Section 5.4 Additional Curve Sketching

Graphing Strategy:

(a) Analyze $f(x)$

i. Find the domain of f.

ii. Find the intercepts.

iii. Find asymptotes.

(b) Analyze $f'(x)$

i. Find the critical values.

ii. Use sign chart for $f'(x)$ and determine intervals of increasing/decreasing.

iii. Find all relative(local) extrema.

(c) Analyze $f''(x)$

i. Use sign chart for $f''(x)$ and find intervals where the graph of the function is cancave up and concave down.

ii. Find all inflection values.

(d) Sketch the graph of f using all of the above information. Plot additional points as needed.

Ex.11) Use the graphing strategy to sketch a graph of $f(x) = x^3 - 27x$.

Ex.12) Use the graphing strategy to sketch a graph of $f(x) = \dfrac{2x^2+11x+14}{x^2-4}$.

Step 1) $f(x) = \dfrac{2x^2+11x+14}{x^2-4} = \dfrac{(x+2)(2x+7)}{(x+2)(x-2)}$

· Domain: $\mathbb{R}$ except $x=-2, 2$

· X-intercept ($y=0$): $0 = \dfrac{\cancel{(x+2)}(2x+7)}{\cancel{(x+2)}(x-2)}$

$\Rightarrow 2x+7=0$

$\therefore x = -\frac{7}{2}$: x-intercp.

· y-intercp ($x=0$): $y = \dfrac{2(0)^2+11(0)+14}{(0)^2-4} = \dfrac{14}{-4} = -\frac{7}{2}$: y-intercp.

· V.A: $x=2$

· hole: $\left(-2, \underbrace{\lim_{x\to-2} f(x)}\right) = \left(-2, -\frac{3}{4}\right)$

$= \lim_{x\to-2} \dfrac{\cancel{(x+2)}(2x+7)}{\cancel{(x+2)}(x-2)} = \lim_{x\to-2} \dfrac{2x+7}{x-2} = \dfrac{2(-2)+7}{(-2)-2} = \dfrac{-4+7}{-4}$

$= \dfrac{3}{-4}$

· H.A: $\lim_{x\to\infty} \dfrac{2x^2+11x+14}{x^2-4} \overset{H.O}{=} \lim_{x\to\infty} \dfrac{2\cancel{x^2}}{\cancel{x^2}} = \dfrac{2}{1} = 2$

$\lim_{x\to-\infty} \dfrac{2x^2+11x+14}{x^2-4} \overset{H.O}{=} \lim_{x\to-\infty} \dfrac{2\cancel{x^2}}{\cancel{x^2}} = \dfrac{2}{1} = 2$

$\therefore$ H.A: $y=2$

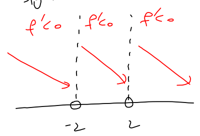
Step 2) $f(x) = \dfrac{\cancel{(x+2)}(2x+7)}{\cancel{(x+2)}(x-2)}$

$f'(x) = \dfrac{(2)\cdot(x-2)-(2x+7)\cdot(1)}{(x-2)^2} = \dfrac{\cancel{2x}-4-\cancel{2x}-7}{(x-2)^2} = \dfrac{-11}{(x-2)^2} = 0$

$\therefore$ No C.N

$\therefore$ P.N: $x=-2, 2$

sign chart for f'



$\therefore$ dec.: $(-\infty,-2), (-2,2), (2,\infty)$

No local extrema.

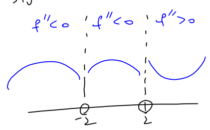
Step 3) $f' = \dfrac{-11}{(x-2)^2} = -11(x-2)^{-2}$

$f''(x) = 22(x-2)^{-3}\cdot(1) = \dfrac{22}{(x-2)^3} = 0$

$\therefore$ No. C.N

P.N: $x=-2, 2$

sign chart for f''

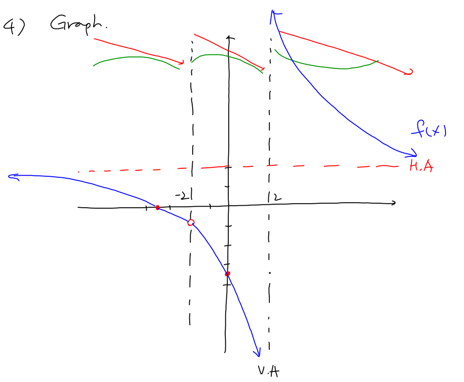


$\therefore$ C.D: $(-\infty,-2), (-2,2)$

C.U: $(2,\infty)$

$\therefore$ No Inf.

Step 4) Graph.



Ex.13) Use the graphing strategy to sketch a graph of $f(x) = (x-2)e^x$.

Step 1) $f(x) = (x-2)\cdot e^x$

- Domain: $\mathbb{R}$
- x-intcp ($y=0$): $0 = (x-2)\underbrace{e^x}_{\neq 0}$

  $\therefore\ x = 2$: x-intcp.

- y-intcp ($x=0$): $y = (0-2)\cdot \underbrace{e^0}_{=1} = -2$

  $\therefore\ y = -2$: y-intcp.

- V.A: None
- H.A: $\lim_{x\to\infty} (x-2)\cdot e^x = \infty$

  $\lim_{x\to -\infty} (x-2)\cdot e^x = 0$ ← we will learn later

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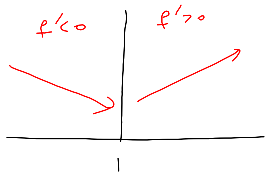
Step 2) $f'(x) = (1)\cdot e^x + (x-2)\cdot e^x$

$= e^x\big(1 + (x-2)\big) = \underbrace{e^x}_{\neq 0,\ >0}(x-1) = 0$

$\therefore\ x = 1$: C.N

Sign chart for f'



$\therefore$ dec.: $(-\infty, 1)$

inc.: $(1, \infty)$

local min: $(1, f(1))$

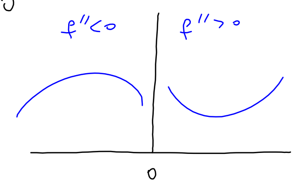
$= (1, -e)$

Step 3) $f'' = e^x\cdot(x-1) + e^x\cdot(1)$

$= e^x\big((x-1) + (1)\big) = \underbrace{e^x}_{\neq 0,\ >0}\cdot x = 0$

$\therefore\ x = 0$: C.N

sign chart for f''

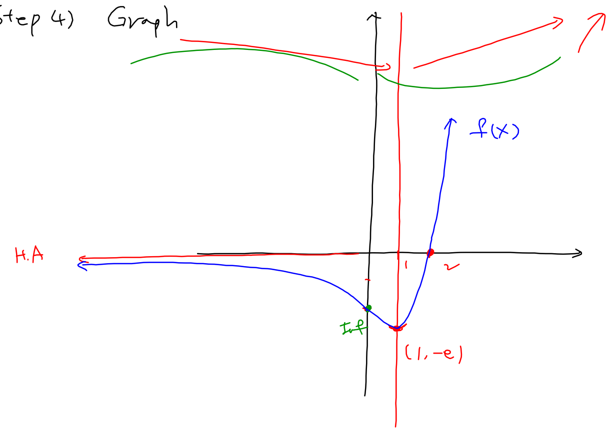


$\therefore$ C.D: $(-\infty, 0)$

C.U: $(0, \infty)$

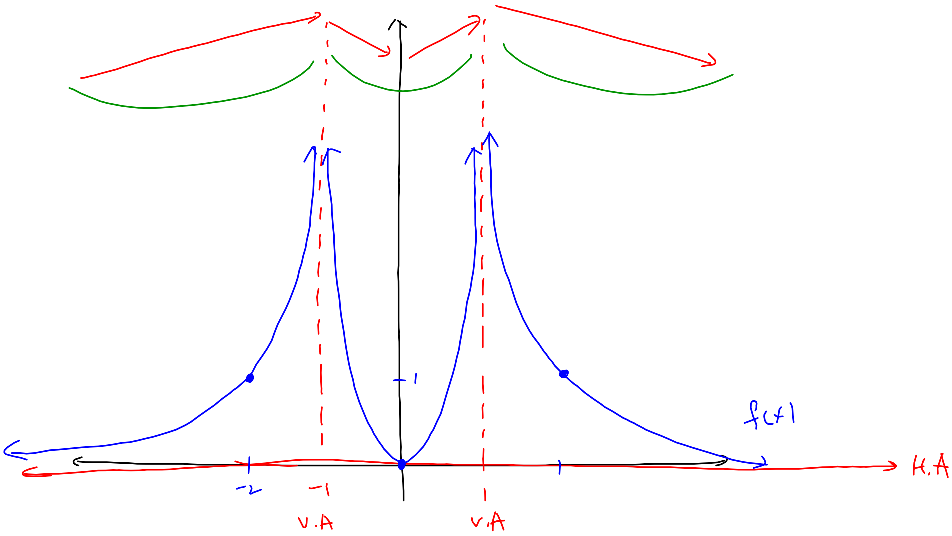
Inf: $(0, f(0)) = (0, -2)$

Step 4) Graph



Ex.14) Use the given information to sketch the graph of f.

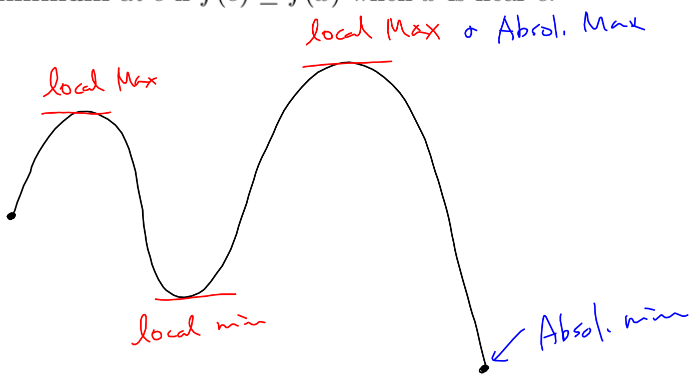
- Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
- Asymptotes: $x = -1$, $x = 1$, $y = 0$
- $f(-2) = 1$, $f(0) = 0$, $f(2) = 1$
- $f'(x) > 0$ on $(-\infty, -1) \cup (0, 1)$
- $f'(x) < 0$ on $(-1, 0) \cup (1, \infty)$
- $f''(x) > 0$ on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$



Section 5.5 Absolute Extrema

Definition. Maximum, Minimum. A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f. A function f has an **absoulte minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f. In this case, we call $f(c)$ the maximum value or minimum value, repectively.

cf) A function f has an **local maximum** at c if $f(c) \geq f(x)$ when x is near c. A function f has an **local minimum** at c if $f(c) \leq f(x)$ when x is near c.



Extreme Value Theorem. A function f that is **continuous** on a **closed interval** $[a, b]$ has both an absolute maximum value and an absolute minimum value on that interval.

Candidates : C.Ns & endpoints

Finding Absolute Extrema on a Closed Interval

1. Check to make certain that f is continuous over $[a, b]$.
2. Find the critical values in the interval (a, b).
3. Evaluate f at the endpoints a and b and at the critical values found in step 2.
4. The largest value obtained from the previous step is the absolute maximum of $f(x)$ on $[a, b]$ and the smallest value obtained is the absolute minimum of $f(x)$ on $[a, b]$.

Ex.1) Find the absolute extrema of the following functions on the given intervals.

a) $f(x) = 2x^3 - 3x^2 - 12x + 24$ on $[1, 4]$

Step 1) C.N

$f'(x) = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$

$= 6(x+1)(x-2) = 0$

$\therefore x = \cancel{-1}, 2$: C.Ns

$\therefore$ Candidates : $x = 1, 2, 4$

$f(1) = 2(1)^3 - 3(1)^2 - 12(1) + 24 = 11$

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 24 = 4$: Absol. min

$f(4) = 2(4)^3 - 3(4)^2 - 12(4) + 24 = 56$: Absol. Max

b) $g(x) = x + \dfrac{4}{x}$ on $[1, 8]$ $= x + 4x^{-1}$

C.N: $g'(x) = 1 - 4x^{-2} = 1 - \dfrac{4}{x^2} = 0$

$\Rightarrow \dfrac{x^2-4}{x^2} = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x+2)(x-2) = 0$

$\therefore x = \cancel{-2}, 2$: C.Ns

$\therefore$ candidates: $x = 1, 2, 8$

$g(1) = 1 + \frac{4}{1} = 5$

$g(2) = 2 + \frac{4}{2} = 4$: Absol. min

$g(8) = 8 + \frac{4}{8} = 8.5$: Absol. Max

c) $h(x) = x^3 + 3x^2 - 9x - 7$ on $[-4, 0)$

C.N: $h'(x) = 3x^2 + 6x - 9$

$= 3(x^2 + 2x - 3)$

$= 3(x-1)(x+3) = 0$

$\therefore x = \cancel{1}, -3$: C.Ns

By Second Derivative Test

$h''(x) = 6x + 6$

$\Rightarrow h''(-3) = 6(-3) + 6 < 0$: $\frown$

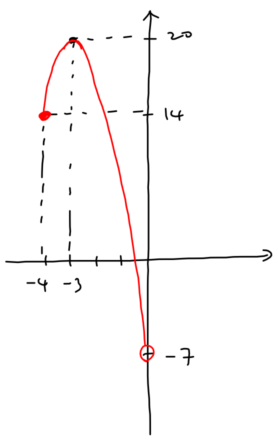
$\therefore$ h has a local Max at $x = -3$.

$\therefore$ local Max : $(-3, h(-3)) = (-3, 20)$

end point:

$h(-4) = 14$

$h(0) = -7$



$\therefore$ Absol. Max.: $y = 20$ at $x = -3$

No Absol. min.

Ex.2) Find the absolute extrema of each function on the given interval.

a) $f(x) = 6x - x^2 + 4$ on $(0, \infty)$

C.N: $f'(x) = 6 - 2x = 0$

$\therefore x = 3$: C.N

By the Second Deriv. Test

$f'' = -2$

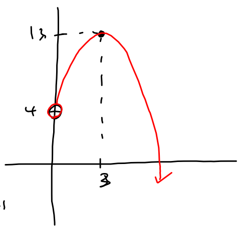
$\Rightarrow f''(3) = -2 < 0$: $\frown$

$\therefore$ f has a local Max at $x=3$

$\therefore$ local Max: $(3, f(3)) = (3, 13)$

end points

$f(0) = 4$



$\therefore$ Absol. Max: $y=13$ at $x=3$

Absol. min: None

b) $g(x) = 2x + \frac{8}{x}$ on $(0, 10)$ $= 2x + 8x^{-1}$

C.N: $g'(x) = 2 - 8x^{-2} = 2 - \frac{8}{x^2} = 0$

$\Rightarrow \frac{2x^2 - 8}{x^2} = 0$

$\Rightarrow 2x^2 - 8 = 0$

$\Rightarrow 2(x^2 - 4) = 0$

$\Rightarrow 2(x+2)(x-2) = 0$

$\therefore x = \cancel{-2}, 2$: C.Ns

By the Second Deriv. Test

$g'' = 16x^{-3} = \frac{16}{x^3}$

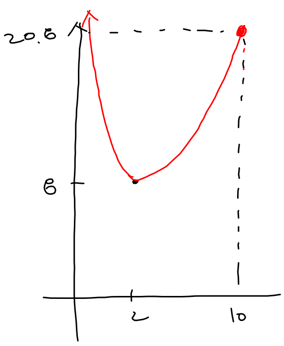
$\Rightarrow g''(2) = \frac{16}{(2)^3} > 0$: $\smile$

$\therefore$ g has a local min at $x=2$

$\therefore$ local min: $(2, g(2)) = (2, 8)$

end points

$g(10) = 2(10) + \frac{8}{(10)} = 20 + \frac{8}{10} = 20.8$



$\therefore$ Absol. min: $y=8$ at $x=2$

Absol. Max: None

Section 5.6 Optimization and Modeling

Strategy for solving Optimization Problems

(a) Introduce variables, look for relationships among these variables, and construct a mathematical model of the form.

(b) Find the critical values of $f(x)$.

(c) Use the procedures developed in Section 5.5 to find the absolute maximum (or minimum) value of $f(x)$ on the interval I and the value(s) of x where this occurs.

(d) Use the solution to answer all questions asked in the problem.

Ex.3) Find two positive numbers whose sum is 60 and whose product is a maximum.

x, y

$x+y=60 \Rightarrow y=60-x$

Q: max product

$\Rightarrow P = x \cdot y$

$\Rightarrow P(x) = x\cdot(60-x) = 60x - x^2$

Find Max of P

C.N: $P' = 60 - 2x = 0$

$\therefore x = 30$: C.N

By the Second Deriv. Test

$P'' = -2$

$\Rightarrow P''(30) = -2 < 0$:

$\therefore$ P has a local Max at $x=30$

$\therefore$ Absol. Max

$\therefore$ Two positive numbers for Max product

: 30, 30