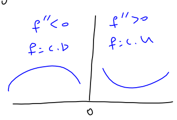
sign chart for f'



$\therefore$ inc.: $(-\infty, -3), (3, \infty)$
dec.: $(-3, 3)$
local Max: $(-3, f(-3)) = (-3, 54)$
local min: $(3, f(3)) = (3, -54)$

Step 3) $f'' = 6x = 0$
$\therefore x = 0$

sign chart for f''



$\therefore$ c.D: $(-\infty, 0)$
c.U: $(0, \infty)$
Inf: $(0, f(0)) = (0, 0)$

Step 4) Sketch the graph