TOWARDS A GEOMETRIC APPROACH TO STRASSEN'S ASYMPTOTIC RANK CONJECTURE

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Abstract. We make a first geometric study of three varieties in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ (for each m), including the Zariski closure of the set of *tight tensors*, the tensors with continuous regular symmetry. Our motivation is to develop a geometric framework for Strassen's *asymptotic rank conjecture* that the asymptotic rank of any tight tensor is minimal. In particular, we determine the dimension of the set of tight tensors. Surprisingly we prove this dimension equals the dimension of the set of *oblique tensors*, a less restrictive class of tensors that Strassen identified as useful for his laser method.

1. Introduction

Strassen's asymptotic rank conjecture (see Conjecture 1.3 below) is a generalization of the famous conjecture that the exponent of matrix multiplication is two. An even further generalization of it is posed as a question in [BCS97, Problem 15.5]. Strassen proved remarkable properties about *tight tensors* defined below that led to the conjecture. Tight tensors were originally defined because of their combinatorial properties that make them useful for Strassen's laser method for proving upper bounds on the exponent of matrix multiplication. The purpose of this paper is to place these conjectures, and additional intermediate questions, in a geometric framework as a first step to comparing them and developing approaches for attacking them with geometric methods.

We make a first geometric study of algebraic varieties defined by three classes of tensors, each characterized by combinatorial properties. We compare these varieties with the well-studied orbit closure of the matrix multiplication tensor and the ambient projective space. We ask questions intermediate to the asymptotic rank conjecture and Problem 15.5 in [BCS97] for these classes of tensors. These classes arise in algebraic complexity theory [Str94], quantum information theory [CVZ18], and geometric invariant theory (more precisely, the study of rational moment polytopes [Bri87, Nes84, Fra02]). We also further the study of the combinatorial properties of these tensors, drawing connections between the original conjecture and its generalizations.

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1.1. **Definitions and Notation.** Throughout the paper, A, B, C denote complex vector spaces respectively of dimension $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Given two tensors $T_1 \in A_1 \otimes B_1 \otimes C_1$ and $T_2 \in A_2 \otimes B_2 \otimes C_2$, one can regard the tensor $T_1 \otimes T_2$ as an element of $(A_1 \otimes A_2) \otimes (B_1 \otimes B_2) \otimes (C_1 \otimes C_2)$. This is called *Kronecker product* of T_1 and T_2 and it is denoted by $T_1 \boxtimes T_2$. Kronecker powers are defined iteratively: for $T \in A \otimes B \otimes C$, let $T^{\boxtimes N} = T^{\boxtimes N-1} \boxtimes T$, which is a tensor in $(A^{\otimes N}) \otimes (B^{\otimes N}) \otimes (C^{\otimes N})$.

A tensor $T \in A \otimes B \otimes C$ is *concise* if the three linear maps $T_A : A^* \to B \otimes C$, $T_B : B^* \to A \otimes C$ and $T_C : C^* \to A \otimes B$ are injective. Kronecker products of concise tensors are concise and in particular if T is concise then $T^{\boxtimes N}$ is concise as well. In order to avoid trivialities, we always work with concise tensors.

The *rank* of $T \in A \otimes B \otimes C$, denoted $\mathbf{R}(T)$, is the smallest integer r such that $T = \sum_{j=1}^{r} u_j \otimes v_j \otimes w_j$ with $u_j \in A$, $v_j \in B$, $w_j \in C$. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that T may be expressed as a limit (in the Euclidean topology) of tensors of rank r. The *asymptotic rank* of T is $\underset{\sim}{\mathbf{R}}(T) = \lim_{N\to\infty} \mathbf{R}(T^{\boxtimes N})^{1/N} = \lim_{N\to\infty} \underline{\mathbf{R}}(T^{\boxtimes N})^{1/N}$. In [Str87] the limits are shown to exist and to be equal.

For every tensor $T \in A \otimes B \otimes C$, we have $\mathbf{R}(T) \geq \underline{\mathbf{R}}(T) \geq \underset{\sim}{\mathbf{R}}(T)$; if T is concise then $\underset{\sim}{\mathbf{R}}(T) \geq \max\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. When equality holds we say T has *minimal* asymptotic rank. Moreover, $\mathbf{R}(T_1 \boxtimes T_2) \leq \mathbf{R}(T_1)\mathbf{R}(T_2)$ and similarly for border rank and asymptotic rank.

Border rank and asymptotic rank are lower semicontinuous under degeneration: Let $G := GL(A) \times GL(B) \times GL(C)$ and let $T, T' \in A \otimes B \otimes C$. We say that T' is a *degeneration* of T if $T' \in \overline{G \cdot T}$, where $\overline{G \cdot T}$ denotes the orbit closure (equivalently in the Zariski or the Euclidean topology) of the tensor T under the natural action of G. One has $\underline{\mathbf{R}}(T') \leq \underline{\mathbf{R}}(T)$ and $\underset{\sim}{\mathbf{R}}(T') \leq \underset{\sim}{\mathbf{R}}(T)$.

Given $m \in \mathbb{N}$, let $[m] := \{1, \ldots, m\}$. Given a subset $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$, let $|\mathcal{S}|$ denote its cardinality. Given a tensor $T = \sum_{ijk} T^{ijk} a_i \otimes b_j \otimes c_k$ with $\{a_i\}$ a basis of A and similarly for $\{b_j\}$ and $\{c_k\}$, the *support* of T in this basis is the set $\mathrm{supp}(T) = \{(i,j,k) : T^{ijk} \neq 0\} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$. We say that a set $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ is concise if the restrictions of the three projections on $[\mathbf{a}]$, $[\mathbf{b}]$ and $[\mathbf{c}]$ to $\mathcal{S}$ are surjective. Generic tensors with concise support are concise.

From a geometric perspective, tightness is a property concerning the stabilizer of T under the action of G: a tensor is tight if the stabilizer of T in G contains a regular semisimple element. The computer science literature (see, e.g., [BCS97, Blä13]) generally works with an equivalent combinatorial definition in terms of the support of T in a preferred basis, as in Definition 1.1. We refer to [Str94] and Section 2.1 for details on the geometric definition and the proof of the equivalence between the two definitions. The combinatorial point of view naturally offers two generalizations, which already appeared in [Str87].

Definition 1.1. A concise subset $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ is called

- *tight* if there exist injective functions $\tau_A : [\mathbf{a}] \to \mathbb{Z}$, $\tau_B : [\mathbf{b}] \to \mathbb{Z}$ and $\tau_C : [\mathbf{c}] \to \mathbb{Z}$ such that $\tau_A(i) + \tau_B(j) + \tau_C(k) = 0$ for every $(i,j,k) \in \mathcal{S}$.
- *oblique* if no two elements of $\mathcal{S}$ are comparable under the partial ordering on $[\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ induced by total orders on $[\mathbf{a}]$,$[\mathbf{b}]$,$[\mathbf{c}]$ (one says $\mathcal{S}$ is an *antichain*);
- *free* if any two $(i_1, j_1, k_1), (i_2, j_2, k_2) \in \mathcal{S}$ differ in at least two entries.

A tensor $T \in A \otimes B \otimes C$ is *tight* (resp. *oblique*, resp. *free*) if there exists a choice of bases $\{a_i\}_{i\in[\mathbf{a}]}, \{b_j\}_{j\in[\mathbf{b}]}, \{c_i\}_{k\in[\mathbf{c}]}$ such that the support $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ of T in the given bases is a tight (resp. oblique, resp. free) subset. In this case, the chosen basis is called a tight (resp. oblique, resp. free) basis. (Sometimes in the computer science literature tightness is defined as a property of a tensor in given coordinates, rather than a property of the tensor.)

1.2. **Questions and Conjectures.** The matrix multiplication tensor $M_{\langle \mathbf{n} \rangle} \in \mathrm{Mat}_{\mathbf{n}}^* \otimes \mathrm{Mat}_{\mathbf{n}}^* \otimes \mathrm{Mat}_{\mathbf{n}}$ is the bilinear map sending two matrices of size $\mathbf{n} \times \mathbf{n}$ to their product. It has the self-reproducing property $M_{\langle \mathbf{n} \rangle}^{\boxtimes N} = M_{\langle \mathbf{n}^N \rangle}$. Moreover, $M_{\langle \mathbf{n} \rangle}$ is tight. The famous conjecture that the exponent of matrix multiplication is two may be phrased in terms of the asymptotic rank:

Conjecture 1.2. *For some (and as a consequence all)* $\mathbf{n} > 1$, $\underset{\sim}{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) = \mathbf{n}^2$, *i.e.,* $M_{\langle \mathbf{n} \rangle}$ *has minimal asymptotic rank.*

Strassen proposed a generalization of Conjecture 1.2 that would imply the following conjecture:

Conjecture 1.3 (Strassen's Asymptotic Rank Conjecture, [Str94])**.** *Let* $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ *be tight and concise. Then* $\underset{\sim}{\mathbf{R}}(T) = m$, *i.e., all concise tight tensors have minimal asymptotic rank.*

The precise conjecture Strassen proposed is stated and discussed in §3.

In [BCS97], the authors asked if tightness is needed in Conjecture 1.3:

Question 1.4 ([BCS97], Problem 15.5)**.** Is $\underset{\sim}{\mathbf{R}}(T) = m$ for all concise $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$? I.e., do all tensors have minimal asymptotic rank?

Every tight tensor is oblique and every oblique tensor is free, see Remarks 2.3 and 2.9. We are unaware of geometric definitions of obliqueness and freeness.

Problem 1.5. Find geometric, i.e., coordinate free, characterizations for obliqueness and freeness.

One could consider questions intermediate to Conjecture 1.3 and Question 1.4 in terms of oblique and free tensors. Explicitly:

Question 1.6. Let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be oblique and concise. Is $\underset{\sim}{\mathbf{R}}(T) = m$? I.e., do all concise oblique tensors have minimal asymptotic rank?

Question 1.7. Let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be free and concise. Is $\underset{\sim}{\mathbf{R}}(T) = m$? I.e., do all concise free tensors have minimal asymptotic rank?

We have the hierarchy of affirmative answers: Question 1.4 $\Rightarrow$ Question 1.7 $\Rightarrow$ Question 1.6 $\Rightarrow$ Conjecture 1.3 $\Rightarrow$ Conjecture 1.2, i.e., $\omega = 2$.

1.3. **How different are the above five questions/conjectures?** We address this question by determining the dimensions of the varieties of tensors to which each conjecture applies.

Let $\overline{Tight}_m$, $\overline{Oblique}_m$ and $\overline{Free}_m$ be the closures (equivalently in the Zariski or Euclidean topology) of the sets of tight, oblique and free tensors respectively. Let

$$\overline{MaMu}_m := \overline{GL(A) \times GL(B) \times GL(C) \cdot M_{\langle \mathbf{n} \rangle}} \subseteq A \otimes B \otimes C$$

with $\mathbf{a} = \mathbf{b} = \mathbf{c} = m = \mathbf{n}^2$. Then Conjecture 1.2 may be rephrased as: If $T \in \overline{MaMu_m}$, then $\underset{\sim}{\mathbf{R}}(T) \leq m$. Similar reformulations of the other questions/conjectures can be given in terms of the varieties $\overline{Tight_m}, \overline{Oblique_m}, \overline{Free_m}$ and finally in terms of the space $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$.

The following result determines the dimensions of the varieties $\overline{MaMu_m}, \overline{Tight_m}, \overline{Oblique_m}, \overline{Free_m}$, providing a first comparison among the sets of tensors to which each of the conjectures mentioned above applies.

Theorem 1.8. *Let $m \geq 2$ and let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$. Then*

(i) if $m = \mathbf{n}^2$, then $\dim \overline{MaMu_m} = 3m^2 - 3m$;

(ii) $\dim \overline{Tight_m} = 3m^2 + \lceil \frac{3}{4} m^2 \rceil - 3m$;

(iii) $\dim \overline{Oblique_m} = 3m^2 + \lceil \frac{3}{4} m^2 \rceil - 3m$;

(iv) $\dim \overline{Free_m} = 4m^2 - 3m$.

Previous to this work, it was not even known if the varieties $\overline{Tight_m}, \overline{Oblique_m}, \overline{Free_m}$ were distinct, i.e., if Conjecture 1.3 and Questions 1.6, 1.7 are all distinct.

The statement of (i) dates back at least to [dG78]. The proofs of the remaining statements are obtained in Section 2 by applying a natural geometric construction (an incidence correspondence) to the explicit maximal supports for each case in Theorem 1.10 below, which is also proved in Section 2.

For any fixed m, one can ask the same questions. Note that for any fixed $\mathbf{n} > 1$, were $\underset{\sim}{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) = \mathbf{n}^2$, it would imply $\omega = 2$, in particular when $\mathbf{n} = 2$, i.e., in the case $m = 4$. Perhaps even more interesting is the case $m = 3$. Despite there being no matrix multiplication tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, by, e.g., [BCS97, Remark 15.44], already Conjecture 1.3 in the case $m = 3$ would imply $\omega = 2$. (The remark points out that were the asymptotic rank of the small Coppersmith-Winograd tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ two, it would imply that $\omega = 2$.) Here the four questions/conjectures reduce to two:

Theorem 1.9. *If $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$, then $A \otimes B \otimes C = \overline{Free_3}$ and $\overline{Tight_3} = \overline{Oblique_3}$, which has codimension 2.*

Regarding the next case, when $\mathbf{a} = \mathbf{b} = \mathbf{c} = 4$, $\overline{Free_4}$ is a variety of codimension 2, $\overline{Tight_4}$ has codimension 16, and $\underline{MaMu_4}$ has codimension 28. Proposition 2.13 shows that the inclusion $\overline{Tight_m} \subseteq \overline{Oblique_m}$ is strict for all $m \geq 4$.

One can measure the strength of each conjecture/question by the dimension of the variety it determines. This measure fails to distinguish Conjecture 1.3 from Question 1.6. If one looks just at the exponent of the leading term (as a function of m) of these dimensions, we see all but Question 1.4 look the same by this coarse measure.

Theorem 1.10. *Let $\mathcal{S} \subseteq [m] \times [m] \times [m]$. Then*

(i) if $\mathcal{S}$ is tight then $|\mathcal{S}| \leq \lceil \frac{3}{4} m^2 \rceil$ and the inequality is sharp;

(ii) if $\mathcal{S}$ is oblique then $|\mathcal{S}| \leq \lceil \frac{3}{4} m^2 \rceil$ and the inequality is sharp;

(iii) if $\mathcal{S}$ is free then $|\mathcal{S}| \leq m^2$ and the inequality is sharp.

Note that with $m = \mathbf{n}^2$, the standard presentation of the matrix multiplication tensor gives $|\mathrm{supp}(M_{\langle \mathbf{n} \rangle})| = \lfloor \frac{3}{2} m^2 \rfloor$. We remark that all the varieties in question are invariant under the action of a torus of dimension equal to the maximum $\mathcal{S}$ in Theorem 1.10.

The sharpness results in Theorem 1.10 follow by exhibiting explicit supports with the desired cardinality. The support described in the proof of Theorem 1.10(i) is used in [LM] to construct the first explicit sequence (depending on m) of tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of border rank greater than $2m$.

In Section 3, we discuss *compressibility* and *slice rank* of tensors. We explain *Strassen's support functionals* and how they motivate Conjecture 1.3. In particular, we prove that tight tensors are far more compressible than generic tensors (Theorem 3.3), which could be taken as evidence to favor Conjecture 1.3 over the other questions.

In Section 4, we establish results on the growth of symmetry groups of tensors under direct sums and Kronecker products. See Theorem 4.1). The dimension of the symmetry group of a tensor is a geometric invariant which is upper semicontinuous under degeneration. In particular, the result of Theorem 4.1(iii) shows that tensors which are generic in terms of dimension of symmetry group (namely having 0-dimensional symmetry group), remain generic under Kronecker product. This can also be interpreted as evidence to favor Conjecture 1.3 over questions 1.6, 1.7 and 1.4. Finally, Theorem 4.1 is motivated by the connection between symmetries of a tensor and the Strassen laser method. We refer to [CGLV19a] for details on the method: we mention here that this technique can be applied to *block tight tensors*, defined implicitly in [BCS97, §15.6] and explicitly in [Lan19, Def. 5.1.4.2], a property weaker than tightness but still implying the tensor has continuous symmetries. The laser method is responsible for the progress on upper bounds for the complexity of matrix multiplication ever since its introduction in [Str87], and it has been most useful for tensors with large symmetry group, as observed in [CGLV19b].

Remark 1.11. Since the intitial arXiv posting of this paper, initial progress on issues raised in this paper has been made: for the important special case of $m = 3$ discussed in Theorem 1.9, in [CGLV19a] numerical evidence that for generic tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, $22 = \underline{\mathbf{R}}(T^{\boxtimes 2}) < \underline{\mathbf{R}}(T)^2 = 25$, which could be taken as positive evidence for an affirmative answer to Question 1.4 when $m = 3$. On the other hand it is shown that there are tight tensors of non-minimal border rank (namely the small Coppersmith-Winograd tensor) such that $\underline{\mathbf{R}}(T^{\boxtimes 2}) = \underline{\mathbf{R}}(T)^2$ and even $\underline{\mathbf{R}}(T^{\boxtimes 3}) = \underline{\mathbf{R}}(T)^3$, which could be taken as negative evidence even for Conjecture 1.3 for general m.

2. Tight, oblique, and free tensors

In this section, we establish information about the sets of tight, oblique and free tensors, and prove Theorems 1.8 and 1.10.

2.1. **Tight tensors.**

Tightness can be characterized in terms of the stabilizer of a tensor in $A \otimes B \otimes C$ under the action of $G = GL(A) \times GL(B) \times GL(C)$. We introduce some useful notation and definitions.

Let $\Phi : GL(A) \times GL(B) \times GL(C) \to GL(A \otimes B \otimes C)$ be the group homomorphism defining the natural action of $GL(A) \times GL(B) \times GL(C)$ on $A \otimes B \otimes C$; Φ has a 2-dimensional kernel $Z_{A,B,C} = \{(\lambda \mathrm{Id}_A, \mu \mathrm{Id}_B, \nu \mathrm{Id}_C) : \lambda\mu\nu = 1\}$, so that $G := (GL(A) \times GL(B) \times GL(C))/Z_{A,B,C}$ can be regarded as a subgroup of $GL(A \otimes B \otimes C)$. The symmetry group of T, denoted G_T, is the stabilizer in G under this action, that is $G_T := \{g \in G : g \cdot T = T\}$.

The differential $d\Phi$ of Φ induces a map at the level of Lie algebras: write $\mathfrak{g}_T$ for the annihilator of a tensor T under the action of $(\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C))/\mathfrak{z}_{A,B,C}$ where $\mathfrak{z}_{A,B,C} \simeq \mathbb{C}^2$ is the Lie algebra of $Z_{A,B,C}$: explicitly $\mathfrak{z}_{A,B,C} = \{(\lambda \mathrm{Id}_A, \mu \mathrm{Id}_B, \nu \mathrm{Id}_C : \lambda + \mu + \nu = 0\}$. Since $\mathfrak{g}_T$ is the Lie algebra of G_T, it determines the continuous symmetries of T, i.e., the connected component of the identity of G_T.

Fix $T \in A \otimes B \otimes C$. Then T is tight if and only if $\mathfrak{g}_T$ contains a regular semisimple element of $(\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C))/\mathfrak{z}_{A,B,C}$. A regular semisimple element is a triple $L = (X, Y, Z)$ which, under some choice of bases, is represented by diagonal matrices X, Y, Z, each of them having distinct (rational) eigenvalues. Equivalently, T is stabilized by a regular semisimple one-parameter subgroup of $(GL(A) \times GL(B) \times GL(C))/Z_{A,B,C}$. Observe that the tightness of T in a given basis only depends on the support of T; in particular, the eigenvalues of the three matrices in $L = (X, Y, Z)$, suitably rescaled, provide the functions τ_A, τ_B, τ_C of Definition 1.1. We refer to [Str91, Str05] for the complete proof that the two characterizations are equivalent.

Example 2.1 (A tight support of cardinality $\lceil \frac{3}{4} m^2 \rceil$)**.** Let $m \geq 0$ be an odd integer and write $m = 2\ell + 1$. Define

$$\mathcal{S}_{\textit{t-max},m} = \{(i,j,k) \in [m] \times [m] \times [m] : i + j + k = 3\ell\}.$$

By Definition 1.1, $\mathcal{S}_{\textit{t-max},m}$ is tight. Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ and let $T \in A \otimes B \otimes C$ be any tensor with support $\mathcal{S}_{\textit{t-max},m}$. Let $L = (U, V, W) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ be the triple of diagonal matrices $U = V = W$ having $i - \ell$ at the i-th diagonal entry, with $i = 0, \ldots, m-1$. Then $L.T = 0$, because for every element $(i,j,k) \in \mathrm{supp}(T)$ we have

$$L.(a_i \otimes b_j \otimes c_k) = [(i-\ell) + (j-\ell) + (k-\ell)] a_i \otimes b_j \otimes c_k = 0.$$

If T has support $\mathcal{S}_{\textit{t-max},m}$, one can write $T = \sum_{jk} T^{jk} a_{3\ell-j-k} \otimes b_j \otimes c_k$. We can represent T as an $m \times m$ matrix whose entries are elements of A; in this case, we have

$$\begin{bmatrix}
 & & & T^{0,\ell}a_{2\ell} & \cdots & T^{0,2\ell-1}a_{\ell+1} & T^{0,2\ell}a_{\ell} \\
 & & \iddots & & & \iddots & T^{1,2\ell}a_{\ell-1} \\
 & \iddots & & & \iddots & & \vdots \\
T^{\ell,0}a_{2\ell} & & & \iddots & & & T^{\ell,2\ell}a_0 \\
\vdots & & \iddots & & & \iddots & \\
T^{2\ell-1,0}a_{\ell+1} & \iddots & & & \iddots & & \\
T^{2\ell,0}a_{\ell} & T^{2\ell,1}a_{\ell-1} & \cdots & T^{2\ell,\ell}a_0 & & &
\end{bmatrix}. \tag{1}$$

Each nonzero entry in this matrix corresponds to an element of $\mathcal{S}_{t\text{-}max,m}$; each of the two triangles of 0's (the top left and the bottom right) consists of $\binom{\ell+1}{2}$ entries. Therefore the number of nonzero entries is $(2\ell+1)^2 - (\ell+1)(\ell) = 3\ell^2 + 3\ell + 1 = \lceil \frac{3}{4}m^2 \rceil$.

If $m = 2\ell$ is even, one obtains a tight support of cardinality $3\ell^2 = \lceil \frac{3}{4}m^2 \rceil$ by erasing the last row and the last column of the matrix and setting a_0 to 0. Geometrically this is equivalent to applying the projection which sends $a_0, b_{2\ell}, c_{2\ell}$ to 0 and the other basis vectors of the odd dimensional spaces to basis vectors of the even dimensional spaces. Explicitly, if one has bases $\{a_0, \dots, a_{2\ell-1}\}$, $\{b_0, \dots, b_{2\ell-1}\}$,$\{c_0, \dots, c_{2\ell-1}\}$ of the spaces A, B, C of dimension 2ℓ, the tight support is determined by the functions $\tau_A(i) = i - \ell + 1$, $\tau_B(j) = \tau_C(j) = j - \ell$.

It turns out that the element L introduced in Example 2.1 is, up to scale, the only non-trivial element of $\mathfrak{g}$ which annihilates a generic tensor with support $\mathcal{S}_{t\text{-}max,m}$, as shown in the following result.

Proposition 2.2. *Let $T \in A \otimes B \otimes C$ be a generic tensor with support $\mathcal{S}_{t\text{-}max,m}$. Then $\dim \mathfrak{g}_T = 1$ and $\mathfrak{g}_T = \langle L \rangle$ where $\langle - \rangle$ denotes the linear span and $L = (U, V, W)$ where U, V, W are diagonal with $u^i_i = v^i_i = w^i_i = i - \ell$.*

Proof. The Theorem of semicontinuity of dimension of the fiber (see e.g., [Sha94, Thm. 1.25]) implies that $\dim \mathfrak{g}_T$ is an upper semicontinuous function. In particular, it suffices to prove the statement for a single element T with support $\mathcal{S}_{t\text{-}max,m}$. Suppose that the coefficients of T are $T^{ijk} = 1$ for every $(i, j, k) \in \mathcal{S}_{t\text{-}max,m}$.

We give the proof in the case $m = 2\ell + 1$ odd. If m is even, the argument is essentially the same, with minor modifications to the index ranges.

Let $d\Phi : \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C) \to \mathfrak{gl}(A \otimes B \otimes C)$ be the differential of the map Φ defined at the beginning of Section 2.1. We show that the annihilator of T under the action of $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ has dimension 3, and coincides with $\langle L \rangle + \ker(d\Phi)$.

Let $(U, V, W) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$; set $u^i_{i'} = 0$ if $i, i' \notin \{0, \dots, 2\ell\}$ and similarly for $v^j_{j'}$ and $w^k_{k'}$. Suppose $(U, V, W) \in \mathfrak{g}_T$, so that every triple (i, j, k) provides a (possibly trivial) equation on the entries of U, V, W as follows

$$(2) \qquad (i, j, k) \qquad u^{i'}_i + v^{j'}_j + w^{k'}_k = 0$$

where i', j', k' are the only integers such that $i' + j + k = i + j' + k = i + j + k' = 3\ell$. Let $\rho = 3\ell - (i + j + k)$; moreover $\rho \in \{-2\ell, \dots, 2\ell\}$ and $\rho = i' - i = j' - j = k' - k$. The equations in (2) can be partitioned into $4\ell + 1$ subsets, indexed by $\rho = -2\ell, \dots, 2\ell$, so that equations in distinct subsets involve disjoint sets of variables. Our goal is to show that the ρ-th set of equations has no nontrivial solutions if $\rho \neq 0$, whereas the 0-th set of equations has exactly a space of solutions of dimension 3 which induces $(U, V, W) \in \langle L \rangle + \ker(d\Phi)$. Indeed, notice that $(U, V, W) \in \langle L \rangle + \ker(d\Phi)$ satisfies all equations in (2).

We consider three separate cases: $\rho = 0, 0 < \rho < \ell$ and $\rho \geq \ell$. The cases $0 > \rho > -\ell$ and $\rho \leq -\ell$ are analogous.

Case $\rho \geq \ell$. First, observe that $u^\rho_0 = v^\rho_0 = w^\rho_0 = 0$. To show this, consider the three equations corresponding to $(i, j, k) = (0, 0, 3\ell - \rho), (0, 3\ell - \rho, 0)$ and $(3\ell - \rho, 0, 0)$, which give the linear system

$$\begin{cases} u_0^\rho + v_0^\rho = 0 \\ u_0^\rho + w_0^\rho = 0 \\ v_0^\rho + w_0^\rho = 0 \end{cases} \tag{3}$$

in the three unknowns $u_0^\rho, v_0^\rho, w_0^\rho$; this linear system has full rank. This shows $u_0^i = v_0^j = w_0^k = 0$ if $i, j, k \geq \ell$.

Now fix q with $\ell > q \geq 1$; we show that $u_q^{\rho+q} = v_q^{\rho+q} = w_q^{\rho+q} = 0$. The equation corresponding to $(i, j, k) = (q, 0, 3\ell - \rho - q)$ is $u_q^{\rho+q} + v_0^\rho = 0$, which provides $u_q^{\rho+q} = 0$ since $v_0^\rho = 0$; similarly $v_q^{\rho+q} = w_q^{\rho+q} = 0$. If $q = \ell$, then $\rho = \ell$ as well (otherwise $u_q^{\ell+q}$ is trivially 0 because $\ell + q > 2\ell$). In this case, the equations corresponding to $(i, j, k) = (0, \ell, \ell), (\ell, 0, \ell), (\ell, \ell, 0)$ provide a linear system similar to (3) which provides $u_\ell^{2\ell} = v_\ell^{2\ell} = w_\ell^{2\ell} = 0$.

Apply a similar argument to the case $\rho \leq -\ell$.

Case $0 < \rho < \ell$**.** We have $\ell \geq 2$, otherwise this case does not occur. First, we show that $u_{2\ell-\rho}^{2\ell} = v_{2\ell-\rho}^{2\ell} = w_{2\ell-\rho}^{2\ell} = 0$. This is obtained in two steps. First consider the three equations corresponding to the indices $(2\ell-\rho+1, \ell-1, 0), (2\ell-\rho+2, \ell-2, 0), (2\ell-\rho+1, \ell-2, 1)$, which are

$$\begin{aligned} &v_{\ell-1}^{\ell-1+\rho} + w_0^\rho = 0, \\ &v_{\ell-2}^{\ell-2+\rho} + w_0^\rho = 0, \\ &v_{\ell-2}^{\ell-2+\rho} + w_1^{\rho+1} = 0; \end{aligned} \tag{4}$$

these provide $v_{\ell-1}^{\ell-1+\rho} + w_1^{\rho+1} = 0$. Now the equation corresponding to the indices $(2\ell-\rho, \ell-1, 1)$, namely $u_{2\ell-\rho}^{2\ell} + v_{\ell-1}^{\ell-1+\rho} + w_1^{\rho+1} = 0$, reduces to $u_{2\ell-\rho}^{2\ell} = 0$; similarly, we have $v_{2\ell-\rho}^{2\ell} = w_{2\ell-\rho}^{2\ell} = 0$.

This provides the base case for an induction argument. If $q \geq 1$, we show that $u_{2\ell-q-\rho}^{2\ell-q} = 0$. This argument is similar to the one before: the three equations corresponding to $(2\ell-\rho-(q-1), \ell+(q-1), 0), (2\ell-\rho+1-(q-1), \ell-1+(q-1), 0), (2\ell-\rho-(q-1), \ell-1+(q-1), 1)$, together with the induction hypothesis, reduce to $v_{\ell+(q-1)}^{\ell+(q-1)+\rho} + w_1^{\rho+1} = 0$. The latter equality, together with the equation corresponding to the indices $(2\ell-q-\rho, \ell+(q-1), 1)$ gives $u_{2\ell-q-\rho}^{2\ell-q} = 0$. Similarly, we have $v_{2\ell-q-\rho}^{2\ell-q}$ for every $q = 0, \ldots, 2\ell-\rho$. We conclude that $u_i^{i+\rho} = v_j^{j+\rho} = w_k^{k+\rho} = 0$ for every i, j, k and every $0 < \rho < \ell$.

Apply a similar argument to the case $0 > \rho > -\ell$.

Case $\rho = 0$**.** We may work modulo $\ker(d\Phi) = \langle(\mathrm{Id}_A, -\mathrm{Id}_B, 0), (\mathrm{Id}_A, 0, -\mathrm{Id}_C)\rangle$. In particular, we may assume V, W satisfy $\mathrm{trace}(V) = \mathrm{trace}(W) = 0$. Consider all equations (ℓ, j, k) so that $j + k = 2\ell$. Adding them up and using the traceless condition, we have $u_\ell^\ell = 0$ and therefore $v_{\ell+q}^{\ell+q} = -w_{\ell-q}^{\ell-q}$ for $q = -\ell, \ldots, \ell$. Let $\xi = u_{\ell+1}^{\ell+1}$. Then for every q, the equation $(\ell+1, \ell+q-1, \ell-q)$ gives $v_{\ell+q-1}^{\ell+q-1} + \xi = -w_{\ell-q}^{\ell-q} = v_{\ell+q}^{\ell+q}$, so that one has $v_{\ell+q}^{\ell+q} = v_\ell^\ell + q\xi$ and similarly $w_{\ell+q}^{\ell+q} = w_\ell^\ell + q\xi$. Since V and W are traceless, we obtain $v_\ell^\ell = w_\ell^\ell = 0$ and $v_{\ell+q}^{\ell+q} = w_{\ell+q}^{\ell+q} = q\xi$. In particular, by adding up the equations for the form $(\ell+q, \ell-q, \ell)$ for $q = -\ell, \ldots, \ell$, we observe that U is traceless as well, and by a similar argument $u_{\ell+q}^{\ell+q} = q\xi$ as well, so that $(U, V, W) = L$. This shows that modulo $\ker(d\Phi)$ we have a $\mathfrak{g}_T = \langle L\rangle$, and this concludes the proof. □

2.2. **Oblique tensors.** Recall that a tensor T is oblique if there are bases such that $\mathrm{supp}(T)$ is an antichain in $[\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ under the partial ordering induced by three total orders on $[\mathbf{a}], [\mathbf{b}], [\mathbf{c}]$. The original definition of oblique considers the three sets $[\mathbf{a}], [\mathbf{b}], [\mathbf{c}]$ with the natural ordering induced by $\mathbb{N}$. Our definition allows reordering in the index ranges of each factor: this does not affect the resulting class of tensors, and provides the following useful fact.

Remark 2.3. Every tight set is oblique. Let $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ be a tight set. After permuting the elements of $[\mathbf{a}], [\mathbf{b}], [\mathbf{c}]$, we may assume that τ_A, τ_B, τ_C are strictly increasing. Suppose $\mathcal{S}$ is not an antichain in $[\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ and let $(i_1, j_1, k_1), (i_2, j_2, k_2) \in \mathcal{S}$ distinct such that $i_1 \leq i_2, j_1 \leq j_2, k_1 \leq k_2$, with at least one strict inequality. Therefore $\tau_A(i_1) + \tau_B(j_1) + \tau_C(k_1) < \tau_A(i_2) + \tau_B(j_2) + \tau_C(k_2)$, in contradiction with the assumption that $\mathcal{S}$ is tight.

In order to give some insights on oblique subsets, we introduce terminology from [Pro82]. To avoid confusion with tensor rank, we use "poset rank" where Proctor uses "rank".

Definition 2.4 ([Pro82])**.** Let $(P, \prec)$ be a poset and let $x, y \in P$. The element x *covers* y if $y \prec x$ and there does not exist $z \in P$ such that $y \prec z \prec x$. A *poset ranked poset* P of *length* r is a poset P with a partition $P = \bigsqcup_{i=0}^{r} P_i$ into $r+1$ *poset ranks* P_i, such that elements in P_i cover only elements in P_{i-1}. A poset ranked poset of length r is *poset rank symmetric* if $|P_i| = |P_{r-i}|$ for $1 \leq i < r/2$. It is *poset rank unimodal* if $|P_1| \leq |P_2| \leq \cdots \leq |P_{h_0}|$ and $|P_{h_0}| \geq |P_{h_0+1}| \geq \cdots \geq |P_{r+1}|$, for some $1 \leq h_0 \leq r+1$.

A poset is *Peck* if it is poset rank symmetric, poset rank unimodal and for every $\ell \geq 1$ no union of ℓ antichains contains more elements than the union of the ℓ largest poset ranks of P.

Example 2.5. For every $\mathbf{a}$, the poset $[\mathbf{a}]$ is poset ranked of length $\mathbf{a} - 1$ and it is Peck.

Using representation-theoretic methods, Proctor [Pro82, Thm. 2] showed that products of Peck posets are Peck posets, with respect to the natural *product ordering* and with poset rank function defined by the sum of the poset rank functions of the factors; in particular $[\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ is Peck according to the induced partial ordering on the product and the poset rank function is given by $h(i, j, k) = i + j + k$.

Remark 2.6. Oblique supports entirely contained in a single poset rank are tight. More explicitly, let $P = [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$. Every oblique tensor T whose support $\mathcal{S}_T$ is an antichain in some poset rank P_h of P is tight. In particular $\mathcal{S}_{t\text{-}max,m}$ coincides with $P_{3\ell}$, with $m = 2\ell+1$ or $m = 2\ell$; using Proctor's terminology, this corresponds to the $\mathfrak{sl}_2$-weight space of weight 0 in the representation $\mathbb{C}^P = \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$ where the factors are regarded as irreducible $\mathfrak{sl}_2$-representations.

The following is a slightly stronger version of Theorem 1.10(ii):

Theorem 2.7. *Let* $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$ *and let* $\mathcal{S} \subset [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ *be oblique. Then*

$$|\mathcal{S}| \leq \begin{cases} \mathbf{ab} - \lfloor \frac{(\mathbf{a}+\mathbf{b}-\mathbf{c})^2}{4} \rfloor & \text{if } \mathbf{a}+\mathbf{b} \geq \mathbf{c} \\ \mathbf{ab} & \text{if } \mathbf{a}+\mathbf{b} \leq \mathbf{c}. \end{cases}$$

Moreover, in all cases there exist $\mathcal{S}$ *such that equality holds.*

Proof. Since $P = [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ is Peck, the cardinality of a maximal antichain is upper bounded by the maximal poset rank subset: since a Peck set is unimodular, the maximal poset rank is the central one, namely $P_{h_{max}} = \{(i, j, k) : i + j + k = h_{max}\}$ where $h_{max} = \lfloor \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-3}{2} \rfloor$ (and equivalently $\lceil \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-3}{2} \rceil$).

If $\mathbf{a}+\mathbf{b}<\mathbf{c}$ then for every $(i,j)\in[\mathbf{a}]\times[\mathbf{b}]$ there exists $k\in[\mathbf{c}]$ such that $i+j+k=h_{max}$, so $|P_{h_{max}}|=\mathbf{ab}$ and the statement of the theorem holds.

Now suppose $\mathbf{a}+\mathbf{b}\geq\mathbf{c}$. Let $\psi:P\to[\mathbf{a}]\times[\mathbf{b}]$ be the projection onto the first two factors. Note that ψ restricted to each P_h is injective because P_h is an antichain. Then $|P_{h_{max}}|=|\psi(P_{h_{max}})|$. We compute the number of elements of $\psi(P_{h_{max}})$. Consider its complement in $[\mathbf{a}]\times[\mathbf{b}]$, that is the set of pairs $(i,j)\in[\mathbf{a}]\times[\mathbf{b}]$ for which there is no $k\in[\mathbf{c}]$ with $i+j+k=h_{max}$. Since $0\leq k\leq\mathbf{c}-1$, these are exactly pairs (i,j) satisfying one of the following conditions:

(i) $i+j\leq h_{max}-(\mathbf{c}-1)-1$, that is $i+j\leq\lfloor\frac{\mathbf{a}+\mathbf{b}-\mathbf{c}-3}{2}\rfloor$;

(ii) $h_{max}\leq i+j-1$, that is $\lfloor\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-1}{2}\rfloor\leq i+j$.

Notice that (i) and (ii) are mutually exclusive. Let $\theta=\lfloor\frac{\mathbf{a}+\mathbf{b}-\mathbf{c}-3}{2}\rfloor$. For every $i=0,\dots,\theta$, and every $j=0,\dots,\theta-i$, we have $i+j\leq\theta$; this gives $1+2+\cdots+(\theta+1)=\binom{\theta+2}{2}$ pairs (i,j) satisfying condition (i). Now, let $i'=\mathbf{a}-1-i$ and $j'=\mathbf{b}-1-j$: condition (ii) can be rephrased as $\mathbf{a}+\mathbf{b}-2-\lfloor\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-1}{2}\rfloor\geq i'+j'$ which in turn becomes $i'+j'\leq\eta$ where $\eta=\lceil\frac{\mathbf{a}+\mathbf{b}-\mathbf{c}-3}{2}\rceil$; this provides $\binom{\eta+2}{2}$ pairs (i',j') which correspond to $\binom{\eta+2}{2}$ pairs (i,j) satisfying (ii). We conclude that the complement of $\psi(P_{h_{max}})$ in $[\mathbf{a}]\times[\mathbf{b}]$ consists of $\binom{\theta+2}{2}+\binom{\eta+2}{2}$ elements. To conclude, observe $\binom{\theta+2}{2}+\binom{\eta+2}{2}=\lfloor\frac{(\mathbf{a}+\mathbf{b}-\mathbf{c})^2}{4}\rfloor$. □

Remark 2.8. The above proof is modeled on the proof of [Str87, Thm. 6.6].

Choosing $\mathbf{a}=\mathbf{b}=\mathbf{c}=m$ in Theorem 2.7, one obtains the bound of Theorem 1.10(ii). Since every tight tensor is oblique, the same bound holds for tight tensors. Since $\mathcal{S}_{t\text{-}max,m}$ from Example 2.1 is a tight support of cardinality $\lceil\frac{3}{4}m^2\rceil$ (which in fact corresponds to a maximal antichain as observed in Remark 2.6), we obtain that the bound is sharp both in the oblique and in the tight case.

2.3. **Free tensors.** We recall that a subset $\mathcal{S}\subseteq[\mathbf{a}]\times[\mathbf{b}]\times[\mathbf{c}]$ is free if any two triples (i,j,k), (i',j',k') in $\mathcal{S}$ differ on at least two entries.

Remark 2.9. Every oblique support is free. Let $\mathcal{S}$ be an oblique support and suppose it is not free. Without loss of generality, $\mathcal{S}$ contains two triples of the form $s_1=(i,j,k_1)$ and $s_2=(i,j,k_2)$ for some k_1,k_2. But then, if $k_1\leq k_2$ then $s_1\leq s_2$ and if $k_2\leq k_1$ then $s_2\leq s_1$, therefore $\mathcal{S}$ is not an antichain, providing a contradiction.

Example 2.10 (A free support of cardinality m^2)**.** We obtain a free support of cardinality m^2 by completing the support $\mathcal{S}_{t\text{-}max,m}$ in a circulant way. More precisely, let $m\geq 0$ be odd with $m=2\ell+1$. Define

$$\mathcal{S}_{f\text{-}max,m}=\{(i,j,k):i+j+k\equiv\ell\mod m\}\subseteq[m]\times[m]\times[m].$$

Notice that in the range where $\ell\leq j+k<3\ell$, then $i=0,\dots,2\ell$ with $i+j+k=3\ell$, recovering the structure of $\mathcal{S}_{t\text{-}max,m}$.

It is immediate from the definition that the cardinality of a free support is at most m^2: indeed, any m^2+1 elements would have at least two triples (i,j,k) with the same (i,j). This observation, together with Example 2.10, completes the proof of Theorem 1.10(iii).

2.4. **Proof of Theorem 1.8.** We first describe the general construction that will be used in the proof.

Fix a vector space V and let $1 \leq \kappa \leq \dim V - 1$. Let $G(\kappa, V)$ denote the Grassmannian of κ-planes through the origin in V and let $\pi_G : \mathcal{T} \to G(\kappa, V)$ denote the tautological subspace bundle of $G(\kappa, V)$, i.e., the vector bundle whose fiber over a κ-dimensional plane $E \in G(\kappa, V)$ is E itself. Let $\pi_V : \mathcal{T} \to V$ denote the projection to V, that is $\pi_V : (E, v) \mapsto v$ for every $E \in G(\kappa, V)$ and $v \in E \subseteq V$.

If $Z \subset G(\kappa, V)$ is a subvariety of dimension z, then $\dim \pi_G^{-1}(Z) = z + \kappa$. Consequently, $\dim \pi_V(\pi_G^{-1}(Z)) \leq z + \kappa$. The action of a group on V naturally induces an action on $G(\kappa, V)$ and the vector bundle $\mathcal{T}$ can be restricted to orbits and orbit-closures of such an action.

We will use this construction in the setting where $V = A \otimes B \otimes C$, and Z is the $GL(A) \times GL(B) \times GL(C)$-orbit closure of the linear space consisting of all tensors with a given support; we refer to such linear space as the *span of a support*.

The variety $\overline{\mathit{Tight}}_m$, (resp. $\overline{\mathit{Oblique}}_m$, $\overline{\mathit{Free}}_m$) is a union of subvarieties of the form $\pi_V(\pi_G^{-1}(Z))$ with $Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E}$ and E is the span of a tight (resp. oblique, free) support in some given bases, regarded as an element of $G(\dim E, A \otimes B \otimes C)$. In particular, we have the following

Lemma 2.11.

$$\dim \overline{\mathit{Tight}}_m = \max \left\{ \dim \pi_V(\pi_G^{-1}(Z)) : \begin{array}{l} Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E} \\ \textit{for some } E \in G(\kappa, A \otimes B \otimes C) \textit{ span of a tight support} \end{array} \right\},$$

and similarly for $\overline{\mathit{Oblique}}_m$ *and* $\overline{\mathit{Free}}_m$.

Proof. Every tight tensor is in the $GL(A) \times GL(B) \times GL(C)$ orbit of a tight tensor in a fixed basis. Moreover, the number of tight supports in a fixed basis is finite. This implies that the irreducible components of the variety $\overline{\mathit{Tight}}_m$ have the form $\pi_V(\pi_G^{-1}(Z))$, where $Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E}$ for some linear space E which is the span of a non-extendable tight support.

Since the number of supports is finite, $\dim \overline{\mathit{Tight}}_m$ is just the dimension of the largest orbits.

The same holds for $\overline{\mathit{Oblique}}_m$ and $\overline{\mathit{Free}}_m$. □

The following lemma gives the dimension of the orbit closure of the span of a concise free support E. Since from Remark 2.3 every tight support is oblique (up to reordering the bases) and from Remark 2.9 every oblique support is free, the same result applies to tight and oblique supports.

Lemma 2.12. *Let* $E \in G(\kappa, A \otimes B \otimes C)$ *be the span of a concise free support and let* $Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E} \subseteq G(\kappa, A \otimes B \otimes C)$. *Then* $\dim Z = \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 - (\mathbf{a} + \mathbf{b} + \mathbf{c})$.

Proof. We show that the affine tangent space to Z at E in the Plucker embedding of $G(\kappa, A \otimes B \otimes C)$ in $\mathbb{P}\Lambda^\kappa(A \otimes B \otimes C)$ has dimension exactly $\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 - (\mathbf{a} + \mathbf{b} + \mathbf{c}) + 1$; in the following, let $\widehat{G}(\kappa, A \otimes B \otimes C) \subseteq \Lambda^\kappa(A \otimes B \otimes C)$ be the cone over $G(\kappa, A \otimes B \otimes C)$. The affine tangent space to Z at E is $\widehat{T}_E Z = \{(\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)) . E\}$, which is naturally a subspace of $\Lambda^\kappa(A \otimes B \otimes C)$. Here E is identified with the element $\bigwedge_{s=1}^{\kappa}(a_{i_s} \otimes b_{j_s} \otimes c_{k_s}) \in \widehat{G}(\kappa, A \otimes B \otimes C)$, where $\{(i_s, j_s, k_s) : s = 1, \ldots, \kappa\}$ is the free support defining E.

Let $(X,Y,Z) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ for three $m \times m$ matrices X,Y,Z. If X,Y,Z are diagonal, then $(X,Y,Z).E = E$ up to scale. Thus $\dim \widehat{T}_E Z \leq \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 - (\mathbf{a}+\mathbf{b}+\mathbf{c}) + 1$. In order to show equality, it suffices to observe that the vectors of the form $(X,Y,Z).E$, where X,Y,Z are three matrices which are all 0 except in a single off-diagonal entry in one of them, are linearly independent and span a subspace of $\Lambda^\kappa(A \otimes B \otimes C)$ which does not contain E; in particular, such a subspace has dimension $\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 - (\mathbf{a}+\mathbf{b}+\mathbf{c})$.

For every $L = (X,Y,Z)$ having exactly one off-diagonal nonzero entry, we observe that $L.E \neq 0$ and that every summand in the expansion of $L.E$ as sum of basis vectors of $\Lambda^\kappa(A \otimes B \otimes C)$ differs from E in exactly one factor: $L.E \neq 0$ follows immediately by freeness, while the second condition is realized whenever E is spanned by basis vectors. In particular, the subspace of $\Lambda^\kappa(A \otimes B \otimes C)$ generated by the $L.E$'s does not contain E.

The same argument shows that the $L.E$'s are linearly independent. Indeed, suppose L_1, L_2 both have exactly one nonzero entry and suppose that $L_1.E$ and $L_2.E$ both have a summand Θ in their expansion as sum of basis vectors of $\Lambda^\kappa(A \otimes B \otimes C)$. Regard Θ as an element of $\widehat{G}(\kappa, A \otimes B \otimes C)$ (it is the wedge product of a set of basis vectors), namely a coordinate κ-plane in $A \otimes B \otimes C$. There are exactly two basis elements $v = a_{i_0} \otimes b_{j_0} \otimes c_{k_0}$, $v' = a_{i_0'} \otimes b_{j_0'} \otimes c_{k_0'}$ such that $v \in E \setminus \Theta$ and $v' \in \Theta \setminus E$ and two of the three factors of v coincide with the corresponding factors of v'. There is a unique element of $L \in \mathfrak{gl}(A) + \mathfrak{gl}(B) + \mathfrak{gl}(C)$ having exactly one off-diagonal entry such that $L.v = v'$, which guarantees $L = L_1 = L_2$. In particular, all the $L.E$'s are linearly independent and this concludes the proof. ☐

In particular, from Lemma 2.12, one immediately obtains $\dim \pi_G^{-1}(Z)$ when Z is the orbit-closure of the span of a concise free support $\mathcal{S}$. If $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$, we have

$$\dim \pi_G^{-1}(Z) = 3m^2 - 3m + |\mathcal{S}|. \tag{5}$$

Equation (5) guarantees that to prove Theorem 1.8, it suffices to determine a tight (resp. oblique, free) support $\mathcal{S}$ such that $|\mathcal{S}| = \lceil \frac{3}{4} m^2 \rceil$ (resp. $\lceil \frac{3}{4} m^2 \rceil$, m^2) and with the property that the projection $\pi_V : \mathcal{T}|_Z \to A \otimes B \otimes C$ is generically finite-to-one. Indeed, if the projection π_V is finite-to-one on $\mathcal{T}|_Z = \pi_G^{-1}(Z)$, we have $\dim \pi_V(\pi_G^{-1}(Z)) = \dim \pi_G^{-1}(Z) = 3m^2 - 3m + |\mathcal{S}|$ and considering $\mathcal{S}$ with $|\mathcal{S}| = \lceil \frac{3}{4} m^2 \rceil$ in the tight case, $|\mathcal{S}| = \lceil \frac{3}{4} m^2 \rceil$ in the oblique case, and $|\mathcal{S}| = m^2$ in the free case, via Lemma 2.11 we obtain the dimensions indicated in Theorem 1.8.

For the tight and oblique cases, we consider $\mathcal{S} = \mathcal{S}_{t\text{-}max,m}$ from Example 2.1, and for the free case we consider $\mathcal{S} = \mathcal{S}_{f\text{-}max,m}$ from Example 2.10.

Tight and Oblique case. Let $Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E} \subseteq G(\lceil \frac{3}{4} m^2 \rceil, A \otimes B \otimes C)$ where $E = \langle \mathcal{S}_{t\text{-}max,m} \rangle$ is the linear space of tensors supported at $\mathcal{S}_{t\text{-}max,m}$. We prove that the fiber of π_V at a generic point of E is 0-dimensional. From Proposition 2.2, we have $\dim G_T = 1$ and in particular the connected component of the identity in G_T is a 1-parameter subgroup which is diagonal in the fixed basis; let Γ_E be this subgroup.

The fiber of $\pi_V|_{\pi_G^{-1}(Z)}$ over a tensor T is the subset of $\mathcal{T}|_Z$ defined by $Y_T = \{(F,T) : F \in Z, T \in F\}$. Our goal is to show that if T is generic, then Y_T is finite. If $(F,T) \in Y_T$, with $F \neq E$, then $F = gE$ for some $g = (g_A, g_B, g_C) \in GL(A) \times GL(B) \times GL(C)$. At least one of g_A, g_B, g_C is not diagonal in the chosen basis, otherwise $gE = E$. The linear space F is a tight support in the bases $g_A(a_i), g_B(b_j), g_C(c_k)$; in particular the one-parameter subgroup $\Gamma_F = g^{-1} \Gamma_E g$ stabilizes every tensor in F and in particular T. We deduce $\Gamma_F \subseteq G_T$. Notice that $\Gamma_F \neq \Gamma_E$, because Γ_F

is not diagonal in the bases a_i, b_j, c_k. Now, Γ_E and Γ_F are two distinct 1-parameter subgroups of G_T, which implies $\dim G_T \geq 2$, in contradiction with Proposition 2.2. This shows that $\pi_V|_{\pi_G^{-1}(Z)}$ is generically finite-to-one.

Free case. Let $Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E} \subseteq G(m^2, A \otimes B \otimes C)$ where $E = \langle \mathcal{S}_{\textit{f-max},m} \rangle$ is the linear space of tensors supported at $\mathcal{S}_{\textit{f-max},m}$. Let T be a tensor in E such that $\operatorname{supp}(T) \subseteq \mathcal{S}_{\textit{t-max},m} \subseteq \mathcal{S}_{\textit{f-max},m}$. The tensor T is tight and the same argument that we followed in the previous case shows that the fiber of π_V is finite at T. By semicontinuity of dimension of the fibers (see e.g., [Sha94, Thm. 1.25]), π_V has 0-dimensional fiber at the generic point of E and therefore $\pi_V|_{\pi_G^{-1}(Z)}$ is generically finite-to-one.

Via equation (5), we now conclude the proof of Theorem 1.8:

$$\begin{aligned}
\dim \overline{\textit{Tight}}_m &= \dim \pi_V(\pi_G^{-1}(Z)) = \dim \pi_G^{-1}(Z) &&= 3m^2 - 3m + |\mathcal{S}_{\textit{t-max},m}| = \\
& &&= 3m^2 - 3m + \lceil \tfrac{3}{4} m^2 \rceil, \\
\dim \overline{\textit{Oblique}}_m &= \dim \pi_V(\pi_G^{-1}(Z)) = \dim \pi_G^{-1}(Z) &&= 3m^2 - 3m + |\mathcal{S}_{\textit{t-max},m}| = \\
& &&= 3m^2 - 3m + \lceil \tfrac{3}{4} m^2 \rceil, \\
\dim \overline{\textit{Free}}_m &= \dim \pi_V(\pi_G^{-1}(Z)) = \dim \pi_G^{-1}(Z) &&= 3m^2 - 3m + |\mathcal{S}_{\textit{f-max},m}| = \\
& &&= 3m^2 - 3m + m^2.
\end{aligned}$$

2.5. **Tight, oblique and free in small dimension and inclusions among classes of tensors.** We saw that every tight tensor is oblique and every oblique tensor is free. The inclusions $\overline{\textit{Oblique}}_m \subseteq \overline{\textit{Free}}_m$ are strict since the two varieties have different dimensions. The varieties $\overline{\textit{Tight}}_m$ and $\overline{\textit{Oblique}}_m$ have the same dimension.

In this subsection we show that $\overline{\textit{Tight}}_3 = \overline{\textit{Oblique}}_3$, and that the inclusion $\overline{\textit{Tight}}_m \subseteq \overline{\textit{Oblique}}_m$ is strict for $m \geq 4$.

Proof of Theorem 1.9. The dimensions follow immediately from Theorem 1.8, so it remains to prove $\overline{\textit{Tight}}_3 = \overline{\textit{Oblique}}_3$. This statement is proved via a computer calculation. There are 144 maximal antichains in $[3] \times [3] \times [3]$; only 80 of these are concise, in the sense that generic tensors with the corresponding support are concise. The group $\mathfrak{S}_3 \times \mathbb{Z}_2$ acts on $[3] \times [3] \times [3]$, where $\mathfrak{S}_3$ permutes the factors and $\mathbb{Z}_2$ maps (i, j, k) to $(2-i, 2-j, 2-k)$. The induced action on subsets of $[3] \times [3] \times [3]$ preserves tight supports and antichains. In particular, without loss of generality, it suffices to prove the statement for an antichain in each orbit of $\mathfrak{S}_3 \times \mathbb{Z}_2$. There are 13 such orbits. The following are representatives for the orbits:

$$
\begin{aligned}
\mathcal{S}_1 &= \{(0,0,2),(0,1,1),(1,0,1),(2,2,0)\},\\
\mathcal{S}_2 &= \{(0,0,2),(0,2,0),(1,1,1),(2,0,0)\},\\
\mathcal{S}_3 &= \{(0,0,2),(0,2,1),(1,1,0),(2,0,1)\},\\
\mathcal{S}_4 &= \{(0,0,2),(0,2,1),(1,2,0),(2,1,1)\},\\
\mathcal{S}_5 &= \{(0,0,2),(0,1,1),(0,2,0),(1,0,1),(2,1,0)\},\\
\mathcal{S}_6 &= \{(0,0,2),(0,1,1),(1,0,1),(1,2,0),(2,1,0)\},\\
\mathcal{S}_7 &= \{(0,0,2),(0,1,1),(1,2,0),(2,0,1),(2,1,0)\},\\
\mathcal{S}_8 &= \{(0,0,2),(0,2,0),(1,1,1),(2,0,1),(2,1,0)\},\\
\mathcal{S}_9 &= \{(0,0,2),(0,2,1),(1,1,1),(2,0,1),(2,2,0)\},\\
\mathcal{S}_{10} &= \{(0,1,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0)\},\\
\mathcal{S}_{11} &= \{(0,0,2),(0,1,1),(0,2,0),(1,0,1),(1,1,0),(2,0,0)\},\\
\mathcal{S}_{12} &= \{(0,0,2),(0,2,1),(1,1,1),(1,2,0),(2,0,1),(2,1,0)\},\\
\mathcal{S}_{13} &= \{(0,1,2),(0,2,1),(1,0,2),(1,1,1),(1,2,0),(2,0,1),(2,1,0)\}.
\end{aligned}
$$

For each of these, we provide the functions τ_A, τ_B, τ_C which guarantee tightness. We record the functions in the following table

	$(\tau_A(0),\tau_A(1),\tau_A(2))$	$(\tau_B(0),\tau_B(1),\tau_B(2))$	$(\tau_C(0),\tau_C(1),\tau_C(2))$
$\mathcal{S}_1$	$(-2,-3,1)$	$(2,1,0)$	$(-1,1,0)$
$\mathcal{S}_2$	$(1,-2,2)$	$(-1,1,0)$	$(-1,1,0)$
$\mathcal{S}_3$	$(-1,2,-2)$	$(1,2,0)$	$(-4,1,0)$
$\mathcal{S}_4$	$(-2,2,-1)$	$(2,-1,0)$	$(-2,2,0)$
$\mathcal{S}_5$	$(-1,-3,2)$	$(1,-1,2)$	$(-1,2,0)$
$\mathcal{S}_6$	$(0,-1,2)$	$(0,-1,2)$	$(-1,1,0)$
$\mathcal{S}_7$	$(2,-3,1)$	$(2,-3,1)$	$(2,1,0)$
$\mathcal{S}_8$	$(2,-2,1)$	$(-2,1,0)$	$(-2,1,0)$
$\mathcal{S}_9$	$(0,1,2)$	$(0,1,2)$	$(-4,-2,0)$
$\mathcal{S}_{10}$	$(-2,2,1)$	$(-2,1,0)$	$(-2,1,0)$
$\mathcal{S}_{11}$	$(-2,1,4)$	$(-2,1,4)$	$(-2,1,4)$
$\mathcal{S}_{12}$	$(-2,1,4)$	$(-2,1,4)$	$(-5,-2,4)$
$\mathcal{S}_{13}$	$(-1,0,1)$	$(-1,0,1)$	$(-1,0,1)$

This shows that every oblique support in $[3]\times[3]\times[3]$ is tight; in particular, every oblique tensor is tight and $\overline{Tight_3} = \overline{Oblique_3}$. □

Proposition 2.13. *Let $T \in A\otimes B\otimes C$ with $\mathbf{a}=\mathbf{b}=\mathbf{c}=4$ be the tensor*

$$
\begin{aligned}
T =& a_0\otimes b_2\otimes c_3 + a_0\otimes b_3\otimes c_2 + a_1\otimes b_0\otimes c_3 + a_1\otimes b_1\otimes c_2 + a_1\otimes b_2\otimes c_1\\
&+a_1\otimes b_3\otimes c_0 + a_2\otimes b_1\otimes c_1 + a_2\otimes b_2\otimes c_0 + a_3\otimes b_0\otimes c_2 + a_3\otimes b_1\otimes c_0.
\end{aligned}
$$

Then T is oblique and not tight.

Proof. The proof of obliqueness is directly by observing that the support

$$\mathcal{S} = \{(0,2,3),(0,3,2),(1,0,3),(1,1,2),(1,2,1),(1,3,0),(2,1,1),(2,2,0),(3,0,2),(3,1,0)\}$$

is an antichain in $[4] \times [4] \times [4]$.

On the other hand T is not tight: a direct calculation shows that its annihilator $\mathfrak{g}_T$ is trivial. □

Relying on the additivity result of Theorem 4.1(i), one obtains that the inclusion $\overline{Tight_m} \subseteq \overline{Oblique_m}$ is strict for every $m \geq 4$. To see this, let T_4 be the tensor of Proposition 2.13 and define $T_m = T_4 \oplus M_{\langle 1 \rangle}^{\oplus m-4}$. Then T_m is oblique but it is not tight.

We conclude this section with a result on border rank of tight tensors. Let $Seg : \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C \to \mathbb{P}(A \otimes B \otimes C)$, $Seg([u],[v],[w]) = [u \otimes v \otimes w]$, be the *Segre embedding*, whose image $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is the variety of rank one tensors. Let $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subseteq \mathbb{P}(A \otimes B \otimes C)$ be the r-th *secant variety* of $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, that is the variety of tensors of border rank at most r.

Proposition 2.14. *Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$. Then*

- $\sigma_m(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subseteq \overline{Tight_m}$. *In other words,* $\underline{\mathbf{R}}(T) \leq m$ *implies* $T \in \overline{Tight_m}$.
- $\sigma_{m+1}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \not\subseteq \overline{Tight_m}$. *In other words, a general tensor T with* $\underline{\mathbf{R}}(T) \geq m+1$ *is not tight.*

Proof. If $r \leq m$, then $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) = \overline{(GL(A) \times GL(B) \times GL(C)) \cdot M_{\langle 1 \rangle}^{\oplus r}}$, where $M_{\langle 1 \rangle}^{\oplus r} = \sum_0^{r-1} a_i \otimes b_i \otimes c_i$. Since $M_{\langle 1 \rangle}^{\oplus r}$ is tight, we have $\sigma_m(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subseteq \overline{Tight_m}$.

Let $T_{std,m} = M_{\langle 1 \rangle}^{\oplus m} + (\sum_1^m a_i) \otimes (\sum_1^m b_i) \otimes (\sum_1^m c_i)$. From the expression one sees that $T_{std,m} \in \sigma_{m+1}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$. A direct calculation shows that the annihilator $\mathfrak{g}_{T_{std,m}}$ is trivial, therefore $T_{std,m}$ is not tight. We conclude $\sigma_{m+1}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \not\subseteq \overline{Tight_m}$. □

3. Compressibility of Tight tensors

In this section we briefly review Strassen's spectral theory and his support functionals in order to state the original version of Conjecture 1.3 and to relate it to the notion of *compressibility* of tensors.

3.1. Strassen's spectral theory.

In [Str86, Str87, Str88, Str91], Strassen proved that asymptotic degeneration of tensors (and in particular the asymptotic rank) is captured by what he named the *asymptotic spectrum of tensors*. What follows is a brief description of the theory, see [CVZ18, Lan19] for extensive discussions. Let $\mathcal{T} = \varinjlim_m \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be the direct limit defined by fixed inclusions of $\mathbb{C}^m \subseteq \mathbb{C}^{m+1}$. The set $\mathcal{T}$ is a semiring under the operations of direct sum and Kronecker product. There is a natural preorder on $\mathcal{T}$ given by *asymptotic degeneration*: $T' \lesssim T$ if there exists a sequence $\{\alpha_N\} \in o(N)$ such that, $T'^{\boxtimes N}$ is a degeneration of $T^{\boxtimes N + \alpha_N}$ for every N. Strassen proved that asymptotic degeneration is controlled by *spectral points*: real-valued semiring homomorphisms which are monotone under degeneration. In symbols, a spectral point is a map $\phi : \mathcal{T} \to \mathbb{R}_+$, such that $\phi(M_{\langle 1 \rangle}) = 1$, $\phi(T_1 \oplus T_2) = \phi(T_1) + \phi(T_2)$, $\phi(T_1 \boxtimes T_2) = \phi(T_1)\phi(T_2)$, and $\phi(T_1) \leq \phi(T_2)$ whenever T_1 is a degeneration of T_2; Strassen proved that $T_1 \lesssim T_2$ if and only if $\phi(T_1) \leq \phi(T_2)$ for all spectral points ϕ. Note that $\underset{\sim}{\mathbf{R}}(T)$ is the smallest r such that $T \lesssim M_{\langle 1 \rangle}^{\oplus r}$ and $\underset{\sim}{\mathbf{R}}(T) = \sup\{\phi(T) : \phi \text{ spectral point}\}$. In this context it is useful to introduce the *asymptotic subrank* of T, denoted $\underset{\sim}{\mathbf{Q}}(T)$, which is the largest r such that $M_{\langle 1\rangle}^{\oplus r}\underset{\sim}{\leq}T$, equivalently $\underset{\sim}{\mathbf{Q}}(T)=\inf\{\phi(T):\phi \text{ spectral point}\}$.

One can restrict the theory to subclasses of tensors which are closed under direct sum and Kronecker product, e.g., the subclasses of tight, oblique or free tensors. Conjecture 1.3 is thus equivalent to positing that for every tight tensor in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$, and for all spectral points ϕ, one has $\phi(T)\leq m$.

3.2. **Compressibility.** A tensor $T\in A\otimes B\otimes C$ is $(\mathbf{a}',\mathbf{b}',\mathbf{c}')$-*compressible* (resp. $(\mathbf{a}',\mathbf{b}',\mathbf{c}')$-*incompressible*) if there exist (resp. do not exist) linear spaces $A'\subset A^*$, $B'\subset B^*$, $C'\subset C^*$, respectively of dimensions $\mathbf{a}',\mathbf{b}',\mathbf{c}'$, such that $T|_{A'\otimes B'\otimes C'}=0$. The *total compressibility* of T is the largest $\mathbf{a}'+\mathbf{b}'+\mathbf{c}'$ such that T is $(\mathbf{a}',\mathbf{b}',\mathbf{c}')$-compressible.

A tensor is ρ-*multicompressible* if it is $(\mathbf{a}',\mathbf{b}',\mathbf{c}')$-compressible for all $\mathbf{a}',\mathbf{b}',\mathbf{c}'\in\mathbb{N}$ such that $\mathbf{a}'+\mathbf{b}'+\mathbf{c}'=\rho$.

A generic tensor in $A\otimes B\otimes C$ with $\mathbf{a}=\mathbf{b}=\mathbf{c}=m$ is $(\mathbf{a}',\mathbf{b}',\mathbf{c}')$-incompressible if $m\leq\frac{(\mathbf{a}')^2+(\mathbf{b}')^2+(\mathbf{c}')^2+\mathbf{a}'\mathbf{b}'\mathbf{c}'}{\mathbf{a}'+\mathbf{b}'+\mathbf{c}'}$ [LM18]. In particular a generic tensor is not $(\sqrt{\frac{m}{3}},\sqrt{\frac{m}{3}},\sqrt{\frac{m}{3}})$-compressible, see [LM18, Ex. 4.3] and consequently it is not $3\sqrt{\frac{m}{3}}$ multi-compressible.

3.3. **Strassen's support functionals: minimal weighted average incompressibility.** Let $\mathcal{F}$ denote the triples of increasing complete flags in A,B,C (and note that $f\in\mathcal{F}$ induces decreasing flags in A^*,B^*,C^*). For $f\in\mathcal{F}$ and $T\in A\otimes B\otimes C$ define

$$\mathrm{incompress}_f(T):=\{(i,j,k)\mid T|_{A_i{}^\perp\otimes B_j{}^\perp\otimes C_k{}^\perp}\neq 0\},$$

the incompressible subspaces in the triple of flags $f\in\mathcal{F}$.

Let θ be a probability distribution on $[3]$, and p a probability distribution on $[\mathbf{a}]\times[\mathbf{b}]\times[\mathbf{c}]$, giving rise to marginal distributions p_1 on $[\mathbf{a}]$, p_2 on $[\mathbf{b}]$ and p_3 on $[\mathbf{c}]$. For a probability distribution p on $[m]$, let $H(p)=-\sum_{j=1}^m p_j\log_2(p_j)$ denote its Shannon entropy. For each θ, Strassen's spectral point $\widehat{\zeta}^\theta$ for oblique tensors may be defined as follows:

$$\log_2(\widehat{\zeta}^\theta(T)):=\min\nolimits_{f\in\mathcal{F}}\max\nolimits_{p\mid\mathrm{supp}(p)\subseteq\mathrm{incompress}_f(T)}\sum_{\alpha=1}^{3}\theta(\alpha)H(p_\alpha). \tag{6}$$

Strassen proved that if T is moreover tight, then $\underset{\sim}{\mathbf{Q}}(T)$ is the infimum of the $\widehat{\zeta}^\theta(T)$. The original form of the asymptotic rank conjecture is essentially that for tight tensors, $\underset{\sim}{\mathbf{R}}(T)$ is the supremum of the $\widehat{\zeta}^\theta(T)$. More precisely:

Conjecture 3.1. [Str94, Conj. 5.3] *For tight tensors, the spectral points are generated by the $\widehat{\zeta}^\theta$'s.*

Definition 3.2. The *slice rank* of $T\in A\otimes B\otimes C$, denoted $\mathrm{slrk}(T)$ is $\mathbf{a}+\mathbf{b}+\mathbf{c}$ minus the total compressibility of T, in other words, the smallest r such that $T=\sum_{i=1}^p a_i\otimes X_i+\sum_{j=1}^q b_j\otimes Y_j+\sum_{k=1}^s c_k\otimes Z_k$ with $a_i\in A$, $b_j\in B$, $c_k\in C$, and $p+q+s=r$.

One defines asymptotic slice rank analogously. In [CVZ18] they show that for tight tensors, the asymptotic slice rank equals the asymptotic subrank.

3.4. **Compressibility of tight tensors.** The following result shows that tight tensors are highly compressible, compared to generic tensors. More precisely, a tight tensor is $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$-compressible for $\mathbf{a}', \mathbf{b}', \mathbf{c}' \approx m/2 \gg \sqrt{\frac{m}{3}}$.

Theorem 3.3. *Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ and let $T \in A \otimes B \otimes C$ be a tight tensor. Then T is $(\lceil m/2 \rceil, \lceil m/2 \rceil, \lfloor m/2 \rfloor)$-compressible and similarly permuting the order of the factors.*

Proof. Let T be expressed in a tight basis and let $\tau_A, \tau_B, \tau_C : \{1, \dots, m\} \to \mathbb{Z}$ be the corresponding increasing injective functions, with $\tau_A + \tau_B + \tau_C$ identically 0 on $\mathrm{supp}(T)$. We impose one additional normalization on τ_A, τ_B, τ_C as follows: we assume $\tau_A(\lfloor m/2 \rfloor) = \tau_B(\lceil m/2 \rceil) = -1$ and if $\tau_C(k) \geq 0$ then $\tau_C(j) \geq 2$: in order to do this, redefine $\tau_A = 3\tau'_A - 3\tau_A(\lfloor m/2 \rfloor) - 1$, $\tau'_B = \tau_B - \tau_B(\lceil m/2 \rceil) - 1$ and $\tau'_C = \tau_C + \tau_A(\lfloor m/2 \rfloor) + \tau_B(\lceil m/2 \rceil) + 2$. Notice that $\tau_A + \tau_B + \tau_C = 0$ if and only if $\tau'_A + \tau'_B + \tau'_C = 0$, so τ_A, τ_B, τ_C define the same tight support as $\tau'_A, \tau'_B, \tau'_C$; moreover $\tau'_A, \tau'_B, \tau'_C$ are increasing, $\tau_A(\lfloor m/2 \rfloor) = \tau_B(\lceil m/2 \rceil) = -1$ and if $\tau_C(k) \geq 0$ then $\tau_C(k) \geq 2$ because $\tau_C \equiv 2 \mod 3$. In fact $\tau_A, \tau_B, \tau_C \equiv 2 \mod 3$ and in particular they are never 0.

Now, we consider two cases:

(i) if $\tau_C(\lceil m/2 \rceil) > 0$, then choose $A' = \langle \alpha_i : i \in \{\lfloor m/2 \rfloor + 1, \dots, m\} \rangle$, $B' = \langle \beta_j : j \in \{\lceil m/2 \rceil, \dots, m\} \rangle$ and $C' = \langle \{\gamma_k : k \in \{\lceil m/2 \rceil, \dots, m\} \rangle$. Notice that $\dim A' = \lceil m/2 \rceil$, $\dim B' = \lfloor m/2 \rfloor + 1$ and $\dim C' = \lfloor m/2 \rfloor + 1$; moreover, the sum of $\tau'_A, \tau'_B, \tau'_C$ on the product of these subsets is lower bounded by $\tau_A(\lfloor m/2 \rfloor + 1) + \tau_B(\lceil m/2 \rceil) + \tau_C(\lfloor m/2 \rfloor) \geq 2 - 1 + 2 = 3 > 0$. This shows $T|_{A' \otimes B' \otimes C'} = 0$ because no elements of $\mathrm{supp}(T)$ appear in this range. In this case T is $(\lceil m/2 \rceil, \lfloor m/2 \rfloor + 1, \lfloor m/2 \rfloor + 1)$-compressible, and in particular $(\lfloor m/2 \rfloor, \lceil m/2 \rceil, \lceil m/2 \rceil)$-compressible.

(ii) if $\tau_C(\lceil m/2 \rceil) < 0$, then choose $A' = \langle \alpha_i : i \in \{1, \dots, \lfloor m/2 \rfloor\} \rangle$, $B' = \langle \beta_j : j \in \{1, \dots, \lceil m/2 \rceil\} \rangle$ and $C' = \langle \{\gamma_k : k \in \{1, \dots, \lceil m/2 \rceil\} \rangle$. Notice that $\dim A' = \lfloor m/2 \rfloor$, $\dim B' = \lceil m/2 \rceil$ and $\dim C' = \lceil m/2 \rceil$; moreover, the sum of $\tau'_A, \tau'_B, \tau'_C$ on the product of these subsets is upper bounded by $\tau_A(\lfloor m/2 \rfloor) + \tau_B(\lceil m/2 \rceil) + \tau_C(\lceil m/2 \rceil) \leq -1 - 1 - 1 = -3$. This shows $T|_{A' \otimes B' \otimes C'} = 0$ because no elements of $\mathrm{supp}(T)$ appear in this range. In this case T is $(\lfloor m/2 \rfloor, \lceil m/2 \rceil, \lceil m/2 \rceil)$-compressible.

□

We show that tight tensors with support equal to $\mathcal{S}_{t\text{-}max,m}$ are highly multicompressible. More precisely, recall that generic tensors are not $3\sqrt{\frac{m}{3}}$-multicompressible, whereas for tensors with support $\mathcal{S}_{t\text{-}max,m}$ we have the following result.

Proposition 3.4. *Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ and let $T \in A \otimes B \otimes C$ be a tight tensor with support $\mathcal{S}_{t\text{-}max,m}$. Then T is $(3\lfloor m/2 \rfloor + 1)$-multicompressible.*

Proof. Recall from Example 2.1, $\tau_A, \tau_B, \tau_C : [m] \to \mathbb{Z}$, with $\tau_A(i) = \tau_B(i) = \tau_C(i) = i - \ell$ if $m = 2\ell + 1$ is odd and with $\tau_A(i) = .i - \ell + 1$, $\tau_B(j) = \tau_C(j) = j - \ell$ if $m = 2\ell$ is even.

Fix $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ with $\mathbf{a}' + \mathbf{b}' + \mathbf{c}' = 3\lfloor m/2 \rfloor + 1 = 3\ell + 1$. We determine $A' \subseteq A^*, B' \subseteq B^*, C' \subseteq C^*$ with $\dim A' = \mathbf{a}'$, $\dim B' = \mathbf{b}'$, $\dim C' = \mathbf{c}'$ such that $T|_{A' \otimes B' \otimes C'} = 0$. Let $A' = \langle \alpha^i : i \in [\mathbf{a}'] \rangle$, $B' = \langle \beta^j : j \in [\mathbf{b}'] \rangle$,$C' = \langle \gamma^k : k \in [\mathbf{c}'] \rangle$.

We claim that $T|_{A' \otimes B' \otimes C'} = 0$. This follows from the fact that $[\mathbf{a}'] \times [\mathbf{b}'] \times [\mathbf{c}'] \cap \mathcal{S}_{t\text{-}max,m} = \emptyset$. If $(i, j, k) \in [\mathbf{a}'] \times [\mathbf{b}'] \times [\mathbf{c}']$, we have $\tau_A(i) + \tau_B(j) + \tau_C(k) \leq i - \ell + 1 + j - \ell + k - \ell \leq \mathbf{a}' - 1 + \mathbf{b}' - 1 + \mathbf{c}' - 1 - 3\ell \leq 3\ell + 1 - 2 - 3\ell = -1$ (here the first inequality is in fact an equality if m is even). In particular, there are no elements $(i,j,k) \in [\mathbf{a}'] \times [\mathbf{b}'] \times [\mathbf{c}']$ such that $\tau_A(i) + \tau_B(j) + \tau_C(k) = 0$. $\square$

However, we observe that highly multicompressible tensors are not necessarily tight:

Example 3.5. This is an example of a $3\lfloor m/2 \rfloor$-multicompressible tensor that is not tight. Let

$$\begin{aligned} T =& a_0 \otimes b_0 \otimes c_0 + a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3 + \\ &+ (a_0 + a_1 + a_2 + a_3) \otimes (b_0 + b_1) \otimes (c_2 + c_3) + (a_1 + a_2 + a_3) \otimes b_2 \otimes (c_2 + c_3) + \\ &+ (a_1 + a_2 + a_3) \otimes b_3 \otimes c_3 + (a_2 + a_3) \otimes b_3 \otimes c_2 \end{aligned}$$

be a tensor in $A \otimes B \otimes C$ with $\mathbf{a} = \mathbf{b} = \mathbf{c} = 4$. A direct calculation shows that T has trivial annihilator $\mathfrak{g}_T$, therefore it is not tight. It is easy to verify that T is 6-multicompressible.

Taking direct sums of copies of the tensor above one obtains highly compressible, not tight tensors in higher dimensions.

3.5. **Additional remarks on compressibility in general.** One can discuss a restricted form of multicompressibility, by letting only the dimensions of two factors vary. In this context we have the following result:

Proposition 3.6. *Let $T \in A \otimes B \otimes C$ with $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$. For every $\mathbf{b}', \mathbf{c}'$ with $\mathbf{b}' + \mathbf{c}' \leq m - \lceil \sqrt{m-1} \rceil$, T is $(1, \mathbf{b}', \mathbf{c}')$-compressible and similarly permuting the roles of the three factors.*

Proof. The result is immediate if T is not concise. Suppose T is concise. Let $\sigma_r(\mathbb{P}B \times \mathbb{P}C) \subseteq \mathbb{P}(B \otimes C)$ denote the subvariety of rank at most r elements in $\mathbb{P}(B \otimes C)$. We have $\dim(\sigma_r(Seg(\mathbb{P}B \times \mathbb{P}C))) = 2rm - r^2 - 1$. By conciseness, the image of the flattening map $T_A : A^* \to B \otimes C$ has dimension m, so its projectivization $\mathbb{P}(T_A(A^*))$ intersects $\sigma_r(Seg(\mathbb{P}B \times \mathbb{P}C))$ when $r = m - \lceil \sqrt{m-1} \rceil$.

So fix $r = m - \lceil \sqrt{m-1} \rceil$ and let $\alpha \in A^*$ such that $[T_A(\alpha)] \in \mathbb{P}(T_A(A^*)) \cap \sigma_r(Seg(\mathbb{P}B \times \mathbb{P}C))$. Then $T_A(\alpha)$ has rank at most $r = m - \lceil \sqrt{m-1} \rceil$.

Choose bases such that $T_A(\alpha) = b_1 \otimes c_1 + \cdots + b_r \otimes c_r$ and let $A' = \langle \alpha \rangle$, $B' = \langle \beta_1, \ldots, \beta_{\mathbf{b}'} \rangle$ and $C' = \langle \gamma_{\mathbf{b}'+1}, \ldots, \gamma_{\mathbf{b}'+\mathbf{c}'} \rangle$. Then $\dim A' = 1$, $\dim B' = \mathbf{b}'$ and $\dim C' = \mathbf{c}'$; we have $T|_{A' \otimes B' \otimes C'} = 0$, so T is $(1, \mathbf{b}', \mathbf{c}')$-compressible. $\square$

More generally, we show that maximally compressible tensors (in the sense of [LM17a]) are also highly multicompressible.

Proposition 3.7. *Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ and let $T \in A \otimes B \otimes C$ be $(m-1, m-1, m-1)$-compressible. Then T is $2m-1$-multicompressible.*

Proof. After fixing bases in A, B, C, we may assume without loss of generality that $T|_{a_0^\perp \otimes b_0^\perp \otimes c_0^\perp} = 0$; in particular T can be written as $T = a_0 \otimes M_A + b_0 \otimes M_B + c_0 \otimes M_C$ where $M_A \in B \otimes C$ and similarly M_B, M_C (and reordering the factors in the second and third summand). Let $\mathbf{a}', \mathbf{b}', \mathbf{c}' \leq m$ with $\mathbf{a}' + \mathbf{b}' + \mathbf{c}' = 2m - 1$. Moreover, we may assume $M_A \in b_0^\perp \otimes c_0^\perp$ because expressing T in the fixed basis, we can include summands including b_0, c_0 in $b_0 \otimes M_B + c_0 \otimes M_C$ and in fact we may fix bases so that $M_A = \sum_1^r b_i \otimes c_i$ for some $r \leq m-1$.

If $\mathbf{a}', \mathbf{b}', \mathbf{c}' < m$, let $A' \subseteq a_0^\perp, B' \subseteq b_0^\perp, C' \subseteq c_0^\perp$, so $T|_{A' \otimes B' \otimes C'} = 0$.

Suppose $\mathbf{a}' = m$; therefore $A' = A^*$ and $\mathbf{b}' + \mathbf{c}' = m-1$. Let $B' = \langle \beta^1, \dots, \beta^{\mathbf{b}'} \rangle$ and $C' = \langle c^{\mathbf{b}'+1}, \dots, c^{m-1} \rangle$. Then $T|_{A' \otimes B' \otimes C'} = 0$. This concludes the proof. □

Proposition 3.7 implies in particular that the Coppersmith-Winograd tensors $T_{cw,q}$ and $T_{CW,q}$, see [CW90], are respectively $(2q+1)$-multicompressible and $(2q+3)$-multicompressible. These are the tensors used in the framework of Strassen's laser method to obtain the current best known upper bound on the asymptotic rank of the matrix multiplication tensor [Sto10, Wil12, Le 14]. In fact, it is known that if Strassen's asymptotic rank conjecture holds for $T_{cw,2}$, then it holds for the matrix multiplication tensor as well, and in particular Conjecture 1.2 is true.

Remark 3.8. Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{n}^2$ and consider $M_{\langle \mathbf{n} \rangle} \in A \otimes B \otimes C$. Then $M_{\langle \mathbf{n} \rangle} \in A \otimes B \otimes C$ is $3\lfloor \mathbf{n}^2/2 \rfloor$-multicompressible. The proof is similar to that of Proposition 3.7. After a change of basis the flattening map $M_{\langle \mathbf{n} \rangle} : A^* \to B \otimes C$ can be written as a $(\mathbf{n} \times \mathbf{n})$-block diagonal matrix of linear forms on A, whose diagonal blocks are all equal to the matrix (α^i_j), see [Lan17, Exercise 2.1.7.4]. In this form, it is easy to see that $M_{\langle \mathbf{n} \rangle}$ is $(\mathbf{n}^2, \mathbf{b}', \mathbf{c}')$-compressible for every $(\mathbf{b}', \mathbf{c}')$ with $\mathbf{b}' + \mathbf{c}' = \mathbf{n}^2$.

At this point, consider $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ with $\mathbf{a}' + \mathbf{b}' + \mathbf{c}' = 3\lfloor \mathbf{n}^2/2 \rfloor$. Notice that $(\mathbf{a}' + \mathbf{b}') + (\mathbf{b}' + \mathbf{c}') + (\mathbf{a}' + \mathbf{c}') \leq 3\mathbf{n}^2$, so at least one among $(\mathbf{a}' + \mathbf{b}'), (\mathbf{b}' + \mathbf{c}'), (\mathbf{a}' + \mathbf{c}')$ is bounded from above by $\mathbf{n}^2$. Suppose $\mathbf{b}' + \mathbf{c}' \leq \mathbf{n}^2$. From the argument above $M_{\langle \mathbf{n} \rangle}$ is $(\mathbf{n}^2, \mathbf{b}', \mathbf{c}')$-compressible and therefore $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$-compressible.

Finally, when $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$, every $T \in A \otimes B \otimes C$ with border rank r is $(3m - r)$-multicompressible [LM18]. For instance, the tensor $T_{std,m}$ defined in the proof of Proposition 2.14 is $(2m-1)$-multicompressible.

3.6. **Combinatorial geometry of tight sets and compressibility.** This subsection discusses a combinatorial approach towards proving compressibility of tight tensors. Theorem 3.3 and Proposition 3.4 may be recovered from Proposition 3.10 below.

The three functions $\tau_A, \tau_B, \tau_C : [m] \to \mathbb{Z}$ define a line arrangement in $\mathbb{R}^2$ as follows. Consider in $\mathbb{R}^3$ (with coordinates (x, y, z)) the following arrangement of planes, consisting of the union of three families of parallel planes, each of them comprising m planes:

$$\widehat{\mathcal{A}} = \bigsqcup_{i=0}^{m-1} \{x = \tau_A(i)\} \cup \bigsqcup_{j=0}^{m-1} \{y = \tau_B(j)\} \cup \bigsqcup_{k=0}^{m-1} \{z = \tau_C(k)\}. \tag{7}$$

Let $\mathcal{A}$ be the intersection of $\widehat{\mathcal{A}}$ with the plane $\Pi = \{x + y + z = 0\} \subset \mathbb{R}^3$. The set $\mathcal{A}$ is an arrangement of three families of parallel lines, each consisting of m lines: with respect to coordinates x, y in Π, the three families of lines are $\mathcal{A} = \bigsqcup_{i=0}^{m-1} \{x = \tau_A(i)\} \cup \bigsqcup_{j=0}^{m-1} \{y = \tau_B(j)\} \cup \bigsqcup_{k=0}^{m-1} \{x + y = -\tau_C(k)\}$; we say that the *x-direction* of $\mathcal{A}$ is the union of the lines with constant x, the y-direction is the union of lines with constant y and the *z-direction* is the union of lines with slope -1. A subset of lines $\mathcal{A}' \subseteq \mathcal{A}$ is called a sub-arrangement if it contains at least one line in each direction.

The set $\{p \in \Pi : p$ belongs to exactly two lines in $\mathcal{A}\}$ is called the *set of double intersection points* of $\mathcal{A}$. The set $\mathfrak{J}(\mathcal{A}) := \{p \in \Pi : p$ belongs to three lines in $\mathcal{A}\}$ is the set of *joints* in $\mathcal{A}$.

Lemma 3.9. *Let T be a tight tensor in a tight basis and let $\mathcal{A}$ be the corresponding arrangement of lines. Then $(\tau_A, \tau_B, \tau_C) : [m]^{\times 3} \to \mathbb{R}^3$ maps* $\mathrm{supp}(T)$ *bijectively to a subset of $\mathfrak{J}(\mathcal{A})$. In particular, if $\mathcal{A}$ is an arrangement with $\mathfrak{J}(A) = \emptyset$ then $T = 0$.*

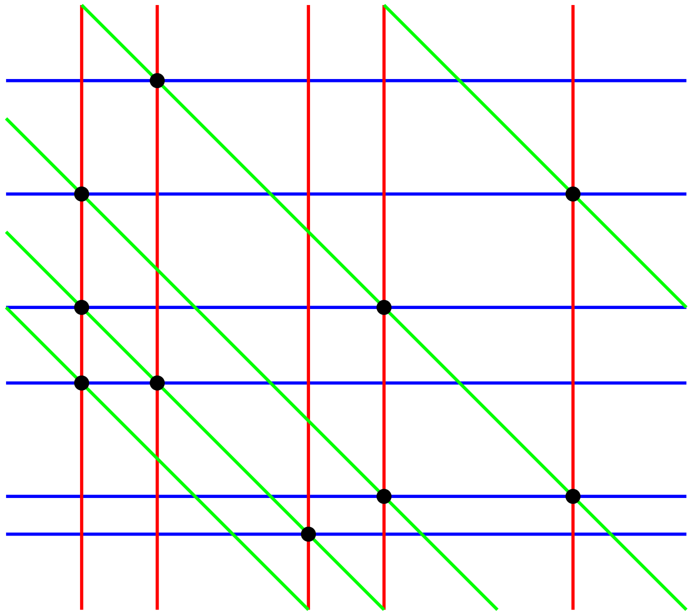

FIGURE 1. An arrangement of lines on the plane Π: the red lines are in the x-direction, the blue lines in the y-direction and the green lines in the z-direction. The joints are marked with black dots.

Proof. If $(i, j, k) \in \mathrm{supp}(T)$ then $\tau_A(i) + \tau_B(j) + \tau_C(k) = 0$, so $(\tau_A(i), \tau_B(j), \tau_C(k)) \in \Pi$ is a point of $\mathcal{A}$ lying on three lines. In particular $(\tau_A(i), \tau_B(j), \tau_C(k)) \in \mathfrak{J}(\mathcal{A})$. □

If T is a tight tensor in a tight basis and $\mathcal{A}$ is the corresponding line arrangement with $\mathrm{supp}(T) \subseteq \mathfrak{J}(\mathcal{A})$, we say that T is *supported* on $\mathcal{A}$. Properties of the support of a tight tensor in a tight basis can be translated into geometric and combinatorial properties of $\mathcal{A}$. For instance, compressibility in given coordinates can be studied combinatorially as follows.

Proposition 3.10. *Let T be a tight tensor and let $\mathcal{A}$ be the corresponding line arrangement. If there exists a sub-arrangement $\mathcal{A}'$ of $\mathcal{A}$ consisting of $\mathbf{a}'$ lines in the x direction, $\mathbf{b}'$ lines in the y direction, and $\mathbf{c}'$ lines in the z direction with $\mathfrak{J}(\mathcal{A}') = \emptyset$, then T is $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$-compressible.*

Proof. After the identification of A, B, C with their duals determined by the choice of bases, let $A' \subseteq A^*$ be the subspace spanned by the basis elements $\{\alpha^i : \{x = \tau_A(i)\} \in \mathcal{A}'\}$ and similarly B' and C'. Then $T' = T|_{A' \otimes B' \otimes C'}$ is tight, and the corresponding arrangement is $\mathcal{A}'$. Since $\mathfrak{J}(\mathcal{A}') = \emptyset$, we conclude by Lemma 3.9. □

4. Propagation of symmetries

Recall that $\Phi : GL(A) \times GL(B) \times GL(C) \to GL(A \otimes B \otimes C)$ defines the natural action of $GL(A) \times GL(B) \times GL(C)$ on $A \otimes B \otimes C$ and G is the image of Φ in $GL(A \otimes B \otimes C)$. Denote by G_T the stabilizer of a tensor T in G and by $\mathfrak{g}_T$ the Lie algebra of G_T, namely the annihilator of T under the Lie algebra action of $\mathfrak{g} = \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)/\mathfrak{z}_{A,B,C}$.

We have the following result on propagation of symmetries.

Theorem 4.1. *Let $T \in A_1 \otimes B_1 \otimes C_1$ and $S \in A_2 \otimes B_2 \otimes C_2$ be concise tensors. Then* *(i) as subalgebras of* $(\mathfrak{gl}(A_1 \oplus A_2) \oplus \mathfrak{gl}(B_1 \oplus B_2) \oplus \mathfrak{gl}(C_1 \oplus C_2)) / \mathfrak{z}_{A_1 \oplus A_2, B_1 \oplus B_2, C_1 \oplus C_2}$

$$\mathfrak{g}_{T \oplus S} = \mathfrak{g}_T \oplus \mathfrak{g}_S;$$

(ii) as a subalgebras of $(\mathfrak{gl}(A_1 \otimes A_2) \oplus \mathfrak{gl}(B_1 \otimes B_2) \oplus \mathfrak{gl}(C_1 \otimes C_2)) / \mathfrak{z}_{A_1 \otimes A_2, B_1 \otimes B_2, C_1 \otimes C_2}$*;*

$$\mathfrak{g}_{T \boxtimes S} \supseteq \mathfrak{g}_T \otimes \mathrm{Id}_{A_2 \otimes B_2 \otimes C_2} + \mathrm{Id}_{A_1 \otimes B_1 \otimes C_1} \otimes \mathfrak{g}_S;$$

(iii) if $\mathfrak{g}_T = 0$ *and* $\mathfrak{g}_S = 0$ *then* $\mathfrak{g}_{T \boxtimes S} = 0$*.*

The containment of (ii) in Theorem 4.1 can be strict, for instance in the case of the matrix multiplication tensor. Additional examples will be provided in [CGLV19a]. We propose the following problem, which addresses the general study of propagation of non-genericity properties under Kronecker powers, in the spirit of Strassen's asymptotic rank conjecture and its generalizations.

Problem 4.2. Characterize tensors $T \in A \otimes B \otimes C$ such that $\mathfrak{g}_T \otimes \mathrm{Id}_{A \otimes B \otimes C} + \mathrm{Id}_{A \otimes B \otimes C} \otimes \mathfrak{g}_T$ is strictly contained in $\mathfrak{g}_{T^{\boxtimes 2}} \in A^{\otimes 2} \otimes B^{\otimes 2} \otimes C^{\otimes 2}$.

Proof of Theorem. 4.1. Throughout the proof, we use the summation convention for which repeated upper and lower indices are to be summed over. The range of the indices is omitted as it should be clear from the context.

Proof of (i). Let $T \in A_1 \otimes B_1 \otimes C_1$ and $S \in A_2 \otimes B_2 \otimes C_2$. Fix bases of $A_1, B_1, C_1, A_2, B_2, C_2$ and write $T = T^{i_1 j_1 k_1} a^{(1)}_{i_1} \otimes b^{(1)}_{j_1} \otimes c^{(1)}_{k_1}$ and $S = S^{i_2 j_2 k_2} a^{(2)}_{i_2} \otimes b^{(2)}_{j_2} \otimes c^{(2)}_{k_2}$. Let $L = (U, V, W) \in \mathfrak{gl}(A_1 \oplus A_2) \oplus \mathfrak{gl}(B_1 \oplus B_2) \oplus \mathfrak{gl}(C_1 \oplus C_2)$. We want to prove that if $L.(T \oplus S) = 0$, then for $\ell = 1, 2$, there is $L_\ell \in \mathfrak{gl}(A_\ell) \oplus \mathfrak{gl}(B_\ell) \oplus \mathfrak{gl}(C_\ell)$ such that $L = L_1 + L_2$ with $L_1.T = 0$ and $L_2.S = 0$. Write $X = X_{11} + X_{12} + X_{21} + X_{22}$ where $X_{11} \in \mathrm{Hom}(A_1, A_1)$ and similarly for the other summands. Consider $X_{21}(T \boxtimes S) = X_{21}(T)$: this is an element of $A_2 \otimes B_1 \otimes C_1$. No other summand of X, nor Y or Z generate a nonzero component in this space. Therefore, $X_{21}(T) = 0$ and by conciseness we deduce $X_{21} = 0$. Similarly $X_{12} = 0$ so that $X = X_{11} + X_{22} \in \mathfrak{gl}(A_1) \oplus \mathfrak{gl}(A_2)$ and similarly for Y and Z. For $\ell = 1, 2$, let $L_\ell = (X_{\ell\ell}, Y_{\ell\ell}, Z_{\ell\ell})$. Then $L = L_1 + L_2$ and $L.(T \boxtimes S) = L_1.T + L_2.S$; notice $L_1.T \in A_1 \otimes B_1 \otimes C_1$ and $L_2.S \in A_2 \otimes B_2 \otimes C_2$ are linearly independent, so if $L.(T \boxtimes S) = 0$, we have $L_1 \in \mathfrak{g}_T$ and $L_2 \in \mathfrak{g}_S$.

Proof of (ii). This is a straightforward consequence of the Leibniz rule. In general if $\mathfrak{g}_1$ acts on a space V_1 and $\mathfrak{g}_2$ acts on a space V_2, then the action of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ on $V_1 \otimes V_2$ is given by the Leibniz rule via $(L_1, L_2) \mapsto L_1 \otimes \mathrm{Id}_{V_2} + \mathrm{Id}_{V_1} \otimes L_2$. If $v_1 \in V_1$ is annihilated by $\mathfrak{g}_1$, and $v_2 \in V_2$ is annihilated by v_2, then $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ annihilates $v_1 \otimes v_2$ via the induced action.

Proof of (iii). Fix bases in all spaces. Let $L = (U, V, W) \in \mathfrak{gl}(A_1 \otimes A_2) \otimes \mathfrak{gl}(B_1 \otimes B_2) \otimes \mathfrak{gl}(C_1 \otimes C_2)$ and write U as an $\mathbf{a}_1\mathbf{a}_2 \times \mathbf{a}_1\mathbf{a}_2$ matrix $u^{i'_1 i'_2}_{i_1 i_2}$, and similarly for V and W. Our goal is to prove that if $L.(T_1 \boxtimes T_2) = 0$, then $L \in \mathfrak{z}_{A_1 \otimes A_2, B_1 \otimes B_2, C_1 \otimes C_2}$.

Write $T_1 = T^{ijk} a^{(1)}_i \otimes b^{(1)}_j \otimes c^{(1)}_k$ and $T_2 = S^{i'j'k'} a^{(2)}_{i'} \otimes b^{(2)}_{j'} \otimes c^{(2)}_{k'}$. The equations for the symmetry Lie algebra $\mathfrak{g}_{T_1 \boxtimes T_2}$ is $L.(T_1 \boxtimes T_2) = 0$; in coordinates, for every $i_1, i_2, j_1, j_2, k_1, k_2$, we have (using the summation convention)

$$u^{i_1 i_2}_{i'_1 i'_2} T^{i'_1 j_1 k_1} S^{i'_2 j_2 k_2} + v^{j_1 j_2}_{j'_1 j'_2} T^{i_1 j'_1 k_1} S^{i_2 j'_2 k_2} + w^{k_1 k_2}_{k'_1 k'_2} T^{i_1 j_1 k'_1} S^{i_2 j_2 k'_2} = 0 \tag{8}$$

Let $U(i_1j_1k_1) \in \mathrm{Hom}(A_2, A_2)$ be the matrix whose (i_2, i_2')-th entry is $u^{i_1i_2}_{i_1'i_2'}T^{i_1'j_1k_1}$ and similarly $V(i_1j_1k_1)$ and $W(i_1j_1k_1)$. Let $L(i_1j_1k_1) = (U(i_1j_1k_1), V(i_1j_1k_1), W(i_1j_1k_1))$. From (8), we have $L(i_1j_1k_1).T_2 = 0$, namely $L(i_1j_1k_1) = \mathfrak{z}_{A_2,B_2,C_2}$, so that $U(i_1j_1k_1) = u(i_1j_1k_1)\mathrm{Id}_{A_2}, V(i_1j_1k_1) = v(i_1j_1k_1)\mathrm{Id}_{B_2}, W(i_1j_1k_1) = w(i_1j_1k_1)\mathrm{Id}_{C_2}$, with $u(i_1j_1k_1) + v(i_1j_1k_1) + w(i_1j_1k_1) = 0$.

In particular, if $i_2 \neq i_2'$, we have $U(i_1j_1k_1)^{i_2}_{i_2'} = 0$ for every i_1, j_1, k_1, which by definition provides $u^{i_1i_2}_{i_1'i_2'}T^{i_1'j_1k_1} = 0$. This implies that the $\mathbf{a}_1 \times \mathbf{a}_1$ matrix $(u^{i_1i_2}_{i_1'i_2'})_{i_2,i_2'}$ satisfies $((u^{i_1i_2}_{i_1'i_2'})_{i_2,i_2'}, 0, 0) \in \mathfrak{g}_{T_1}$. By conciseness, this implies $(u^{i_1i_2}_{i_1'i_2'})_{i_2,i_2'} = 0$, and therefore $u^{i_1i_2}_{i_1'i_2'} = 0$ for every i_1, i_1' and every $i_2 \neq i_2'$. By exchanging the role of the two tensors, we obtain that $u^{i_1i_2}_{i_1'i_2'} = 0$ for every $i_1 \neq i_1'$ and every i_2, i_2'. We deduce that U is diagonal. Similar argument applies to V and W.

Consequently, for each fixed $i_1, i_2, j_1, j_2, k_1, k_2$, (8) reduces to (with no summation) $T^{i_1j_1k_1}S^{i_2j_2k_2}(u^{i_1i_2}_{i_2i_2} + v^{j_1j_2}_{j_1j_2} + w^{k_1k_2}_{k_1k_2}) = 0$. Since our choice of bases is arbitrary, we may assume that $T^{i_1j_1k_1} \neq 0 \neq S^{i_2j_2k_2}$. Then taking different values of k_1, k_2 and fixing i_1, i_2, j_1, j_2, we see all the $w^{k_1k_2}_{k_1k_2}$ must be equal and similarly for U and V. This shows that $U = \lambda\mathrm{Id}_{A_1\otimes A_2}$, $V = \mu\mathrm{Id}_{B_1\otimes B_2}$ and $W = \mathrm{Id}_{C_1\otimes C_2}$. By evaluating (8) one last time, we see $\lambda = \mu = \nu$ that is $L \in \mathfrak{z}_{A_1\otimes A_2, B_1\otimes B_2, C_1\otimes C_2}$. □

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