MATH 142 Business Calculus

Fall 2019, WEEK 7

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Week 7 Section 5.2, 5.3, 5.4 Second Derivative, Limits at Infinity, Additional Curve Sketching

Section 5.2 The Second Derivative

Definition. Second Derivative: Given a function $y = f(x)$, the **second derivative**, denoted by $f''(x)$, is defined to be the derivative of the first derivative. Thus,

$$f''(x) = \frac{d}{dx}(f'(x))$$

Ex.1) Find the first and second derivative of the following functions:

a) $f(x) = 2x^3 - 14x^2 + 3x - 16$

b) $y = 3x^2 \ln x$

Definition. Concave Up and Down:

(a) We say that the graph of f is concave up ($\smile$) on (a, b) if $f'(x)$ is increasing on (a, b).

(b) We say that the graph of f is concave down ($\frown$) on (a, b) if $f'(x)$ is decreasing on (a, b).

Test for Concavity

(a) If $f''(x) > 0$ on (a, b), then the graph of f is concave up ($\smile$) on (a, b).

(b) If $f''(x) < 0$ on (a, b), then the graph of f is concave down ($\frown$) on (a, b).

Definition. Inflection Point: A point $(c, f(c))$ on the graph of f is an **inflection point** and c is and **inflection value** if $f(c)$ is defined and the concavity of the graph of f changes at $(c, f(c))$.

Locating Inflection Points:

(a) Determine the values of x where the second derivative is zero or where the second derivative is undefined.

(b) Place these values on a number line and create a sign chart for the second derivative.

(c) The point is an inflection point if the second derivative changes sign and if the x-value is in the domain of $f(x)$.

Ex.2) Given $f(x) = -x^3 + 2x^2 - 3x + 9$ determine the intervals where $f'(x)$ is increasing and decreasing.

Ex.3) Determine the intervals where the functions below are concave up and concave down and locate any inflection points.

a) $f(x) = x^4 - 6x^2$

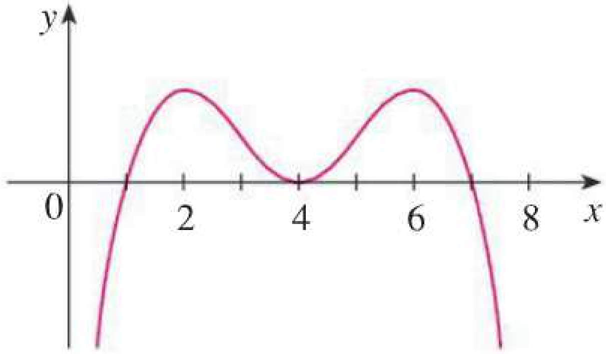
b) $f(x) = \ln(x^2 + 6x + 13)$

Ex.4) Suppose the function $f(x)$ has a domain of all real numbers except $x = 1$. The second derivative of $f(x)$ is shown below,

$$f''(x) = \frac{2x+4}{(x-1)^4}.$$

Determine the intervals where the function is concave up and concave down and locate any inflection points.

Ex.5) The graph of the second derivative f'' of a function f is shown below.

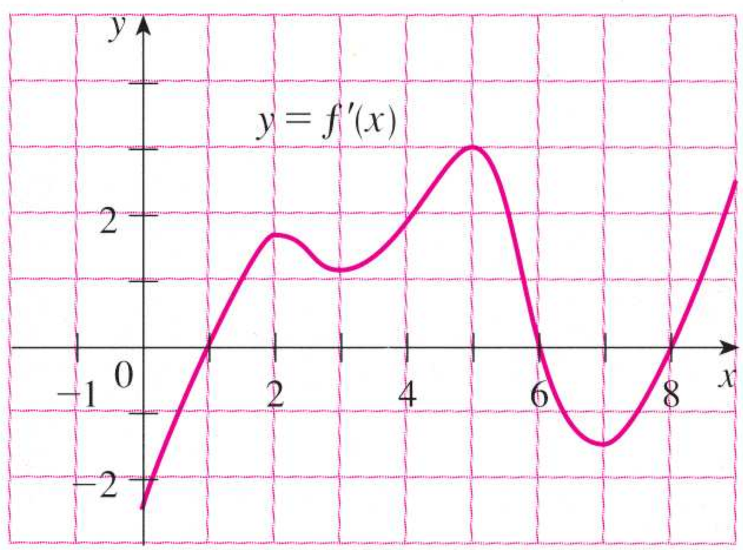


(a) On what intervals is f concave upward?

(b) On what intervals is f concave downward?

(c) Stat the x-coordinates of the points of inflection.

Ex.6) The graph of the derivative f' of a function f is shown below



a) the interval(s) on which $f(x)$ is increasing.

b) the interval(s) on which $f(x)$ is decreasing.

c) At what values of x does f have a local maximum or minimum?

d) the interval(s) on which $f'(x)$ is increasing.

e) the interval(s) on which $f'(x)$ is decreasing.

f) the x-value of the inflection point(s) of $f(x)$.

g) Sketh a graph of f.

Second Derivative Test

Suppose that f is defined on (a, b), $f'(c) = 0$, and $c \in (a, b)$.

(a) If $f''(c) > 0$, then $f(c)$ is a relative (local) minimum.

(b) If $f''(c) < 0$, then $f(c)$ is a relative (local) maximum.

Ex.7) Determine where the local extrema of $f(x) = -x^3 + 12x - 5$ occur (and classify) using the Second Derivative Test for Local Extrema.

Section 5.3 Limits at Infinity

Definition. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Definition. Horizontal Asymptote: The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

Limit of Power Function at Infinity

If p is a positive real number,

$$\lim_{x \to \pm\infty} \frac{1}{x^p} = 0.$$

Ex.8) Find the limits:

a) $\lim_{x\to\infty} x$

b) $\lim_{x\to-\infty} x$

c) $\lim_{x\to\infty} (x - x^2)$

d) $\lim_{x\to-\infty} (x - x^3)$

e) $\lim_{x\to\infty} \frac{1}{x}$

f) $\lim_{x\to-\infty} \frac{1}{x}$

g) $\lim_{x\to\infty} \frac{1}{x^4}$

h) $\lim_{x\to\infty} \frac{7x+12}{x^2+10x+5}$

i) $\lim_{x\to\infty} \frac{7x^3+4x}{2x^3-x^2+3}$

j) $\lim\limits_{x\to-\infty} \dfrac{3x+4}{2x+1}$

k) $\lim\limits_{x\to-\infty} \dfrac{x^2+7x+12}{x+5}$

l) $\lim\limits_{t\to\infty} \dfrac{t^4-t^2+1}{t^5+t^3-t}$

m) $\lim\limits_{x\to-\infty} \dfrac{x^4+2x+3}{x(x^2-1)}$

n) $\lim\limits_{x\to\infty} \dfrac{\sqrt{1+4x^2}}{4+x}$

o) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{1+4x^2}}{4+x}$

p) $\lim_{x\to\infty} \frac{3}{1+e^{-5x}}$

q) $\lim_{x\to-\infty} \frac{3}{1+e^{-5x}}$

r) $\lim_{x\to\infty} \frac{e^{3x}-2e^{-3x}}{e^{3x}+e^{-3x}}$

s) $\lim_{x\to-\infty} \frac{e^{3x}-2e^{-3x}}{e^{3x}+e^{-3x}}$

Finding limits at infinity for a rational function, $f(x)$: Look for the highest degree of x:

(a) If it is in the denominator, then $\lim_{x\to\pm\infty} f(x) = 0$.

(b) If it is in the numberator, then $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

(c) If the degree of the polynomial in the numberator and denominator is the same then $\lim_{x\to\pm\infty} f(x) =$ ratio of the leading coefficients.

Finding the Vertical Asymptote and Horizontal Asymptote.

(a) Vertical asymptote: undefined point but if it could be cancelled, it is not vertical asymptote but hole.

(b) Horizontal asymptote: use infinite limit $x \to \infty$ and $x \to -\infty$.

Ex.9) Find all horizontal and vertical asymptotes of the functions below.

a) $f(x) = \dfrac{3x+2}{x-4}$

b) $g(x) = \dfrac{x^2+9}{x}$

c) $h(x) = \dfrac{x+3}{x^2+7x+12}$

Ex.10) Find all horizontal and vertical asymptotes of $f(x) = \dfrac{6e^x}{1 - 4e^x}$.

Section 5.4 Additional Curve Sketching

Graphing Strategy:

(a) Analyze $f(x)$

i. Find the domain of f.

ii. Find the intercepts.

iii. Find asymptotes.

(b) Analyze $f'(x)$

i. Find the critical values.

ii. Use sign chart for $f'(x)$ and determine intervals of increasing/decreasing.

iii. Find all relative(local) extrema.

(c) Analyze $f''(x)$

i. Use sign chart for $f''(x)$ and find intervals where the graph of the function is cancave up and concave down.

ii. Find all inflection values.

(d) Sketch the graph of f using all of the above information. Plot additional points as needed.

Ex.11) Use the graphing strategy to sketch a graph of $f(x) = x^3 - 27x$.

Ex.12) Use the graphing strategy to sketch a graph of $f(x) = \dfrac{2x^2 + 11x + 14}{x^2 - 4}$.

Ex.13) Use the graphing strategy to sketch a graph of $f(x) = (x-2)e^x$.

Ex.14) Use the given information to sketch the graph of f.

- Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
- Asymptotes: $x = -1$, $x = 1$, $y = 0$
- $f(-2) = 1$, $f(0) = 0$, $f(2) = 1$
- $f'(x) > 0$ on $(-\infty, -1) \cup (0, 1)$
- $f'(x) < 0$ on $(-1, 0) \cup (1, \infty)$
- $f''(x) > 0$ on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$