Cartan for Beginners:
Differential Geometry via Moving Frames and Exterior Differential Systems

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Contents

Chapter 1

Moving Frames and Exterior Differential Systems

In this chapter we motivate the use of differential forms to study problems in geometry and partial differential equations. We begin with familiar material: the Gauss and mean curvature of surfaces in $\mathbb{E}^3$ in §1.1, and Picard's Theorem for local existence of solutions of ordinary differential equations in §1.2. We continue in §1.2 with a discussion of a simple system of partial differential equations, and then in §1.3 rephrase it in terms of differential forms, which facilitates interpreting it geometrically. We also state the Frobenius Theorem.

In §1.4, we review curves in $\mathbb{E}^2$ in the language of moving frames. We generalize this example in §§1.5–1.6, describing how one studies submanifolds of homogeneous spaces using moving frames, and introducing the Maurer-Cartan form. We give two examples of the geometry of curves in homogeneous spaces: classifying holomorphic mappings of the complex plane under fractional linear transformations in §1.7, and classifying curves in $\mathbb{E}^3$ under Euclidean motions (i.e., rotations and translations) in §1.8. We also include exercises on plane curves in other geometries.

In §1.9, we define exterior differential systems and integral manifolds. We prove the Frobenius Theorem, give a few basic examples of exterior differential systems, and explain how to express a system of partial differential equations as an exterior differential system using jet bundles.

Throughout this book we use the summation convention: unless otherwise indicated, summation is implied whenever repeated indices occur up and down in an expression.

1.1. Geometry of surfaces in $\mathbb{E}^3$ in coordinates

Let $\mathbb{E}^3$ denote Euclidean three-space, i.e., the affine space $\mathbb{R}^3$ equipped with its standard inner product.

Given two smooth surfaces $S, S' \subset \mathbb{E}^3$, when are they "equivalent"? For the moment, we will say that two surfaces are (locally) equivalent if there exist a rotation and translation taking (an open subset of) S onto (an open subset of) S'.

Figure 1. Are these two surfaces equivalent?

It would be impractical and not illuminating to try to test all possible motions to see if one of them maps S onto S'. Instead, we will work as follows:

Fix one surface S and a point $p \in S$. We will use the Euclidean motions to put S into a normalized position in space with respect to p. Then any other surface S' will be locally equivalent to S at p if there is a point $p' \in S'$ such that the pair (S', p') can be put into the same normalized position as (S, p).

The implicit function theorem implies that there always exist coordinates such that S is given locally by a graph $z = f(x, y)$. To obtain a normalized position for our surface S, first translate so that $p = (0, 0, 0)$, then use a rotation to make T_pS the xy-plane, i.e., so that $z_x(0, 0) = z_y(0, 0) = 0$. We will call such coordinates *adapted to p*. At this point we have used up all our freedom of motion except for a rotation in the xy-plane.

If coordinates are adapted to p and we expand $f(x, y)$ in a Taylor series centered at the origin, then functions of the coefficients of the series that are invariant under this rotation are differential invariants.

In this context, a (Euclidean) *differential invariant* of S at p is a function I of the coefficients of the Taylor series for f at p, with the property that, if we perform a Euclidean change of coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where A is a rotation matrix and a, b, c are arbitrary constants, after which S is expressed as a graph $\tilde{z} = \tilde{f}(\tilde{x}, \tilde{y})$ near p, then I has the same value when computed using the Taylor coefficients of $\tilde{f}$ at p. Clearly a necessary condition for (S, p) to be locally equivalent to (S', p') is that the values of differential invariants of S at p match the values of the corresponding invariants of S' at p'.

For example, consider the Hessian of $z = z(x, y)$ at p:

$$\text{Hess}_p = \begin{pmatrix} z_{xx} & z_{yx} \\ z_{xy} & z_{yy} \end{pmatrix}\Big|_p . \tag{1.1.1}$$

Assume we are have adapted coordinates to p. If we rotate in the xy plane, the Hessian gets conjugated by the rotation matrix. The quantities

$$\begin{aligned} K_0 &= \det(\text{Hess}_p) = (z_{xx}z_{yy} - z_{xy}^2)\,|_p, \\ H_0 &= \tfrac{1}{2}\text{trace}(\text{Hess}_p) = \tfrac{1}{2}(z_{xx} + z_{yy})\,|_p\,. \end{aligned} \tag{1.1.2}$$

are differential invariants because the determinant and trace of a matrix are unchanged by conjugation by a rotation matrix. Thus, if we are given two surfaces S, S' and we normalize them both at respective points p and p' as above, a necessary condition for there to be a rigid motion taking p' to p such that the Taylor expansions for the two surfaces at the point p coincide is that $K_0(S) = K_0(S')$ and $H_0(S) = H_0(S')$.

The formulas (1.1.2) are only valid at one point, and only after the surface has been put in normalized position relative to that point. To calculate K and H as functions on S it would be too much work to move each point to the origin and arrange its tangent plane to be horizontal. But it is possible to adjust the formulas to account for tilted tangent planes (see §2.10). One then obtains the following functions, which are differential invariants under Euclidean motions of surfaces that are locally described as graphs $z = z(x, y)$:

$$\begin{aligned} K(x,y) &= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}, \\ H(x,y) &= \frac{1}{2}\frac{(1 + z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1 + z_x^2)z_{yy}}{(1 + z_x^2 + z_y^2)^{\frac{3}{2}}}, \end{aligned} \tag{1.1.3}$$

respectively giving the *Gauss* and *mean* curvature of S at $p = (x, y, z(x, y))$.

Exercise 1.1.0.1: By locally describing each surface as a graph, calculate the Gauss and mean curvature functions for a sphere of radius R, a cylinder of radius r (e.g., $x^2+y^2 = r^2$) and the smooth points of the cone $x^2+y^2 = z^2$.

Once one has found invariants for a given submanifold geometry, one may ask questions about submanifolds with special invariants. For surfaces in $\mathbb{E}^3$, one might ask which surfaces have K constant or H constant. These can be treated as questions about solutions to certain partial differential equations (PDE). For example, from (1.1.3) we see that surfaces with $K \equiv 1$ are locally given by solutions to the PDE

$$z_{xx}z_{yy} - z_{xy}^2 = (1 + z_x^2 + z_y^2)^2. \tag{1.1.4}$$

We will soon free ourselves of coordinates and use moving frames and differential forms. As a provisional definition, a *moving frame* is a smoothly varying basis of the tangent space to $\mathbb{E}^3$ defined at each point of our surface. In general, using moving frames one can obtain formulas valid at every point analogous to coordinate formulas valid at just one preferred point. In the present context, the Gauss and mean curvatures will be described at all points by expressions like (1.1.2) rather than (1.1.3); see §2.1.

Another reason to use moving frames is that the method gives a uniform procedure for dealing with diverse geometric settings. Even if one is originally only interested in Euclidean geometry, other geometries arise naturally. For example, consider the *warp* of a surface, which is defined to be $(k_1 - k_2)^2$, where the k_j are the eigenvalues of (1.1.1). It turns out that this quantity is invariant under a larger change of coordinates than the Euclidean group, namely *conformal* changes of coordinates, and thus it is easier to study the warp in the context of conformal geometry.

Regardless of how unfamiliar a geometry initially appears, the method of moving frames provides an algorithm to find differential invariants. Thus we will have a single method for dealing with conformal, Hermitian, projective and other geometries. Because it is familiar, we will often use the geometry of surfaces in $\mathbb{E}^3$ as an example, but the reader should keep in mind that the beauty of the method is its wide range of applicability. As for the use of differential forms, we shall see that when we express a system of PDE as an exterior differential system, the geometric features of the system—i.e., those which are independent of coordinates—will become transparent.

1.2. Differential equations in coordinates

The first questions one might ask when confronted with a system of differential equations are: Are there any solutions? If so, how many?

In the case of a single ordinary differential equation (ODE), here is the answer:

Theorem 1.2.0.2 (Picard[1])**.** *Let $f(x,u) : \mathbb{R}^2 \to \mathbb{R}$ be a function with f and f_u continuous. Then for all $(x_0, u_0) \in \mathbb{R}^2$, there exist an open interval $I \ni x_0$ and a function $u(x)$ defined on I, satisfying $u(x_0) = u_0$ and the differential equation*

$$\frac{du}{dx} = f(x,u). \tag{1.2.1}$$

Moreover, any other solution of this initial value problem must coincide with this solution on I.

[1] See, e.g., [?], p.423

In other words, for a given ODE there exists a solution defined near x_0 and this solution is unique given the choice of a constant u_0. Thus for an ODE for one function of one variable, we say that *solutions depend on one constant*. More generally, Picard's Theorem applies to systems of n first-order ODE's involving n unknowns, where solutions depend on n constants.

The graph in $\mathbb{R}^2$ of any solution to (1.2.1) is tangent at each point to the vector field $X = \frac{\partial}{\partial x} + f(x,u)\frac{\partial}{\partial u}$. This indicates how determined ODE systems generalize to the setting of differentiable manifolds (see). If M is a manifold and X is a vector field on M, then a solution to the *system defined by* X is an immersed curve $c : I \to M$ such that $c'(t) = X_{c(t)}$ for all $t \in I$. (This is also referred to as an *integral curve* of X.) Away from singular points, one is guaranteed existence of local solutions to such systems and can even take the solution curves as coordinate curves:

Theorem 1.2.0.3 (Flowbox coordinates[2]). *Let M be an m-dimensional C^∞ manifold, let $p \in M$, and let $X \in \Gamma(TM)$ be a smooth vector field which is nonzero at p. Then there exists a local coordinate system $(x^1, \ldots, x^m)$, defined in a neighborhood U of p, such that $\frac{\partial}{\partial x^1} = X$.*

Consequently, there exists an open set $V \subset U \times \mathbb{R}$ on which we may define the *flow of* X, $\phi : V \to M$, by requiring that for any point $q \in U$, $\frac{\partial}{\partial t}\phi(q,t) = X|_{\phi(q,t)}$ The flow is given in flowbox coordinates by

$$(x^1, \ldots, x^m, t) \mapsto (x^1 + t, x^2, \ldots, x^m).$$

With systems of PDE, it becomes difficult to determine the appropriate initial data for a given system (see for examples). We now examine a simple PDE system, first in coordinates, and then later (in §**??**) using differential forms.

Example 1.2.0.4. Consider the system for $u(x,y)$ given by

$$\begin{aligned} u_x &= A(x,y,u), \\ u_y &= B(x,y,u), \end{aligned} \tag{1.2.2}$$

where A, B are given smooth functions. Since (1.2.2) specifies both partial derivatives of u, at any given point $p = (x,y,u) \in \mathbb{R}^3$ the tangent plane to the graph of a solution passing through p is uniquely determined.

In this way, (1.2.2) defines a smoothly-varying field of two-planes on $\mathbb{R}^3$, just as the ODE (1.2.1) defines a field of one-planes (i.e., a line field) on $\mathbb{R}^2$. For (1.2.1), Picard's Theorem guarantees that the one-planes "fit together" to form a solution curve through any given point. For (1.2.2), existence of solutions amounts to whether or not the two-planes "fit together".

[2]See, e.g., [**?**] vol. I, p.205

We can attempt to solve (1.2.2) in a neighborhood of $(0,0)$ by solving a succession of ODE's. Namely, if we set $y = 0$ and $u(0,0) = u_0$, Picard's Theorem implies that there exists a unique function $\tilde{u}(x)$ satisfying

$$\frac{d\tilde{u}}{dx} = A(x, 0, \tilde{u}), \qquad \tilde{u}(0) = u_0. \tag{1.2.3}$$

After solving (1.2.3), hold x fixed and use Picard's Theorem again on the initial value problem

$$\frac{du}{dy} = B(x, y, u), \qquad u(x, 0) = \tilde{u}(x). \tag{1.2.4}$$

This determines a function $u(x, y)$ on some neighborhood of $(0,0)$. The problem is that this function may not satisfy our original equation.

Whether or not (1.2.4) actually gives a solution to (1.2.2) depends on whether or not the equations (1.2.2) are "compatible" as differential equations. For smooth solutions to a system of PDE, compatibility conditions arise because mixed partials must commute, i.e., $(u_x)_y = (u_y)_x$. In our example,

$$\begin{aligned}
(u_x)_y &= \frac{\partial}{\partial y} A(x, y, u) = A_y(x, y, u) + A_u(x, y, u)\frac{\partial u}{\partial y} = A_y + BA_u, \\
(u_y)_x &= B_x + AB_u,
\end{aligned}$$

so setting $(u_x)_y = (u_y)_x$ reveals a "hidden equation", the compatibility condition

$$A_y + BA_u = B_x + AB_u. \tag{1.2.5}$$

We will prove in §1.9 that the commuting of second-order partials in this case implies that all higher-order mixed partials commute as well, so that there are no further hidden equations. In other words, if (1.2.5) is an identity in x, y, u, then solving the ODE's (1.2.3) and (1.2.4) in succession gives a solution to (1.2.2), and solutions depend on one constant.

Exercise 1.2.0.5: Show that, if (1.2.5) is an identity, then one gets the same solution by first solving for $\tilde{u}(y) = u(0, y)$.

If (1.2.5) is not an identity, there are several possibilities. If u appears in (1.2.5), then it gives an equation which every solution to (1.2.2) must satisfy. Given a point $p = (0, 0, u_0)$ at which (1.2.5) is not an identity, and such that the implicit function theorem may be applied to (1.2.5) to determine $u(x, y)$ near $(0,0)$, then only this solved-for u can be the solution passing through p. However, it still may not satisfy (1.2.2), in which case there is no solution through p.

If u does not appear in (1.2.5), then it gives a relation between x and y, and there is no solution defined on an open set around $(0,0)$.

Remark 1.2.0.6. For more complicated systems of PDE, it is not as easy to determine if all mixed partials commute. The Cartan-Kähler Theorem (see Chapters 5 and 7) will provide an algorithm which tells us when to stop checking compatibilities.

Exercises 1.2.0.7:

1. Consider this special case of Example 1.2.0.4:

$$u_x = A(x, y),$$
$$u_y = B(x, y),$$

where A and B satisfy $A(0, 0) = B(0, 0) = 0$. Verify that solving the initial value problems (1.2.3)–(1.2.4) gives

$$u(x, y) = u_0 + \int_{s=0}^{x} A(s, 0)ds + \int_{t=0}^{y} B(x, t)dt. \tag{1.2.6}$$

Under what condition does this function u satisfy (1.2.2)? Verify that the resulting condition is equivalent to (1.2.5) in this special case.

2. Rewrite (1.2.6) as a line integral involving the 1-form

$$\omega := A(x, y)dx + B(x, y)dy,$$

and determine the condition which ensures that the integral is independent of path.

3. Determine the space of solutions to (1.2.2) in the following special cases:
   (a) $A = -\frac{x}{u}$, $B = -\frac{y}{u}$.
   (b) $A = B = \frac{x}{u}$.
   (c) $A = -\frac{x}{u}$, $B = y$.

1.3. Introduction to differential equations without coordinates

Example 1.2.0.4 revisited. Instead of working on $\mathbb{R}^2 \times \mathbb{R}$ with coordinates $(x, y) \times (u)$, we will work on the larger space $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ with coordinates $(x, y) \times (u) \times (p, q)$, which we will denote $J^1(\mathbb{R}^2, \mathbb{R})$, or J^1 for short. This space, called the *space of* 1*-jets of mappings from* $\mathbb{R}^2$ *to* $\mathbb{R}$, is given additional structure and generalized in §1.9.

Let $u : U \to \mathbb{R}$ be a smooth function defined on an open set $U \subset \mathbb{R}^2$. We associate to u the surface in J^1 given by

$$u = u(x, y),\ p = u_x(x, y),\ q = u_y(x, y). \tag{1.3.1}$$

which we will refer to as the *lift* of u. The graph of u is the projection of the lift (1.3.1) in J^1 to $\mathbb{R}^2 \times \mathbb{R}$.

We will eventually work on J^1 without reference to coordinates. As a step in that direction, consider the differential forms

$$\theta := du - pdx - qdy, \quad \Omega := dx \wedge dy$$

defined on J^1. Suppose $i : S \hookrightarrow J^1$ is a surface such that $i^*\Omega \neq 0$ at each point of S. Since dx, dy are linearly independent 1-forms on S, we may use x, y as coordinates on S, and the surface may be expressed as

$$u = u(x, y),\ p = p(x, y),\ q = q(x, y).$$

Suppose $i^*\theta = 0$. Then

$$i^*du = pdx + qdy.$$

On the other hand, since u restricted to S is a function of x and y, we have

$$du = u_x dx + u_y dy.$$

Because dx, dy are independent on S, these two equations imply that $p = u_x$ and $q = u_y$ on S. Thus, surfaces $i : S \hookrightarrow J^1$ such that $i^*\theta = 0$ and $i^*\Omega$ is nonvanishing correspond to lifts of maps $u : U \to \mathbb{R}$.

Now consider the 3-fold $j : \Sigma \hookrightarrow J^1$ defined by the equations

$$p = A(x, y, u), \qquad q = B(x, y, u).$$

Let $i : S \hookrightarrow \Sigma$ be a surface such that $i^*\theta = 0$ and $i^*\Omega$ is nonvanishing. Then the projection of S to $\mathbb{R}^2 \times \mathbb{R}$ is the graph of a solution to (1.2.2). Moreover, all solutions to (1.2.2) are the projections of such surfaces, by taking S as the lift of the solution.

Thus we have a correspondence

$$\text{solutions to (1.2.2)} \Leftrightarrow \text{ surfaces } i : S \hookrightarrow \Sigma \text{ such that } i^*\theta \equiv 0 \text{ and } i^*\Omega \neq 0.$$

On such surfaces, we also have $i^*d\theta \equiv 0$, but

$$\begin{aligned} d\theta &= -dp \wedge dx - dq \wedge dy, \\ j^*d\theta &= -(A_x dx + A_y dy + A_u du) \wedge dx - (B_x dx + B_y dy + B_u du) \wedge dy, \\ i^*d\theta &= (A_y - B_x + A_u B - B_u A) i^*(\Omega). \end{aligned}$$

(To obtain the second line we use the defining equations of Σ and to obtain the third line we use $i^*(du) = Adx + Bdy$.) Because $i^*\Omega \neq 0$, the equation

$$A_y - B_x + A_u B - B_u A = 0 \tag{1.3.2}$$

must hold on S. This is precisely the same as the condition (1.2.5) obtained by checking that mixed partials commute.

If (1.3.2) does not hold identically on Σ, then it gives another equation which must hold for any solution. But since $\dim \Sigma = 3$, in that case (1.3.2) already describes a surface in Σ. If there is any solution surface S, it must be an open subset of the surface in Σ given by (1.3.2). This surface will only be a solution if θ pulls back to be zero on it. If (1.3.2) is an identity, then we may use the Frobenius Theorem (see below) to conclude that through any point of Σ there is a unique solution S (constructed, as in §1.2, by solving a succession of ODE's). In this sense, (1.3.2) implies that all higher partial derivatives commute.

We have now recovered our observations from §1.2.

The general game plan for treating a system of PDE as an *exterior differential system* (EDS) will be as follows:

One begins with a "universal space" (J^1 in the above example) where the various partial derivatives are represented by independent variables. Then one restricts to the subset Σ of the universal space defined by the system of PDE by considering it as a set of equations among independent variables. Solutions to the PDE correspond to submanifolds of Σ on which the variables representing what we want to be partial derivatives actually are partial derivatives. These submanifolds are characterized by the vanishing of certain differential forms.

These remarks will be explained in detail in §1.9.

Picard's Theorem revisited. On $\mathbb{R}^2$ with coordinates (x, u), consider $\theta = du - f(x, u)dx$. Then there is a one-to-one correspondence between solutions of the ODE (1.2.1) and curves $c : \mathbb{R} \to \mathbb{R}^2$ such that $c^*(\theta) = 0$ and $c^*(dx)$ is nonvanishing.

More generally, the flowbox coordinate theorem 1.2.0.3 implies:

Theorem 1.3.0.8. *Let M be a C^∞ manifold of dimension m, and let $\theta^1, \dots, \theta^{m-1} \in \Omega^1(M)$ be pointwise linearly independent in some open neighborhood $U \subset M$. Then through $p \in U$ there exists a curve $c : \mathbb{R} \to U$, unique up to reparametrization, such that $c^*(\theta^j) = 0$ for $1 \leq j \leq m-1$.*

(For a proof, see [**?**].)

The Frobenius Theorem. In §1.9 we will prove the following result, which is a generalization, both of Theorem 1.3.0.8 and of the asserted existence of solutions to Example 1.2.0.4 when (1.2.5) holds, to an existence theorem for certain systems of PDE:

Theorem 1.3.0.9 (Frobenius Theorem, first version)**.** *Let Σ be a C^∞ manifold of dimension m, and let $\theta^1, \dots, \theta^{m-n} \in \Omega^1(\Sigma)$ be pointwise linearly independent. If there exist 1-forms $\alpha^i_j \in \Omega^1(\Sigma)$ such that $d\theta^j = \alpha^j_i \wedge \theta^i$ for all j, then through each $p \in \Sigma$ there exists a unique n-dimensional manifold $i : N \hookrightarrow \Sigma$ such that $i^*(\theta^j) = 0$ for $1 \leq j \leq m-n$.*

In order to motivate our study of exterior differential systems, we reword the Frobenius Theorem more geometrically as follows: Let Σ be an

m-dimensional manifold such that through each point $x \in \Sigma$ there is an n-dimensional subspace $E_x \subset T_x\Sigma$ which varies smoothly with x (such a structure is called a *distribution*). We consider the problem of finding submanifolds $X \subset \Sigma$ such that $T_xX = E_x$ for all $x \in X$.

Consider ${E_x}^\perp \subset T_x^*\Sigma$. Let θ_x^a, $1 \leq a \leq m-n$, be a basis of ${E_x}^\perp$. We may choose the θ_x^a to vary smoothly to obtain $m-n$ linearly independent forms $\theta^a \in \Omega^1(\Sigma)$. Let $\mathcal{I} = \{\theta^1, \ldots, \theta^{m-n}\}$ denote the *differential ideal* they generate in $\Omega^*(\Sigma)$ (see §**??**). The submanifolds X tangent to the distribution E are exactly the n-dimensional submanifolds $i : N \hookrightarrow \Sigma$ such that $i^*(\alpha) = 0$ for all $\alpha \in \mathcal{I}$. Call such a submanifold an *integral manifold* of $\mathcal{I}$.

To find integral manifolds, we already know that if there are any, their tangent space at any point $x \in \Sigma$ is already uniquely determined, namely it is E_x. The question is whether these n-planes can be "fitted together" to obtain an n-dimensional submanifold. This information is contained in the derivatives of the θ^a's, which indicate how the n-planes "move" infinitesimally.

If we are to have $i^*(\theta^a) = 0$, we must also have $d(i^*\theta^a) = i^*(d\theta^a) = 0$. If there is to be an integral manifold through x, or even an n-plane $E_x \subset T_x\Sigma$ on which $\alpha\mid_{E_x} = 0$, $\forall \alpha \in \mathcal{I}$, the equations $i^*(d\theta^a) = 0$ cannot impose any additional conditions, i.e., we must have $d\theta^a|_{E_x} = 0$ because we already have a unique n-plane at each point $x \in \Sigma$. To recap, for all a we must have

$$d\theta^a = \alpha_1^a \wedge \theta^1 + \ldots + \alpha_{m-n}^a \wedge \theta^{m-n} \tag{1.3.3}$$

for some $\alpha_b^a \in \Omega^1(\Sigma)$, because the forms θ_x^a span ${E_x}^\perp$.

Notation 1.3.0.10. Suppose $\mathcal{I}$ is an ideal and ϕ and ψ are k-forms. Then we write $\phi \equiv \psi \operatorname{mod} \mathcal{I}$ if $\phi = \psi + \beta$ for some $\beta \in \mathcal{I}$.

Let $\{\theta^1, \ldots, \theta^{m-n}\} \subset \Omega^*(\Sigma)$ denote the algebraic ideal generated by $\theta^1, \ldots, \theta^{m-n}$ (see §**??**). Now (1.3.3) may be restated as

$$d\theta^a \equiv 0 \text{ mod } \{\theta^1, \ldots, \theta^{m-n}\}. \tag{1.3.4}$$

The Frobenius Theorem states that this necessary condition is also sufficient:

Theorem 1.3.0.11 (Frobenius Theorem, second version)**.** *Let $\mathcal{I}$ be a differential ideal generated by the linearly independent 1-forms $\theta^1, \ldots, \theta^{m-n}$ on an m-fold Σ, i.e., $\mathcal{I} = \{\theta^1, \ldots, \theta^{m-n}\}$. Suppose $\mathcal{I}$ is also generated algebraically by $\theta^1, \ldots, \theta^{m-n}$, i.e., $\mathcal{I} = \{\theta^1, \ldots, \theta^{m-n}\}$. Then through any $p \in \Sigma$ there exists an n-dimensional integral manifold of $\mathcal{I}$. In fact, in a sufficiently small neighborhood of p there exists a coordinate system $y^1, \ldots, y^m$ such that $\mathcal{I}$ is generated by $dy^1, \ldots, dy^{m-n}$.*

We postpone the proof until §1.9.

Definition 1.3.0.12. We will say a subbundle $I \subset T^*\Sigma$ is *Frobenius* if the ideal generated algebraically by sections of I is also a differential ideal. We will say a distribution $\Delta \subset \Gamma(T\Sigma)$ is *Frobenius* if $\Delta^\perp \subset T^*\Sigma$ is Frobenius. Equivalently (see Exercise 1.3.0.13.2 below), Δ is Frobenius if $\forall X, Y \in \Delta$, $[X, Y] \in \Delta$, where $[X, Y]$ is the Lie bracket.

If $\{\theta^a\}$ fails to be Frobenius, not all hope is lost for an n-dimensional integral manifold, but we must restrict ourselves to the subset $j : \Sigma' \hookrightarrow \Sigma$ on which (1.3.4) holds, and see if there are n-dimensional integral manifolds of the ideal generated by $j^*(\theta^a)$ on Σ'. (This was what we did in the special case of Example 1.2.0.4.)

Exercises 1.3.0.13:

1. Which of the following ideals are Frobenius?
$$\mathcal{I}_1 = \{dx^1, x^2dx^3 + dx^4\}$$
$$\mathcal{I}_2 = \{dx^1, x^1dx^3 + dx^4\}$$
2. Show that the differential forms and vector field conditions for being Frobenius are equivalent, i.e., $\Delta \subset \Gamma(T\Sigma)$ satisfies $[\Delta, \Delta] \subseteq \Delta$ if and only if $\Delta^\perp \subset T^*\Sigma$ satisfies $d\theta \equiv 0 \mod \Delta^\perp$ for all $\theta \in \Gamma(\Delta^\perp)$.
3. On $\mathbb{R}^3$ let $\theta = Adx + Bdy + Cdz$, where $A = A(x, y, z)$, etc. Assume the differential ideal generated by θ is Frobenius, and explain how to find a function $f(x, y, z)$ such that the differential systems $\{\theta\}$ and $\{df\}$ are equivalent.

1.4. Introduction to geometry without coordinates: curves in $\mathbb{E}^2$

We will return to our study of surfaces in $\mathbb{E}^3$ in . To see how to use moving frames to obtain invariants, we begin with a simpler problem.

Let $\mathbb{E}^2$ denote the oriented Euclidean plane. Given two parametrized curves $c_1, c_2 : \mathbb{R} \to \mathbb{E}^2$, we ask two questions: When does there exist a Euclidean motion $A : \mathbb{E}^2 \to \mathbb{E}^2$ (i.e., a rotation and translation) such that $A(c_1(\mathbb{R})) = c_2(\mathbb{R})$? And, when do there exist a Euclidean motion $A : \mathbb{E}^2 \to \mathbb{E}^2$ and a constant c such that $A(c_1(t)) = c_2(t + c)$ for all t?

Figure 2. Are these two curves equivalent?

Instead of using coordinates at a point, we will use an *adapted frame* , i.e., for each t we take a basis of $T_{c(t)}\mathbb{E}^2$ that is "adapted" to Euclidean geometry. This geometry is induced by the group of Euclidean motions—the changes of coordinates of $\mathbb{E}^2$ preserving the inner product and orientation—which we will denote by $ASO(2)$.

In more detail, the group $ASO(2)$ consists of transformations of the form

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto \begin{pmatrix} t^1 \\ t^2 \end{pmatrix} + R \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \tag{1.4.1}$$

where $R \in SO(2)$ is a rotation matrix. It can be represented as a matrix Lie group by writing

$$ASO(2) = \left\{ M \in GL(3,\mathbb{R}) \,\middle|\, M = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & R \end{pmatrix},\ \mathbf{t} \in \mathbb{R}^2, R \in SO(2) \right\}. \tag{1.4.2}$$

Then its action on $\mathbb{E}^2$ is given by $\mathbf{x} \mapsto M\mathbf{x}$, where we represent points in $\mathbb{E}^2$ by $\mathbf{x} = {}^\top\begin{pmatrix} 1 & x^1 & x^2 \end{pmatrix}$.

We may define a mapping from $ASO(2)$ to $\mathbb{E}^2$ by

$$\begin{pmatrix} 1 & 0 \\ x & R \end{pmatrix} \mapsto x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1.4.3}$$

which takes each group element to the image of the origin under the transformation (1.4.1). The fiber of this map over every point is a left coset of $SO(2) \subset ASO(2)$, so $\mathbb{E}^2$, as a manifold, is the quotient $ASO(2)/SO(2)$. Furthermore, $ASO(2)$ may be identified with the bundle of oriented orthonormal bases of $\mathbb{E}^2$ by identifying the columns of the rotation matrix $R = (e_1, e_2)$ with an oriented orthonormal basis of $T_x\mathbb{E}^2$, where x is the basepoint given by (1.4.3). (Here we use the fact that for a vector space V, we may identify V with T_xV for any $x \in V$.)

Returning to the curve $c(t)$, we choose an oriented orthonormal basis of $T_{c(t)}\mathbb{E}^2$ as follows: A natural element of $T_{c(t)}\mathbb{E}^2$ is $c'(t)$, but this may not be of unit length. So, we take $e_1(t) = c'(t)/|c'(t)|$, and this choice also determines $e_2(t)$. Of course, to do this we must assume that the curve is regular:

Definition 1.4.0.14. A curve $c(t)$ is said to be *regular* if $c'(t)$ never vanishes. More generally, a map $f : M \to N$ between differentiable manifolds is regular if df is everywhere defined and of rank equal to $\dim M$.

What have we done? We have constructed a map to the Lie group $ASO(2)$ as follows:

$$\begin{aligned} C : \mathbb{R} &\to ASO(2), \\ t &\mapsto \begin{pmatrix} 1 & 0 \\ c(t) & (e_1(t), e_2(t)) \end{pmatrix}. \end{aligned}$$

We will obtain differential invariants of our curve by differentiating this mapping, and taking combinations of the derivatives that are invariant under Euclidean changes of coordinates.

Consider $v(t) = |c'(t)|$, called the *speed* of the curve. It is invariant under Euclidean motions and thus is a differential invariant. However, it is only an invariant of the mapping, not of the image curve (see Exercise 1.4.0.15.2). The speed measures how much (internal) distance is being distorted under the mapping c.

Consider $\frac{de_1}{dt}$. We must have $\frac{de_1}{dt} = \lambda(t)e_2(t)$ for some function $\lambda(t)$ because $|e_1(t)| \equiv 1$ (see Exercise 1.4.0.15.1 below). Thus $\lambda(t)$ is a differential invariant, but it again depends on the parametrization of the curve. To determine an invariant of the image alone, we let $\tilde{c}(t)$ be another parametrization of the same curve. We calculate that $\tilde{\lambda}(t) = \frac{\tilde{v}(t)}{v(t)}\lambda(t)$, so we set $\kappa(t) = \frac{\lambda(t)}{v(t)}$. This $\kappa(t)$, called the *curvature* of the curve, measures how much c is infinitesimally moving away from its tangent line at $c(t)$.

A necessary condition for two curves $c, \tilde{c}$ to have equivalent images is that there exists a diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that $\kappa(t) = \tilde{\kappa}(\psi(t))$. It will follow from Corollary 1.6.0.31 that the images of curves are locally classified up to congruence by their curvature functions, and that parametrized curves are locally classified by κ, v.

Exercises 1.4.0.15:

1. Let V be a vector space with a nondegenerate inner product $\langle , \rangle$. Let $v(t)$ be a curve in V such that $F(t) := \langle v(t), v(t)\rangle$ is constant. Show that $v'(t) \perp v(t)$ for all t. Show the converse is also true.
2. Suppose that c is regular. Let $s(t) = \int_0^t |c'(\tau)|d\tau$ and consider c parametrized by s instead of t. Since s gives the length of the image of $c : [0, s] \to \mathbb{E}^2$, s is called an *arclength parameter*. Show that in this preferred parametrization, $\kappa(s) = |\frac{de_1}{ds}|$.
3. Show that $\kappa(t)$ is constant iff the curve is an open subset of a line (if $\kappa = 0$) or circle of radius $\frac{1}{\kappa}$.
4. Let $c(t) = (x(t), y(t))$ be given in coordinates. Calculate $\kappa(t)$ in terms of $x(t), y(t)$ and their derivatives.
5. Calculate the function $\kappa(t)$ for an ellipse. Characterize the points on the ellipse where the maximum and minimum values of $\kappa(t)$ occur.
6. Can $\kappa(t)$ be unbounded if $c(t)$ is the graph of a polynomial?

[*Osculating circles*]

(a) Calculate the equation of a circle passing through three points in the plane.

(b) Calculate the equation of a circle passing through two points in the plane and having a given tangent line at one of the points.

Parts (a) and (b) may be skipped; the exercise proper starts here:

(c) Show that for any curve $c \subset \mathbb{E}^2$, at each point $x \in c$ one can define an *osculating circle* by taking the limit of the circle through the three points $c(t), c(t_1), c(t_2)$ as $t_1, t_2 \to t$. (A line is defined to be a circle of infinite radius.)

(d) Show that one gets the same circle if one takes the limit as $t \to t_1$ of the circle through $c(t)$, $c(t_1)$ that has tangent line at $c(t)$ parallel to $c'(t)$.

(e) Show that the radius of the osculating circle is $1/\kappa(t)$.

(f) Show that if $\kappa(t)$ is monotone, then the osculating circles are nested. ⊚

1.5. Submanifolds of homogeneous spaces

Using the machinery we develop in this section and §1.6, we will answer the questions about curves in $\mathbb{E}^2$ posed at the beginning of §1.4. The quotient $\mathbb{E}^2 = ASO(2)/SO(2)$ is an example of a *homogeneous space*, and our answers will follow from a general study of classifying maps into homogeneous spaces.

Definition 1.5.0.16. Let G be a Lie group, H a closed Lie subgroup, and G/H the set of left cosets of H. Then G/H is naturally a differentiable manifold with the induced differentiable structure coming from the quotient map (see [**?**], Theorem II.3.2). The space G/H is called a *homogeneous space*.

Definition 1.5.0.17 (Left and right actions)**.** Let G be a group that acts on a set X by $x \mapsto \sigma(g)(x)$. Then σ is called a *left action* if $\sigma(a) \circ \sigma(b) = \sigma(ab)$, or a *right action* if $\sigma(a) \circ \sigma(b) = \sigma(ba)$,

For example, the action of G on itself by left-multiplication is a left action, while left-multiplication by g^{-1} is a right action.

A homogeneous space G/H has a natural (left) G-action on it; the subgroup stabilizing $[e]$ is H, and the stabilizer of any point is conjugate to H. Conversely, a manifold X is a homogeneous space if it admits a smooth transitive action by a Lie group G. If H is the isotropy group of a point $x_0 \in X$, then $X \simeq G/H$, and x_0 corresponds to $[e] \in G/H$, the coset of the identity element. (See [**?, ?**] for additional facts about homogeneous spaces.)

In the spirit of Klein's *Erlanger Programm* (see [**?**, **?**] for historical accounts), we will consider G as the group of *motions* of G/H. We will study the geometry of submanifolds $M \subset G/H$, where two submanifolds $M, M' \subset G/H$ will be considered *equivalent* if there exists a $g \in G$ such that $g(M) = M'$.

To determine necessary conditions for equivalence we will find differential invariants as we did in §1.1 and §1.4. (Note that we need to specify whether we are interested in invariants of a mapping or just of the image.) After finding invariants, we will then interpret them as we did in the exercises in §1.4.

We will derive invariants for maps $f : M \to G/H$ by constructing *lifts* $F : M \to G$ as we did for curves in $\mathbb{E}^2$.

Definition 1.5.0.18. A *lift* of a mapping $f : M \to G/H$ is defined to be a map $F : M \to G$ such that the following diagram commutes:

$$\begin{tikzcd} & G \arrow[d] \\ M \arrow[ur, "F"] \arrow[r, "f"'] & G/H \end{tikzcd}$$

Given a lift F of f, any other lift $\tilde{F} : M \to G$ must be of the form

$$\tilde{F}(x) = F(x)a(x) \tag{1.5.1}$$

for some map $a : M \to H$.

By associating the value of the lift $F(x)$ with its action on G/H, we may think of choosing a lift to G as analogous to putting a point p in a normalized position, as we did in §1.1.

Given $f : M \to G/H$, we will choose lifts adapted to the infinitesimal geometry. To explain what this statement means, we first remark that in the situations we will be dealing with, the fiber at $x \in G/H$ of the fibration $\pi : G \to G/H$ admits the interpretation of being a subset of the space of *framings* or bases of T_xG/H. Since H fixes the point $[e] \in G/H$, its infinitesimal action on tangent vectors gives a representation $\rho : H \to GL(T_{[e]}G/H)$, called the *isotropy representation* [**?**]. Now fix a reference basis $(v_1, \dots, v_n)$ of $T_{[e]}G/H$. We identify the H-orbit of this basis with $\pi^{-1}([e])$. Similarly, at other points $x \in G/H$ we have a group conjugate to H acting on T_xG/H.

Thus, a choice of lift may be considered as a choice of framing of G/H along M, and we will make choices that reflect the geometry of M. For example, if $\dim M = n$, we may require the first n basis vectors of T_xG/H to be tangent to M. In the above example of curves in $\mathbb{E}^2$, we also normalized the length of the first basis vector e_1 to be constant.

Once a unique lift is determined, differentiating that lift will provide differential invariants. This is because we can classify maps into G, up to equivalence under left multiplication, using the *Maurer-Cartan form*.

1.6. The Maurer-Cartan form

If you need to brush up on matrix Lie groups and Lie algebras, this would be a good time to consult §**??** and §**??**.

Definition 1.6.0.19. Let $G \subseteq GL(n, \mathbb{R})$ be a matrix Lie group with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$, and let $g = (g^i_j)$ be the matrix-valued function which embeds G into the vector space $M_{n\times n}$ of $n\times n$ matrices with real entries, with differential $dg_a : T_aG \to T_{g(a)}M_{n\times n} \simeq M_{n\times n}$. We define the *Maurer-Cartan form* of G as

$$\omega_a = g(a)^{-1} dg_a.$$

This is often written

$$\omega = g^{-1} dg.$$

The Maurer-Cartan form is $M_{n\times n}$-valued. In fact, in Exercise 1.6.0.23 you will show that it takes values in $\mathfrak{g} \subset M_{n\times n}$, i.e., $\omega_a(v) \in \mathfrak{g}$ for all $v \in T_aG$.

Example 1.6.0.20. Consider $G = SO(2) \subset GL(2, \mathbb{R})$. We may parametrize $SO(2)$ by

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

Then

$$\omega = g^{-1} dg = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix}.$$

Definition 1.6.0.21. A differential form $\alpha \in \Omega^k(G)$ is *left-invariant* if for all $a \in G$, we have $L_a^*(\alpha_g) = \alpha_{a^{-1}g}$, where $L_a : G \to G$ is the diffeomorphism $g \mapsto ag$. (We similarly define left-invariant vector fields and k-vector fields.)

Note that a left-invariant form α is uniquely determined by α_g for any $g \in G$. In this way, the set of left-invariant k-forms may be identified with $\Lambda^k T_g^* G$.

Given an arbitrary Lie group G, we will let $\mathfrak{g}$ denote its Lie algebra, which may be identified with T_eG or with the space of left-invariant vector fields. We generalize the definition of the Maurer-Cartan form to this situation as follows:

Definition 1.6.0.22. The *Maurer-Cartan form* ω of G is the unique left-invariant $\mathfrak{g}$-valued 1-form on G such that $\omega|_e : T_eG \to \mathfrak{g}$ is the identity map.

Exercise 1.6.0.23: Show that the two definitions agree when G is a matrix Lie group. In particular, the Maurer-Cartan form of a matrix Lie group is $\mathfrak{g}$-valued.

Remark 1.6.0.24. When $F : M \to G$ is a lift of a map $f : M \to G/H$, and G is a matrix Lie group, the change in the pullback of the Maurer-Cartan form resulting from a change of lift (1.5.1) is

$$\tilde{F}^*(\omega) = a^{-1}F^*(\omega)a + a^{-1}da. \tag{1.6.1}$$

For an abstract Lie group, the analogous formula is

$$\tilde{F}^*(\omega) = \mathrm{Ad}_{a^{-1}}(F^*\omega) + a^*\omega.$$

Definition 1.6.0.25. Let $\omega = (\omega^i_k)$ and $\eta = (\eta^i_k)$ be matrices whose entries are elements of a vector space V, so that $\omega, \eta \in V \otimes M_{n\times n}$. Define their *matrix wedge product* $\omega \wedge \eta \in \Lambda^2 V \otimes M_{n\times n}$ by

$$(\omega \wedge \eta)^i_j := \omega^i_k \wedge \eta^k_j.$$

More generally, for $\omega \in \Lambda^k V \otimes M_{n\times n}, \eta \in \Lambda^j V \otimes M_{n\times n}$ the same formula yields $\omega \wedge \eta \in \Lambda^{k+j} V \otimes M_{n\times n}$.

One thing that makes the Maurer-Cartan form ω especially useful to work with is that its exterior derivative may be computed algebraically as follows: If G is a matrix Lie group, then

$$d\omega = d(g^{-1}) \wedge dg.$$

To compute $d(g^{-1})$, consider the identity matrix $e = (\delta^i_j)$ as a constant map $G \to M_{n\times n}$ and note that it is the product of two nonconstant functions:

$$0 = d(e) = d(g^{-1}g) = d(g^{-1})g + g^{-1}dg.$$

So, $d(g^{-1}) = -g^{-1}(dg)g^{-1}$ and

$$d\omega = -g^{-1}(dg)g^{-1} \wedge dg = -(g^{-1}dg) \wedge (g^{-1}dg) = -\omega \wedge \omega.$$

Summary 1.6.0.26. On a matrix Lie group G, the Maurer-Cartan form ω defined by $\omega = g^{-1}dg$ is a left-invariant $\mathfrak{g}$-valued 1-form and satisfies the *Maurer-Cartan equation*:

$$d\omega = -\omega \wedge \omega. \tag{1.6.2}$$

Definition 1.6.0.27. If ω, θ are two $\mathfrak{g}$-valued 1-forms, define the $\mathfrak{g}$-valued 2-form $[\omega, \theta]$ by

$$[\omega, \theta](X, Y) = [\omega(X), \theta(Y)] + [\omega(Y), \theta(X)].$$

The Maurer-Cartan equation holds on an abstract Lie group G in the following form:

$$d\omega = -\frac{1}{2}[\omega, \omega]. \tag{1.6.3}$$

As mentioned above, the Maurer-Cartan form will be our key to classifying maps into homogeneous spaces of G. We first show how it classifies maps into G:

Theorem 1.6.0.28 (Cartan). *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form ω. Let M be a manifold on which there exists a $\mathfrak{g}$-valued 1-form ϕ satisfying $d\phi = -\phi \wedge \phi$. Then for any point $x \in M$ there exist a neighborhood U of x and a map $f : U \to G$ such that $f^*\omega = \phi$. Moreover, any two such maps f_1, f_2 must satisfy $f_1 = L_a \circ f_2$ for some fixed $a \in G$.*

Corollary 1.6.0.29. *Given maps $f_1, f_2 : M \to G$, then $f_1^*\omega = f_2^*\omega$ if and only if $f_1 = L_a \circ f_2$ for some fixed $a \in G$.*

Proof of Corollary. Let $\phi = f_1^*\omega$. □

Proof of Theorem 1.6.0.28. This is a good opportunity to use the Frobenius Theorem.

On $\Sigma = M^n \times G$, let π, ρ denote the projections to each factor and let $\theta = \pi^*(\phi) - \rho^*(\omega)$. Write $\theta = (\theta^i_j)$, and let $I \subset T^*\Sigma$ be the subbundle spanned by the forms θ^i_j. Submanifolds of dimension n to which these forms pull back to be zero are graphs of maps $f : M \to G$ such that $\phi = f^*\omega$. We check the conditions given in the Frobenius Theorem. Calculating derivatives (and omitting the pullback notation), we have

$$\begin{aligned} d\theta &= -\phi \wedge \phi + \omega \wedge \omega \\ &= -\phi \wedge \phi + (\theta - \phi) \wedge (\theta - \phi) \\ &\equiv 0 \bmod I. \end{aligned}$$

Thus, the system is Frobenius and there is a unique n-dimensional integral manifold through any $(x, g) \in \Sigma$.

Suppose f_1, f_2 are two different solutions. Say $f_1(x) = g$. Let $a = gf_2(x)^{-1}$. Then the graph of $f = L_a \circ f_2$ passes through (x, g) and $f^*\omega = \phi$. By uniqueness, it follows that $f_1 = L_a \circ f_2$. □

Remark 1.6.0.30. If we assume in Theorem 1.6.0.28 that M is connected and simply-connected, then the desired map f may be extended to all of M [**?**].

We may apply Theorem 1.6.0.28 to classify curves in $\mathbb{E}^2$. In this case, the pullback of the Maurer-Cartan form of $ASO(2) \subset GL(3, \mathbb{R})$ under the lift constructed in §1.4 takes the simple form

$$F^*\omega = \begin{pmatrix} 0 & 0 & 0 \\ dt & 0 & -\kappa\, dt \\ 0 & \kappa\, dt & 0 \end{pmatrix}, \tag{1.6.4}$$

where t is an arclength parameter.

Corollary 1.6.0.31. *For curves $c, \tilde{c} \subset \mathbb{E}^2$, if $\kappa(t) = \tilde{\kappa}(t+a)$ for some constant a, then $c, \tilde{c}$ are congruent.*

Exercises 1.6.0.32:

1. Let $CO(2)$ be the matrix Lie group parametrized by
$$g(\theta, t) = \begin{pmatrix} t\cos\theta & -t\sin\theta \\ t\sin\theta & t\cos\theta \end{pmatrix}, \qquad t \in (0, \infty).$$
Explicitly compute the Maurer-Cartan form and verify the Maurer-Cartan equation for $CO(2)$.
2. Verify (1.6.1).
3. Verify (1.6.4) and complete the proof of Corollary 1.6.0.31.
4. Show that (1.6.3) coincides with (1.6.2) when G is a matrix Lie group.
5. Let $\mathfrak{g}$ be a vector space with basis X_B, $1 \leq B \leq \dim \mathfrak{g}$, and a multiplication given by $[X_A, X_B] = c^C_{AB} X_C$ on the basis and extended linearly. Determine necessary and sufficient conditions on the constants c^A_{BC} implying that, with this bracket, $\mathfrak{g}$ is a Lie algebra.
6. On a Lie group G with Maurer-Cartan form ω, show that
$$d\omega_e(X, Y) = [X, Y].$$
Conclude that $d\omega = 0$ iff G is abelian.
7. On a Lie group G with Lie algebra $\mathfrak{g}$, let $\{e_A\}$ be a basis of $\mathfrak{g}$ and write $\omega = \omega^A e_A$. (Note that each 1-form ω^A is left-invariant.) Write $d\omega^A = -\tilde{C}^A_{BC}\omega^B \wedge \omega^C$, where $\tilde{C}^A_{BC} = -\tilde{C}^A_{CB}$. Show that the coefficients $\tilde{C}^A_{BC}$ are constants, and determine the set of equations that these constants must satisfy because $d^2 = 0$. Relate these equations to your answer to problem 5.

1.7. Plane curves in other geometries

Equivalence of holomorphic mappings under fractional linear transformations. Here is an example of a study of curves in a less familiar homogeneous space, the complex projective line $\mathbb{CP}^1$. To find differential invariants in such situations, we generally seek a uniquely defined lift that renders the pullback of the Maurer-Cartan form as simple as possible. Then, after finding differential invariants, we interpret them.

Definition 1.7.0.33. A *fractional linear transformation* (FLT) is a map $\mathbb{CP}^1 \to \mathbb{CP}^1$ given in terms of homogeneous coordinates $\top[x, y]$ by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right], \qquad \text{with } ad - bc = 1.$$

The group of FLT's is $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm \mathrm{Id}\}$, which acts transitively on $\mathbb{CP}^1$. Thus, $\mathbb{CP}^1$ is a homogeneous space $PSL(2, \mathbb{C})/P$, where

$$P = \left[\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right] \in PSL(2, \mathbb{C})$$

is the stabilizer of $\top[1, 0]$. Although $PSL(2, \mathbb{C})$, as presented, is not a matrix Lie group, we may avoid problems by localizing as follows:

If $\Delta \subset \mathbb{C} \subset \mathbb{CP}^1$ is a domain, then $PSL(2, \mathbb{C})$ acts on maps $f : \Delta \to \mathbb{C}$ by

$$f \mapsto \frac{af + b}{cf + d}.$$

(Since we will be working locally, there is no harm in considering f as a map to $\mathbb{C}$; then to think of f as a map to $\mathbb{CP}^1$, write it as $\top[f, 1]$.) Suppose $f, g : \Delta \to \mathbb{C}$ are two holomorphic maps with nonzero first derivatives. When are these locally equivalent via a fractional linear transformation, i.e., when does $g = A \circ f$ for some FLT A? (One can ask the same question in the real category for analytic maps $f, g : (0, 1) \to \mathbb{RP}^1$, and the answer will be the same.)

Note that in this example the target is of the same dimension as the source of the mapping, so we cannot expect an analogue of curvature, but there will be an analogue of speed.

The coordinate approach to getting invariants would be to use an FLT to normalize the map at some point z_0, say by requiring $f(z_0) = 0$, $f'(z_0) = 1$ and $f''(z_0) = 0$. Since this is exactly the extent of normalization that $PSL(2, \mathbb{C})$ can achieve, then $f'''(z_0)$ must be an invariant. Of course, this is valid only at the point z_0.

Instead we construct a lift to $PSL(2, \mathbb{C})$, which we will treat as $SL(2, \mathbb{C})$ in order to be working with a matrix Lie group. As a first try, let

$$F = \begin{pmatrix} f & -1 \\ 1 & 0 \end{pmatrix}.$$

where the projection to $\mathbb{CP}^1$ is the equivalence class of the first column. Any other lift $\tilde{F}$ of f must be of the form $\tilde{F}(z) = F(z)A(z)$, where

$$A(z) = \begin{pmatrix} a(z) & b(z) \\ 0 & a^{-1}(z) \end{pmatrix}, \qquad a(z) \neq 0.$$

We want to pick functions a, b to obtain a new lift whose Maurer-Cartan form is as simple as possible. We have

$$\begin{aligned}\tilde{F}^{-1}d\tilde{F} &= A^{-1}F^{-1}dFA + A^{-1}dA \\ &= \left\{\begin{pmatrix} a^{-1} & -b \\ 0 & a\end{pmatrix}\begin{pmatrix} 0 & 0 \\ -f' & 0\end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix} a^{-1} & -b \\ 0 & a\end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & -a'a^{-2}\end{pmatrix}\right\} dz \\ &= \begin{pmatrix} abf' + a^{-1}a' & a^{-2}ba' + a^{-1}b' + b^2 f' \\ -a^2 f' & -(abf' + a^{-1}a')\end{pmatrix} dz.\end{aligned}$$

Choose $a(z) = \pm 1/\sqrt{|f'(z)|}$ with the same sign as $f'(z)$. Then $\tilde{F}^{-1}d\tilde{F}$ takes the form

$$\begin{pmatrix} * & * \\ \mp 1 & *\end{pmatrix} dz.$$

(This is the analogue of setting $f'(z_0) = 1$ in the coordinate approach.) The function b is still free. We use it to set the diagonal term in the pullback of the Maurer-Cartan form to zero, i.e., to set $abf' + a^{-1}a' = 0$. This implies

$$b = -\frac{a'}{a^2 f'} = \pm\frac{f''}{2|f'|^{3/2}}.$$

Now our lift is unique and of the form

$$\begin{pmatrix} 0 & \frac{1}{2}\mathcal{S}_f(z) \\ \mp 1 & 0\end{pmatrix} dz,$$

where

$$\mathcal{S}_f(z) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is a differential invariant, called the *Schwarzian derivative* [**?**]. The ambiguity of the $\pm$ is due to the fact that $G = PSL(2,\mathbb{C})$, not $SL(2,\mathbb{C})$.

Exercises 1.7.0.34:

1. Show that if f is an FLT then $S_f \equiv 0$. So, just as the curvature of a curve in $\mathbb{R}^2$ measures the failure of a curve to be a line, $\mathcal{S}_f(z)$ is an infinitesimal measure of the failure of a holomorphic map to be an FLT. Since FLT's map circles to circles, $\mathcal{S}_f$ may be thought of as measuring how much circles are being distorted under f.
2. Calculate S_f for $f = ae^{bz}$, and $f = x^n$. How to these compare asymptotically? What does this say about how circles are distorted as one goes out to infinity?

Exercises on curves in other plane geometries.

Exercises 1.7.0.35:

1. *(Curves in the special affine plane)* We consider the geometry of curves that are equivalent up to translations and area-preserving

linear transformations of $\mathbb{R}^2$. These transformations are given by the matrix group

$$ASL(2,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix} \middle| \, x \in \mathbb{R}^2, A \in SL(2,\mathbb{R}) \right\},$$

acting on $\mathbb{R}^2$ in the same way as $ASO(2)$ acts in §1.4. Since the origin is fixed by the subgroup $SL(2,\mathbb{R})$, in this context we will relabel $\mathbb{R}^2$ as the *special affine plane* $S\mathbb{A}^2 = ASL(2,\mathbb{R})/SL(2,\mathbb{R})$.

(a) Find differential invariants for curves in $S\mathbb{A}^2$. (As with the Euclidean case, one can consider invariants of a parametrized curve or invariants of just the image curve.)

(b) What are the image curves with invariants zero? The image curves with constant invariants?

(c) Let κ_A denote the differential invariant that distinguishes image curves. Interpret $\kappa_A(t)$ in terms of an osculating curve, as we did with the osculating circles to a curve in §1.4.

(d) The preferred frame will lead to a unique choice of e_2. Give a geometric interpretation of e_2. ⊚

2. *(Curves in the projective plane)* Carry out the analogous exercise for curves in the projective plane $\mathbb{P}^2 = GL(3)/P$, where P is the subgroup preserving a line. Show that the curves with zero invariants are the projective lines and plane conics. Derive the *Monge equation* $((y'')^{\frac{-2}{3}})''' = 0$, characterizing plane conics, by working in a local adapted coordinate system. Note that one may do this exercise over $\mathbb{R}$ or $\mathbb{C}$.
3. Carry out the analogous exercise for curves in the conformal plane $ACO(2)/CO(2)$, where equivalence is up to translations, rotations and dilations.
4. Carry out the analogous exercise for curves in $\mathbb{L}^2 = ASO(1,1)/SO(1,1)$, where $SO(1,1)$ is the subgroup of $GL(2,\mathbb{R})$ preserving the quadratic form $Q = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note that there will be three distinct types of curves: *spacelike* curves, where $Q(c'(t), c'(t)) > 0$; *timelike* curves, where $Q(c'(t), c'(t)) < 0$; and *lightlike* curves, where $Q(c'(t), c'(t)) = 0$. What are the curves with constant invariants?

1.8. Curves in $\mathbb{E}^3$

The group $ASO(3)$ and its Maurer-Cartan form. The group $ASO(3)$ is the set of transformations of $\mathbb{E}^3$ of the form $\mathbf{x} \mapsto \mathbf{t} + R\mathbf{x}$, i.e.,

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix} + R \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

where $R \in SO(3)$ is a rotation matrix. Like $ASO(2)$, it may be represented as a matrix Lie group by writing

$$(1.8.1) \quad ASO(3) = \left\{ M \in GL(4,\mathbb{R}) \,\middle|\, M = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & R \end{pmatrix},\ \mathbf{t} \in \mathbb{R}^3, R \in SO(3) \right\}.$$

The action on $\mathbb{E}^3$ is given by $\mathbf{x} \mapsto M\mathbf{x}$, where we represent points in $\mathbb{E}^3$ by $\mathbf{x} = {}^t\begin{pmatrix} 1 & x^1 & x^2 & x^3 \end{pmatrix}$.

Having expressed $ASO(3)$ as in (1.8.1), we may express an arbitrary element of its Lie algebra $\mathfrak{aso}(3)$ as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ x^1 & 0 & -x^2_1 & -x^3_1 \\ x^2 & x^2_1 & 0 & -x^3_2 \\ x^3 & x^3_1 & x^3_2 & 0 \end{pmatrix}, \qquad x^i, x^i_j \in \mathbb{R}.$$

In this presentation, the Maurer-Cartan form of $ASO(3)$ is

$$(1.8.2) \qquad \omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix},$$

where $\omega^i, \omega^i_j \in \Omega^1(ASO(3))$. Recall from §1.6 that the forms ω^i, ω^i_j are left-invariant, and are a basis for the space of left-invariant 1-forms on $ASO(3)$.

We identify $ASO(3)$ with the space of *oriented orthonormal frames* of $\mathbb{E}^3$, as follows. Denote $g \in ASO(3)$ by a 4-tuple of vectors (warning: this is not a presentation as a matrix Lie group),

$$(1.8.3) \qquad g = (x, e_1, e_2, e_3),$$

where $x \in \mathbb{E}^3$ corresponds to translation by x, and $\{e_1, e_2, e_3\}$ is an oriented orthonormal basis of $T_x\mathbb{E}^3$ which corresponds to the rotation $R = (e_1, e_2, e_3) \in SO(3)$.

With this identification, we obtain geometric interpretations of the left-invariant forms. Substituting (1.8.3) and (1.8.2) into $dg = g\,\omega$, and considering the first column, gives

$$(1.8.4) \qquad dx = e_i\omega^i.$$

Thus, ω^i has the geometric interpretation of measuring the infinitesimal motion of the point x in the direction of e_i. More precisely, if $x(t) \in \mathbb{E}^3$ lifts to

$$C(t) = (x(t), e_1(t), e_2(t), e_3(t)) \in ASO(3),$$

then $\omega^i(C'(t)) = \langle x'(t), e_i\rangle$. Similarly, ω^i_j measures the infinitesimal motion of e_j toward e_i, because the other columns of $dg = g\,\omega$ show that

$$(1.8.5) \qquad de_j = e_i\omega^i_j.$$

That these motions are infinitesimal rotations is reflected in the relation $\omega^i_j = -\omega^j_i$, as illustrated by the following picture:

Recall from that a form $\alpha \in \Omega^1(P)$ on a bundle $\pi : P \to M$ is *semi-basic* for π if $\alpha(v) = 0$ for all $v \in \ker \pi_*$.

Proposition 1.8.0.36. *The forms* ω^i*,* $1 \le i \le 3$*, are semi-basic for the projection* $ASO(3) \to \mathbb{E}^3$*.*

Proof. Let $C(t) = (x(t), e_1(t), e_2(t), e_3(t)) \subset ASO(3)$ be a curve in a fiber. We need to show that $\omega^i(C'(t)) = 0$. If $C(t)$ stays in one fiber, then $\frac{dx}{dt} = 0$, but equation (1.8.5) shows $\frac{dx}{dt} = C'(t) \lrcorner\, dx = \omega^j(C'(t)) e_j(t)$. The result follows because the e_j are linearly independent. $\square$

Differential invariants of curves in $\mathbb{E}^3$. We find differential invariants of a regular curve $c : \mathbb{R} \to \mathbb{E}^3$. For simplicity, we only consider the image curve, so we can and will assume $|c'(t)| \equiv 1$. Consequently, we have $c'' \perp c'$ (see Exercise 1.4.0.15.1). To obtain a lift $C : \mathbb{R} \to ASO(3)$ we may take $e_1(t) = c'(t)$, $e_2(t) = c''(t)/|c''(t)|$ and this determines $e_3(t)$. Our adaptations have the effect that $C^*(\omega^1)$ is nonvanishing and $C^*(\omega^2) = C^*(\omega^3) = 0$. In terms of the Maurer-Cartan form, we have:

$$
\begin{aligned}
&d(x(t), e_1(t), e_2(t), e_3(t)) \\
&= (x(t), e_1(t), e_2(t), e_3(t))\ C^* \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ 0 & \omega^2_1 & 0 & -\omega^3_2 \\ 0 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix}.
\end{aligned}
\tag{1.8.6}
$$

Exercise 1.8.0.37: Show that $C^*(\omega^3_1) = 0$.

All forms pulled back to $\mathbb{R}$ will be multiples of ω^1, as $\omega^1 = dt$ furnishes a basis of $T^*\mathbb{R}^1$ at each point. (We continue our standard abuse of notation, writing ω^1 instead of $C^*(\omega^1)$.) So, we may write $\omega^2_1 = \kappa(t)\omega^1$ and $\omega^3_2 = \tau(t)\omega^1$, where $\kappa(t), \tau(t)$ are functions called the *curvature* and *torsion* of the curve. Traditionally one writes $e_1 = T$, $e_2 = N$, $e_3 = B$; then (1.8.6) yields the *Frenet equations*

$$d(T, N, B) = (T, N, B) \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} dt.$$

Curves with $\kappa \equiv 0$ are lines, and we may think of κ as a measurement of the failure of the curve to be a line. Curves with $\tau \equiv 0$ lie in a plane, and we may think of τ as measuring the failure of a curve to lie in a plane. In contrast to the example of plane curves, we needed a third-order invariant (the torsion) to determine a unique lift in this case.

Theorem 1.3.0.8 implies that one can specify any functions $(\kappa(t), \tau(t))$, and there will be a curve having these as curvature and torsion (because on $ASO(3) \times \mathbb{R}$ the forms $\omega^1 - d(t), \omega^3, \omega^2, \omega^2_1 - \kappa(t)\omega^1, \omega^3_1, \omega^3_2 + \tau(t)\omega^1$ satisfy the hypotheses of the theorem). If the functions are nowhere vanishing the curve will be unique up to congruence (see the exercises below).

Remark 1.8.0.38. Defining N as the unit vector in the direction of $c''(t)$ means that κ cannot be negative, and N (along with the binormal B and the torsion) is technically undefined at inflection points along the curve (i.e., points where $c''(t) = 0$). However, it is still possible to smoothly extend the frame (T, N, B) across inflection points, while satisfying the Frenet equations for smooth functions $(\kappa(t), \tau(t))$ where κ is allowed to change sign (see the discussion in [**?**]). Such frames are sometimes called *generalized Frenet frames*, and it is in this sense that ODE existence theorems provide a framed curve with given curvature and torsion functions.

Exercises 1.8.0.39:

1. Using Corollary 1.6.0.29, show that if $c, \tilde{c}$ are curves with $\kappa(t) = \tilde{\kappa}(t)$, $\tau(t) = \tilde{\tau}(t)$, then $c, \tilde{c}$ differ by a rigid motion.
2. Show that a curve $c \subset \mathbb{R}^3$ has constant κ and τ if and only if there exists a line $l \subset \mathbb{E}^3$ with the property that every normal line of c intersects l orthogonally. (A normal line is the line through $c(t)$ in the direction of $N(t)$.)
3. *(Bertrand curves)* In the previous exercise we characterized curves with constant invariants. Here we study the next simplest case, when there is a linear relation among the curvature and torsion, i.e., constants a, b, c such that $a\kappa(t) + b\tau(t) = c$ for all t.
   (a) Show that if such a linear relation holds, then there exists a second curve $\overline{c}(t)$ with the same normal line as $c(t)$ for all t.
   (b) Show moreover that the distance between the points $c(t)$ and $\overline{c}(t)$ is constant. ⊚
   (c) Characterize the curves c where there exists more than one curve $\overline{c}$ with this property.

4. Derive invariants for curves in $\mathbb{E}^n$. How many derivatives does one need to take to obtain a complete set of invariants?
5. *(Curves on spheres)* Show that a curve c with $\kappa, \tau \neq 0$ is contained in a sphere if and only if $\rho^2 + \sigma^2$ is constant, where $\rho = 1/\kappa$ and $\sigma = \rho'/\tau$. ⊚
6. Let $\mathbb{L}^3 = ASO(2,1)/SO(2,1)$, where $SO(2,1)$ is the subgroup of $GL(3,\mathbb{R})$ preserving
$$Q = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$
Find differential invariants of curves in $\mathbb{L}^3$. (As before, curves may be spacelike, timelike, or lightlike.) What are the curves with constant invariants?

1.9. Exterior differential systems and jet spaces

In §1.3, we saw how a system of ODE or PDE could be replaced by an ideal of differential forms, and solutions became submanifolds on which the forms pulled back to be zero. We now formalize this perspective, defining exterior differential systems with and without independence condition.

Exterior differential systems with independence condition.

Definition 1.9.0.40. An *exterior differential system with independence condition* on a manifold Σ consists of a differential ideal $\mathcal{I} \subset \Omega^*(\Sigma)$ and a differential n-form $\Omega \in \Omega^n(\Sigma)$ defined up to scale. This Ω, or its equivalence class $[\Omega]$ up to scale, is called the *independence condition.*

(See §**??** for a discussion of differential ideals.)

Definition 1.9.0.41. An *integral manifold* (or solution) of the system $(\mathcal{I}, \Omega)$ is an immersed n-fold $f : M^n \to \Sigma$ such that $f^*(\alpha) = 0 \ \forall \alpha \in \mathcal{I}$ and $f^*(\Omega) \neq 0$ at each point of M.

We also define the notion of an infinitesimal solution:

Definition 1.9.0.42. Let V be a vector space, and let $G(n, V)$ denote the Grassmannian of n-planes through the origin in V (see). We say $E \in G(n, T_x\Sigma)$ is an *integral element* of $(\mathcal{I}, \Omega)$ if $\Omega|_E \neq 0$ and $\alpha|_E = 0 \ \forall \alpha \in \mathcal{I}$. We let $\mathcal{V}_n(\mathcal{I}, \Omega)_x$ denote the space of integral elements of $(\mathcal{I}, \Omega)$ at $x \in \Sigma$.

Integral elements are the potential tangent spaces to integral manifolds, in the sense that the integral manifolds of an exterior differential system are the immersed submanifolds $M \subset \Sigma$ such that T_xM is an integral element for all $x \in M$.

Exercise 1.9.0.43: Let $\mathcal{I}^n = \mathcal{I} \cap \Omega^n(\Sigma)$. Show that $\mathcal{V}_n(\mathcal{I}, \Omega)_x = \{E \in G_n(T_x\Sigma) \mid \alpha|_E = 0 \ \forall \alpha \in \mathcal{I}^n\}$.

We now explain how to rephrase any system of PDE as an exterior differential system with independence condition using the language of jet bundles.

Jets.

Definition 1.9.0.44. Let t be a coordinate on $\mathbb{R}$ and let $k \geq 0$. Two differentiable maps $f, g : \mathbb{R} \to \mathbb{R}$ with $f(0) = g(0) = 0$ are said to have the same *k-jet* at 0 if

$$\frac{df}{dt}(0) = \frac{dg}{dt}(0),\ \frac{d^2 f}{dt^2}(0) = \frac{d^2 g}{dt^2}(0), \ldots, \frac{d^k f}{dt^k}(0) = \frac{d^k g}{dt^k}(0).$$

Let M, N are differentiable manifolds and $f, g : M \to N$ be two maps. Then f and g are said to have the same k-jet at $p \in M$ if

i. $f(p) = g(p) = q$, and

ii. for all maps $u : \mathbb{R} \to M$ and $v : N \to \mathbb{R}$ with $u(0) = p$, the differentiable maps $v \circ f \circ u$ and $v \circ g \circ u$ have the same k-jet at 0.

Exercise 1.9.0.45: Show that to determine if $f, g : M \to N$ have the same k-jet at p, it is sufficient to check derivatives up to order k with respect to coordinate directions in any pair of local coordinate systems around p and q.

Having the same k-jet at p is an equivalence relation on smooth maps. We denote the equivalence class of f by $j^k_p(f)$, the space k-jets where p maps to q by $J^k_{pq}(M, N)$ and the space of all k-jets of all maps from M to N by $J^k(M, N)$. This is a smooth manifold, with local coordinates as follows:

Suppose M has local coordinates x^i and N local coordinates u^a. Then $J^k(M, N)$ has coordinates x^i, u^a, p^a_i, $p^a_{ij}, , \ldots ,, p^a_{i_1,\ldots,i_k}$. We will abbreviate this as (x^i, u^a, p^a_I), where I is a multi-index of length up to k whose entries range between 1 and dim N. Then the point $j^k_{x_0}(f) \in J^k(M, N)$ has coordinates x^i_0, $u^a = f^a(x_0)$ and $p^a_I = \frac{\partial^{|I|} f^a}{\partial x^I}(x_0)$ for $1 \leq |I| \leq k$.

Furthermore, because we assume f is smooth, we may take the entries in I to be nondecreasing. For example, coordinates on $J^2(\mathbb{R}^2, \mathbb{R})$ would be x^1, x^2,$u^1, p^1_1, p^1_2, p^1_{11}, p^1_{12}$ and p^1_{22}.

Exercise 1.9.0.46: Calculate the dimensions of (a) $J^2(\mathbb{R}^3, \mathbb{R})$, (b) $J^3(\mathbb{R}^2, \mathbb{R}^2)$, (c) $J^k(\mathbb{R}^n, \mathbb{R}^n)$.

Note that $T_x^*M = J^1_{x,0}(M,\mathbb{R})$, $T_xM = J^1_{0,x}(\mathbb{R},M)$, and, in general, $J^k(M,N)$ is a bundle over M (as well as over $M \times N$). Any map $f : M \to N$ induces a section $p \mapsto j^k_p(f)$ of this bundle, called the *lift of the graph* of f.

Canonical contact systems. On $J^k(M,N)$ there is a canonical EDS with independence condition, called the *contact system*, whose integral manifolds are the lifts of graphs of maps $f : M \to N$ to $J^k(M,N)$. We now describe this system in the local coordinates used above.

Let $\Omega := dx^1 \wedge \ldots \wedge dx^n$ and let $\mathcal{I}$ be the ideal generated differentially by the 1-forms

$$
\begin{aligned}
\theta^a &:= du^a - p^a_i dx^i,\\
\theta^a_i &:= dp^a_i - p^a_{ij} dx^j,\\
&\ \vdots\\
\theta^a_{i_1,\ldots,i_{k-1}} &:= dp^a_{i_1,\ldots,i_{k-1}} - p^a_{i_1,\ldots,i_k} dx^{i_k},
\end{aligned}
\tag{1.9.1}
$$

which we will call *contact forms*. (Note the summation on i_k.) We will use multi-index notation to abbreviate the forms in (1.9.1) as

$$\theta^a_I := dp^a_I - p^a_{Ij} dx^j.$$

The system $(\mathcal{I},\Omega)$ on $J^k(M.N)$ is defined globally and is independent of the coordinates chosen. It is called the canonical *contact system* on $J^k(M,N)$. Its integral manifolds are exactly the lifts of graphs $\Gamma_f = \{(x, f(x)) \mid x \in M\} \subset M \times N$ of mappings $f : M \to N$ to $J^k(M,N)$. To see this, let $i : X \hookrightarrow J^k$ be an n-dimensional integral manifold with local coordinates $x^1,\ldots,x^n$. On X, $u = u(x^1,\ldots,x^n)$, $p^a_i = p^a_i(x^1,\ldots,x^n)$, etc., and $i^*(\theta^a) = 0$ implies that $p^a_i = \frac{\partial u^a}{\partial x^i}$ for all $1 \le i \le n$. Similarly, the vanishing of the other forms in the ideal force the other jet coordinates to be the higher derivatives of u.

How to express any PDE system as an EDS with independence condition. Given a kth-order system of PDE for maps $f : \mathbb{R}^n \to \mathbb{R}^s$,

$$F^r\left(x^i, u^a, \frac{\partial^{|I|}u^a}{\partial x^I}\right) = 0, \qquad 1 \le r \le R, \quad 1 \le |I| \le k, \tag{1.9.2}$$

we define a submanifold $\Sigma \subset J^k$ by the equations $F^r(x^i, u^a, p^a_I) = 0$. The lifts of solutions of (1.9.2) are precisely the integral manifolds of the pullback of the contact system to Σ. Note that Ω tells us what the independent variables should be.

Standard abuse of notation. Given an inclusion $i : M \hookrightarrow \Sigma$, instead of writing $i^*(\theta) = 0$ or $i^*(\Omega) \neq 0$ we will often simply say respectively "$\theta = 0$ on M"or "$\Omega \neq 0$ on M".

Exterior differential systems. We generalize our notion of exterior differential systems with independence condition as follows:

Definition 1.9.0.47. An *exterior differential system* on a manifold Σ is a differential ideal $\mathcal{I} \subset \Omega^*(\Sigma)$. An *integral manifold* of the system $\mathcal{I}$ is an immersed submanifold $f : M \to \Sigma$ such that $f^*(\alpha) = 0 \ \forall \alpha \in \mathcal{I}$.

Note that for an exterior differential system, not only do we do not specify the analog of independent variables, but we do not even specify a required dimension for integral manifolds.

We define a k-dimensional *integral element* of $\mathcal{I}$ at $x \in \Sigma$ to be an $E \in G(k, T_x\Sigma)$ such that $\alpha|_E = 0 \ \forall \alpha \in \mathcal{I}$. Let $\mathcal{V}_k(\mathcal{I})_x$ denote the space of k-dimensional integral elements to $\mathcal{I}$ at x.

Exercise 1.9.0.48: Let $\mathcal{I} = \{x^1 dx^2, dx^3\}$ be an exterior differential system on $\mathbb{R}^3$. Calculate $\mathcal{V}_1(\mathcal{I})_{(1,1,1)}$, $\mathcal{V}_1(\mathcal{I})_{(0,0,0)}$, $\mathcal{V}_2(\mathcal{I})_{(1,1,1)}$ and $\mathcal{V}_2(\mathcal{I})_{(0,0,0)}$.

Let $\Omega = adx^1 + bdx^2 + edx^3$, where a, b, c are constants, not all zero. Calculate $\mathcal{V}_1(\mathcal{I}, \Omega)_{(1,1,1)}$ and $\mathcal{V}_1(\mathcal{I}, \Omega)_{(0,0,0)}$.

Proof of the Frobenius Theorem. We now prove Theorem 1.3.0.11, which we restate as follows:

Theorem 1.9.0.49 (Frobenius Theorem, second version)**.** *Let $\mathcal{I}$ be a differential ideal generated by the linearly independent* 1*-forms $\theta^1, \dots, \theta^{m-n}$ on an m-fold Σ, i.e., $\mathcal{I} = \{\theta^1, \dots, \theta^{m-n}\}$. Suppose $\mathcal{I}$ is also generated algebraically by $\theta^1, \dots, \theta^{m-n}$, i.e., $\mathcal{I} = \{\theta^1, \dots, \theta^{m-n}\}$. Then through any $p \in \Sigma$ there exists an n-dimensional integral manifold of $\mathcal{I}$. In fact, in a sufficiently small neighborhood of p there exists a coordinate system $y^1, \dots, y^m$ such that $\mathcal{I}$ is generated by $dy^1, \dots, dy^{m-n}$.*

Proof. We follow the proof in [**?**], as that proof will get us used to calculations with differential forms.

We proceed by induction on n. If $n = 1$, then the distribution defines a line field and we are done by Theorem 1.3.0.8. Assume that the theorem is true up to $n - 1$, and we will show that it is true for n.

Let $x : M \to \mathbb{R}$ be a smooth function such that $\theta^1 \wedge \ldots \wedge \theta^{m-n} \wedge dx \neq 0$ on a neighborhood U of p, and consider the ideal $\mathcal{I}' = \{\theta^1, \dots, \theta^{m-n}, dx\}$. Since $\mathcal{I} = \{\theta^1, \dots, \theta^{m-n}\}$ is Frobenius, $\mathcal{I}'$ is also Frobenius. By our induction hypothesis, there exist local coordinates $(y^1, \dots, y^m)$ such that $\mathcal{I}' = \{dy^1, \dots, dy^{m-n+1}\}$.

At this point we have an $(n-1)$-dimensional integral manifold of $\mathcal{I}'$ (hence, also of $\mathcal{I}$) passing through p, obtained by setting $y^1, \dots, y^{m-n+1}$ equal to the appropriate constants. We want to enlarge it to an n-dimensional integral manifold.

Let $1 \leq i, j \leq n - m$. We may write

$$dx = a_i dy^i + a_{m-n+1} dy^{m-n+1},$$
$$\theta^i = c^i_j dy^j + c^i_{m-n+1} dy^{m-n+1},$$

where $a_i, a_{m-n+1}, c^i_j, c^i_{m-n+1}$ are smooth functions. Without loss of generality, we may assume $\partial x/\partial y^{m-n+1} \neq 0$, so we may rewrite the second line as

$$\theta^i = \tilde{c}^i_j dy^j + f^i dx$$

for some smooth functions $\tilde{c}^i_j, f^i$. The matrix of functions $\tilde{c}^i_j$ is invertible at each point in a (possibly smaller) neighborhood $\tilde{U}$ of p, so locally we may take a new set of generators for $\mathcal{I}$, of the form

$$\tilde{\theta}^i = dy^i + e^i dx$$

for some smooth functions e^i, and with $\theta^i = \tilde{c}^i_j \tilde{\theta}^j$. Then $d\tilde{\theta}^i = de^i \wedge dx$ and, since $\mathcal{I}$ is Frobenius,

$$de^i \wedge dx \equiv 0 \operatorname{mod}\{\tilde{\theta}^i\}.$$

Hence

$$de^i = a dx + b^i_j dy^j$$

for some functions a, b^i_j. In particular, the e^i are functions of the y^j and x only, and it follows that the $\tilde{\theta}^i$ are defined in terms of the variables $y^1, \dots, y^{m-n+1}$ only.

Let $V \subset \tilde{U}$ be the submanifold through p obtained by setting y^{m-n+2} through y^m constant. Then $\mathcal{I}|_V$ is a codimension-one Frobenius system. Hence there are coordinates $(\tilde{y}^1, \dots, \tilde{y}^{m-n+1})$ on V that are functions of the $y^1, \dots, y^{m-n+1}$, such that $\mathcal{I}|_V$ is generated by $d\tilde{y}^1, \dots, d\tilde{y}^{m-n}$. These relationships extend to $\tilde{U}$, so that

$$(\tilde{y}^1, \dots, \tilde{y}^{m-n+1}, y^{m-n+2}, \dots, y^m)$$

is the desired coordinate system. □

Symplectic manifolds, contact manifolds and their EDS's. What follows are two examples of classical exterior differential systems and a complete local description of their integral manifolds.

Symplectic manifolds. Let $\Sigma = \mathbb{R}^{2n}$ with coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ and let

$$\phi = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{1.9.3}$$

Consider the exterior differential system $\mathcal{I} = \{\phi\}$.

Exercises 1.9.0.50:

1. Show that at any point $\frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}$ is an integral element of $\mathcal{I}$.

2. Show that any graph $y^j = f^j(x^1, \ldots, x^n)$ of the form $y^j = f^j(x^j)$ for each j is an integral manifold.

We claim there are no $(n+1)$-dimensional integral elements for $\mathcal{I}$. First, it is easy to check that ϕ is *nondegenerate*, i.e., $\phi(v, w) = 0$ for all vectors $w \in T_p\Sigma$ only if $v = 0$. Now suppose $E = \{v_1, \ldots, v_{n+1}\}$ were an integral element at some point $p \in \Sigma$. Then the nondegeneracy of ϕ implies that the forms $\alpha_j = v_j \lrcorner \phi \in T_p^*\Sigma$ are linearly independent. However, this is impossible since $\alpha_i|_E = 0$ for every α_i.

Exercise 1.9.0.51: Alternatively, show that there are no $(n+1)$-dimensional integral elements to $\mathcal{I}$ by relating ϕ to the standard inner product $\langle , \rangle$ on $\mathbb{R}^{2n}$. Namely, let $\phi(v, w) = \langle v, Jw \rangle$, where J is the standard complex structure defined in Exercise **??**. Then, if E is a integral element, show that $\langle , \rangle$ must be degenerate on $E \cap J(E)$.

An even-dimensional manifold with closed nondegenerate 2-form is called a *symplectic manifold*, and the 2-form is called a *symplectic form*. The following theorem shows that the above example on $\mathbb{R}^{2n}$ is quite general.

Notation 1.9.0.52. For $\omega \in \Omega^2(M)$, we will write ω^r for the r-fold wedge product $\omega \wedge \omega \wedge \cdots \wedge \omega$ of ω with itself.

Theorem 1.9.0.53 (Darboux)**.** *Suppose a closed 2-form $\omega \in \Omega^2(M^n)$ is such that $\omega^r \neq 0$ but $\omega^{r+1} = 0$ in some neighborhood $U \subset M$. Then there exists a coordinate system $w^1, \ldots, w^n$, possibly in a smaller neighborhood, such that*

$$\omega = dw^1 \wedge dw^2 + \ldots + dw^{2r-1} \wedge dw^{2r}.$$

In particular, ω takes the form (1.9.3) *when $n = 2r$.*

Darboux's Theorem implies that all symplectic manifolds are locally equivalent, in contrast to Riemannian manifolds (see 2.6.0.88). Globally this is not at all the case, and the study of the global geometry of symplectic manifolds is an active area of research (see [**?**], for example).

Example 1.9.0.54. Given any differentiable manifold M, the cotangent bundle T^*M is canonically a symplectic manifold.

Let $\pi : T^*M \to M$ be the projection and let $\alpha \in \Omega^1(T^*M)$ be the *canonical 1-form* defined as follows: $\alpha(v)_{(x,u)} = u\left(\pi_*(v)\right)$, where $x \in M$ and $u \in T_x^*M$. If M has local coordinates x^i, then T^*M has local coordinates (x^i, y_j) such that if $u \in T_x^*M$ then $u = \sum_j y_j(x, u) dx^j$. So, in these coordinates $\alpha = \sum_j y_j dx^j$. Hence, $\omega = d\alpha$ is a symplectic form on T^*M.

Contact manifolds.

Exercise 1.9.0.55: Consider the contact system on $J^1(\mathbb{R}^n,\mathbb{R}) = \mathbb{R}^{2n+1}$ with coordinates $(z, x^1, \dots, x^n, y^1, \dots, y^n)$. Here $\theta = dz - \Sigma_i y^i dx^i$ generates the contact system $\mathcal{I} = \{\theta\}$

(a) Show that at any point $\frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}$ is an integral element.

(b) Show that any graph $z = h(x^1, \dots, x^n), y^j = f^j(x^1, \dots, x^n)$ such that $f^j = \partial h/\partial x^j$ is an integral manifold. (In fact, all n-dimensional integral manifolds are locally of this form.)

(c) Show that there are no $(n+1)$-dimensional integral elements for $\mathcal{I}$.

Again, this example is general:

Theorem 1.9.0.56 (Pfaff). *Let M be a manifold of dimension $n+1$, let $\theta \in \Omega^1(M)$ and $\mathcal{I} = \{\theta\}$. Let $r \in \mathbb{N}$ be such that $(d\theta)^r \wedge \theta \neq 0$ but $(d\theta)^{r+1} \wedge \theta = 0$ in some neighborhood $U \subset M$. Then there exists a coordinate system $w^0, \dots, w^n$, possibly in a smaller neighborhood, such that $\mathcal{I}$ is locally generated by*

$$\tilde{\theta} = dw^0 + w^{r+1}dw^1 + \dots + w^{2r}dw^r$$

(i.e., θ is a nonzero multiple of $\tilde{\theta}$ on U). In fact, there exist coordinates $y^0, \dots, y^n$ such that

$$\theta = \begin{cases} y^0dy^1 + y^2dy^3 + \dots + y^{2r}dy^{2r+1} & \text{if } (d\theta)^{r+1} \neq 0, \\ dy^1 + y^2dy^3 + \dots + y^{2r}dy^{2r-1} & \text{if } (d\theta)^{r+1} = 0, \end{cases}$$

on U.

If $n = 2r+1$, then the Pfaff Theorem implies that M is locally diffeomorphic to the jet bundle $J^1(\mathbb{R}^r,\mathbb{R})$ and $\tilde{\theta}$ is the pullback of the standard contact form. Thus, r-dimensional integral manifolds of $\mathcal{I}$ are given in the coordinates $w^0, \dots, w^n$ by

$$\begin{aligned} w^0 &= f(w^1, \dots, w^r), \\ w^{r+1} &= \frac{\partial f}{\partial w^1}, \\ &\vdots \\ w^{2r} &= \frac{\partial f}{\partial w^r}. \end{aligned}$$

A 1-form θ on a $(2n+1)$-dimensional manifold Σ is called a *contact form* if it is as nondegenerate as possible, i.e., if $\theta \wedge (d\theta)^n$ is nonvanishing.

Since we use the one form θ to define an EDS, we really only care about it up to multiplication by a nonvanishing function. A *contact manifold* is defined to be a manifold with a contact form, defined up to scale. This generalizes the contact structure on $J^1(M, \mathbb{R})$, our first example of a contact manifold.

Example 1.9.0.57. The projectivized tangent bundle $\mathbb{P}TM$ may be given the structure of a contact manifold, by taking the distribution $\alpha^\perp \subset T(TM)$ and projecting to $\mathbb{P}TM$.

Exercise 1.9.0.58 (Normal form for degenerate contact forms)**:** On $\mathbb{R}^3$, consider a 1-form θ such that $\theta \wedge d\theta = f\Omega$, where Ω is a volume form and f is a function such that $df|_p \neq 0$ whenever $f(p) = 0$. Show that there are coordinates (x, y, z), in a neighborhood of any such point, such that $\theta = dz - y^2 dx$.

Chapter 2

Euclidean Geometry and Riemannian Geometry

In this chapter we return to the study of surfaces in Euclidean space $\mathbb{E}^3 = ASO(3)/SO(3)$. Our goal is not just to understand Euclidean geometry, but to develop techniques for solving equivalence problems for submanifolds of arbitrary homogeneous spaces. We begin with the problem of determining if two surfaces in $\mathbb{E}^3$ are locally equivalent up to a Euclidean motion. More precisely, given two immersions $f, \tilde{f} : U \to \mathbb{E}^3$, where U is a domain in $\mathbb{R}^2$, when do there exist a local diffeomorphism $\phi : U \to U$ and a fixed $A \in ASO(3)$ such that $\tilde{f} \circ \phi = A \circ f$? Motivated by our results on curves in 1, we first try to find a complete set of *Euclidean differential invariants* for surfaces in $\mathbb{E}^3$, i.e., functions $I_1, \ldots, I_r$ that are defined in terms of the derivatives of the parametrization of a surface, with the property that $f(U)$ differs from $\tilde{f}(U)$ by a Euclidean motion if and only if $(\tilde{f} \circ \phi)^* I_j = f^* I_j$ for $1 \leq j \leq r$.

In §2.1 we derive the Euclidean differential invariants Gauss curvature K and mean curvature H using moving frames. Unlike with curves in $\mathbb{E}^3$, for surfaces in $\mathbb{E}^3$ there is not always a unique lift to $ASO(3)$, and we are led to define the *space of adapted frames*. (Our discussion of adapted frames for surfaces in $\mathbb{E}^3$ is later generalized to higher dimensions and codimensions in §2.5.) We calculate the functions H, K for two classical classes of surfaces in §2.2; developable surfaces and surfaces of revolution, and discuss basic properties of these surfaces.

Scalar-valued differential invariants turn out to be insufficient (or at least not convenient) for studying equivalence of surfaces and higher-dimensional submanifolds, and we are led to introduce *vector bundle valued invariants.* This study is motivated in §2.4 and carried out in §2.5, resulting in the definitions of the first and second fundamental forms, I and II. In §2.5 we also interpret II and Gauss curvature, define the Gauss map and derive the Gauss equation for surfaces.

Relations between intrinsic and extrinsic geometry of submanifolds of Euclidean space are taken up in §2.6, where we prove Gauss's theorema egregium, derive the Codazzi equation, discuss frames for C^∞ manifolds and Riemannian manifolds, and prove the fundamental lemma of Riemannian geometry. We include many exercises about connections, curvature, the Laplacian, isothermal coordinates and the like. We conclude the section with the fundamental theorem for hypersurfaces.

In §2.7 and §2.8 we discuss two topics we will need later on, space forms and curves on surfaces. In §2.9 we discuss and prove the Gauss-Bonnet and Poincaré-Hopf theorems. We conclude this chapter with a discussion of non-orthonormal frames in §2.10, which enables us to finally prove the formula (1.1.3) and show that surfaces with H identically zero are minimal surfaces.

The geometry of surfaces in $\mathbb{E}^3$ is studied further in §**??** and throughout Chapters 5–7. Riemannian geometry is discussed further in .

2.1. Gauss and mean curvature via frames

Guided by Cartan's Theorem 1.6.0.29, we begin our search for differential invariants of immersed surfaces $f : U^2 \to \mathbb{E}^3$ by trying to find a lift $F : U \to ASO(3)$ which is adapted to the geometry of $M = f(U)$. The most naïve lift would be to take

$$F(p) = \begin{pmatrix} 1 & 0 \\ f(p) & \mathrm{Id} \end{pmatrix}.$$

Any other lift $\tilde{F}$ is of the form

$$\tilde{F} = F \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$$

for some map $R : U \to SO(3)$.

Let $x = f(p)$; then $T_x\mathbb{E}^3$ has distinguished subspaces, namely $f_*(T_pU)$ and its orthogonal complement. We use our rotational freedom to adapt to this situation by requiring that e_3 always be normal to the surface, or equivalently that $\{e_1, e_2\}$ span T_xM. This is analogous to our choice of coordinates at our preferred point in 1, but is more powerful since it works on an open set of points in U.

We will call a lift such that e_3 is normal to T_xM a *first-order adapted lift*, and continue to denote such lifts by F. Our adaptation implies that

$$F^*(\omega^3) = 0, \tag{2.1.1}$$

$$F^*(\omega^1 \wedge \omega^2) \neq 0 \text{ at each point.} \tag{2.1.2}$$

The equation $dx = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3$ (see (1.8.4)) shows that (2.1.1) can be interpreted as saying that x does not move in the direction of e_3 to first order, and (2.1.2) implies that to first order x may move independently towards e_1 and e_2.

Let $\pi : \mathcal{F}^1 \to U$ denote the bundle whose fiber over $x \in U$ is the set of oriented orthonormal bases (e_1, e_2, e_3) of $T_{f(x)}\mathbb{E}^3$ such that $e_3 \perp T_{f(x)}M$. The first-order adapted lifts are exactly the sections of $\mathcal{F}^1$. By fixing a reference frame at the origin in $\mathbb{E}^3$, $ASO(3)$ may be identified as the bundle of all oriented orthonormal frames of $\mathbb{E}^3$, and $\mathcal{F}^1$ is a subbundle of $f^*(ASO(3))$. Throughout this chapter we will not distinguish between U and M when the distinction is unimportant. In particular, we will usually consider $\mathcal{F}^1$ as a bundle over M.

Consequences of our adaptation. Thanks to the Maurer-Cartan equation (1.6.2), we may calculate the derivatives of the left-invariant forms on $ASO(3)$ algebraically:

$$\begin{aligned} &d\begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix} \\ &\quad = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix}. \end{aligned}$$

Write $i : \mathcal{F}^1 \hookrightarrow ASO(3)$ for the inclusion map. By our definition of $\mathcal{F}^1$, $i^*\omega^3 = 0$, and hence

$$0 = i^*(d\omega^3) = -i^*(\omega^3_1 \wedge \omega^1 + \omega^3_2 \wedge \omega^2). \tag{2.1.3}$$

By (2.1.2), $i^*\omega^1$ and $i^*\omega^2$ are independent, and we can apply the Cartan Lemma **??** to the right hand side of (2.1.3). We obtain

$$i^*\begin{pmatrix} \omega^3_1 \\ \omega^3_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} i^*\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, \tag{2.1.4}$$

where $h_{ij} = h_{ji}$ are some functions defined on $\mathcal{F}^1$. This $h = (h_{ij})$ is analogous to the Hessian at the origin in (1.1.1), but it has the advantage of being defined on all of $\mathcal{F}^1$.

Given an adapted lift $F : U \to \mathcal{F}^1$, we have

$$F^* \begin{pmatrix} \omega^3_1 \\ \omega^3_2 \end{pmatrix} = h_F\, F^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

where $h_F = F^*(h)$. We now determine the invariance of h_F. Since F is uniquely defined up to a rotation in the tangent plane to M, all other possible adapted lifts are of the form

$$\widetilde{F} = F \begin{pmatrix} 1 & & \\ & R & \\ & & 1 \end{pmatrix} = Fr, \tag{2.1.5}$$

where $R : U \to SO(2)$ is an arbitrary smooth function.

We compare $\widetilde{F}^*(\omega) = \widetilde{F}^{-1} d\widetilde{F}$ with $F^*(\omega) = F^{-1}dF$:

$$\begin{aligned} \widetilde{F}^{-1} d\widetilde{F} &= (Fr)^{-1} d(Fr) = r^{-1}(F^{-1}dF)r + r^{-1}F^{-1}F dr \\ &= \begin{pmatrix} 1 & & \\ & R^{-1} & \\ & & 1 \end{pmatrix} F^* \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & R & \\ & & 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & & \\ & R^{-1}dR & \\ & & 0 \end{pmatrix}. \end{aligned}$$

In particular,

$$\widetilde{F}^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = R^{-1} F^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, \qquad \widetilde{F}^*(\omega^3_1, \omega^3_2) = F^*(\omega^3_1, \omega^3_2)R.$$

Since $R^{-1} = {}^tR$, we conclude that

$$h_{\tilde{F}} = R^{-1} h_F R. \tag{2.1.6}$$

Thus, the properties of h_F that are invariant under conjugation by a rotation matrix are invariants of the mapping f. The functions $\frac{1}{2}\,\mathrm{trace}(h_F)$ and $\det(h_F)$ generate the ideal of functions on h_F that are invariant under (2.1.6). They are respectively called the *mean curvature* (first defined by Sophie Germain), denoted by H, and the *Gauss curvature* (first defined by a mathematician with better p.r.), denoted by K. We see immediately that for two surfaces to be congruent it is necessary that they must have the same Gauss and mean curvature functions at corresponding points, thus recovering our observations of §1.1.

Another perspective. Instead of working with lifts to $\mathcal{F}^1$, one could work with $h : \mathcal{F}^1 \to S^2\mathbb{R}^2$ directly, calculating how h varies as one moves in the fiber.

Let k_1, k_2 denote the eigenvalues of h; for the sake of definiteness, say $k_1 \geq k_2$. These are called the *principal curvatures* of $M \subset \mathbb{E}^3$, and are also differential invariants. However, H, K are more natural invariants, because, e.g., the Gauss curvature plays a special role in the intrinsic geometry of the surface; see for example Theorem 2.6.0.78 below.

Note also that if M is smooth, then H, K are smooth functions on M while k_1 and k_2 may fail to be differentiable at points where $k_1 = k_2$, which are called *umbilic points*.

Exercises 2.1.0.59:

1. Let $\tilde{F}$ be as in (2.1.5) and let $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Calculate $\tilde{F}^*\omega_1^2$ in terms of θ and $F^*\omega_1^2$.
2. Show that H, K are invariants of the image of f. ⊚
3. Express k_1, k_2 in terms of H and K. ⊚

Example 2.1.0.60 (Surfaces with $H = K = 0$)**.** If $H = K = 0$, then the matrix h is zero and ω_1^3, ω_2^3 vanish. So, on $\Sigma = ASO(3)$ define $\mathcal{I} = \{\omega^3, \omega_1^3, \omega_2^3\}$ with independence condition $\Omega = \omega^1 \wedge \omega^2$.

Suppose $f : U \to \mathbb{E}^3$ gives a surface M with H, K identically zero. Such a surface is (as you may already have guessed) a subset of a plane. For, if $F : U \to ASO(3)$ is a first-order adapted frame for a surface with $H = K = 0$, then $F(U)$ will be an integral surface of $(\mathcal{I}, \Omega)$. Note that $de_3 = -\omega_1^3 e_1 - \omega_2^3 e_2 = 0$, so e_3 is constant for such lifts. Therefore, for all $x \in M$ there is a fixed vector e_3 such that $e_3 \perp T_x M$, and thus M is contained in a plane perpendicular to e_3.

2.2. Calculation of H and K for some examples

The Helicoid. Let $\mathbb{R}^2$ have coordinates (s, t), fix a constant $a > 0$ and consider the mapping $f : \mathbb{R}^2 \to \mathbb{E}^3$ defined by

$$f(s,t) = (s\cos t, s\sin t, at).$$

The image surface is called the *helicoid*.

Exercise 2.2.0.61: Draw the helicoid. Note that the z-axis is contained in the surface, as is a horizontal line emanating out from each point on the z-axis, and this line rotates as we move up the z-axis.

We compute a first-order adapted frame for the helicoid. Note that

$$\begin{aligned} f_s &= (\cos(t), \sin(t), 0), \\ f_t &= (-s\sin(t), s\cos(t), a). \end{aligned}$$

so $\langle f_s, f_t\rangle = 0$ and we may take

$$
\begin{aligned}
e_1 &= \frac{f_s}{|f_s|} = (\cos(t), \sin(t), 0),\\
e_2 &= \frac{f_t}{|f_t|} = \frac{1}{\sqrt{s^2+a^2}}(-s\sin(t), s\cos(t), a),\\
e_3 &= e_1 \times e_2 = \frac{1}{\sqrt{s^2+a^2}}(a\sin(t), -a\cos(t), s).
\end{aligned}
\tag{2.2.1}
$$

Since $df = f_s ds + f_t dt = f^*(\omega^1)e_1 + f^*(\omega^2)e_2$, we obtain (omitting the f^* from the notation here and in what follows)

$$
\begin{aligned}
\omega^1 &= ds,\\
\omega^2 &= (s^2+a^2)^{\frac{1}{2}}dt.
\end{aligned}
\tag{2.2.2}
$$

Next, we calculate

$$
\begin{aligned}
de_3 = &\Big(-s(s^2+a^2)^{-3/2}(a\sin(t), -a\cos(t), s) + ((s^2+a^2)^{-1/2}(0,0,1)\Big)\, ds\\
&+ (s^2+a^2)^{-1/2}(a\cos(t), a\sin(t), 0)dt.
\end{aligned}
$$

So, using (2.2.1), (2.2.2), we obtain

$$
\begin{aligned}
\omega^3_1 &= -a(s^2+a^2)^{-1}\omega^2,\\
\omega^3_2 &= -a(s^2+a^2)^{-1}\omega^1,
\end{aligned}
$$

and conclude that $H(s,t) \equiv 0$ and $K(s,t) = -\frac{a^2}{(s^2+a^2)^2}$.

Surfaces with H identically zero are called *minimal surfaces* and are discussed in more detail in §2.10 and §**??**.

Developable surfaces. A surface $M^2 \subset \mathbb{E}^3$ is said to be *developable* if it is describable as (a subset of) the union of tangent rays to a curve. (Developable surfaces are also called *tangential surfaces.*)

Let $c : \mathbb{R} \to \mathbb{E}^3$ be a regular parametrized curve, and consider the surface $f : \mathbb{R}^2 \to \mathbb{E}^3$ defined by $(u, v) \mapsto c(u) + vc'(u)$. Since

$$df = (c'(u) + vc''(u))du + c'(u)dv,$$

we see that f is regular, i.e., df is of maximal rank, when $c''(u)$ is linearly independent from $c'(u)$ and $v \neq 0$. (We will assume $v > 0$.) Note that the tangent space, spanned by $c'(u), c''(u)$, is independent of v.

Assume that c is parametrized by arclength. Then $\langle c'(u), c''(u)\rangle = 0$, and we can take

$$
\begin{aligned}
e_1(u,v) &= c'(u),\\
e_2(u,v) &= c''(u)/||c''(u)||.
\end{aligned}
$$

Using $df = \omega^1 e_1 + \omega^2 e_2$, we calculate

$$\omega^1 = du + dv,$$
$$\omega^2 = v\kappa(u)du,$$

where $\kappa(u)$ is the curvature of c.

Note that our frame is the same as if we were to take an adapted framing of c (as in §1.8), so we have

$$de_3 = (-\tau(u)e_2)du.$$

Thus,

$$\omega_1^3 = 0, \qquad \omega_2^3 = \frac{\tau(u)}{\kappa(u)v}\omega^2,$$

showing that $H(u,v) = \frac{\tau(u)}{2\kappa(u)v}$ and $K \equiv 0$.

Surfaces with K identically zero are called *flat*, and we study their geometry more in §2.4.

Developable surfaces are also examples of *ruled surfaces* (as is the helicoid). A surface is *ruled* if through any point of the surfaces there passes a straight line (or line segment) contained in the surface.

Surfaces of revolution. Let $U \subset \mathbb{R}^2$ be an open set with coordinates u, v and let $f : U \to \mathbb{E}^3$ be a map of the form

$$x(u,v) = r(v)\cos(u),$$
$$y(u,v) = r(v)\sin(u),$$
$$z(u,v) = t(v),$$

where r, t are smooth functions. The resulting surface is called a *surface of revolution* because it is constructed by rotating a generating curve (e.g., in the xz-plane) about the z-axis. Call the image M.

Assuming that the generating curve is regular, we can choose v to be an arclength parameter, so that $(r'(v))^2 + (t'(v))^2 = 1$. Let

$$\begin{aligned} e_1 &= (-\sin u, \cos u, 0), \\ e_2 &= (r'(v)\cos u, r'(v)\sin u, t'(v)). \end{aligned} \tag{2.2.3}$$

Note that $e_j \in \Gamma(U, f^*(T\mathbb{E}^3))$.

Exercises 2.2.0.62:

1. (a) Show that e_1, e_2 in (2.2.3) is an orthonormal basis of $T_{f(u,v)}M$.
   (b) Calculate e_3 such that e_1, e_2, e_3 is an orthonormal basis of $T_{f(u,v)}\mathbb{E}^3$.
2. Considering this frame as a lift $F : U \to \mathcal{F}^1$, calculate the pullback of the Maurer-Cartan forms in terms of du, dv.
3. Calculate the Gauss and mean curvature functions of M.

4. Consider the surfaces of revolution generated by the following data. In each case, describe the surface geometrically. (Take time out to draw some pictures and have fun!) Calculate H, K and describe their asymptotic behavior.
   (a) $r(v) = \text{constant}$, $t(v) = v$.
   (b) $r(v) = av$, $t(v) = bv$, where $a^2 + b^2 = 1$.
   (c) $r(v) = \cos v$, $t = \sin v$.
   (d) The generating curve in the xz plane is a parabola, e.g. $x - bz^2 = c$.
   (e) The generating curve is a hyperbola, e.g. $x^2 - bz^2 = c$.
   (f) The generating curve is an ellipse, e.g. $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$.
5. Find all surfaces of revolution with $K \equiv 0$. Give a geometric construction of these surfaces. ◎
6. Find all surfaces of revolution with $K \equiv 1$ that intersect the $x - y$ plane perpendicularly. (Your answer should involve an integral and the choice of one arbitrary constant.) Which of these are complete? ◎

2.3. Darboux frames and applications

Recall that $k_1 \geq k_2$ are the eigenvalues of the second fundamental form matrix h. Away from umbilic points (points where $k_1 = k_2$), k_1 and k_2 are smooth functions, and we may further adapt frames by putting h in the form

$$h = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

because a real symmetric matrix is always diagonalizable by a rotation matrix. In this case F is uniquely determined. We will call such a framing a *Darboux* or *principal framing*.

Notation. In general, we will express the derivative of a function u on a framed surface as $du = u_1\omega^1 + u_2\omega^2$, where $u_j = e_j(u)$. In particular, write $dk_j = k_{j,1}\omega^1 + k_{j,2}\omega^2$ to represent the derivatives of the k_j in Exercises 2.3.0.63 below.

Exercises 2.3.0.63:

1. Let $c \subset \mathbb{E}^2$ be the curve defined by intersecting M with the plane through x parallel to e_1, e_3. Show that the curvature of c at x is k_1.
2. Calculate $F^*(\omega_1^2)$ in a Darboux framing as a function of the principal curvatures and their derivatives.

3. Suppose that X_1, X_2 are vector fields on V such that $f_* X_i = e_i$. Show that
$$[X_1, X_2] = -\frac{k_{1,2}}{k_1 - k_2} X_1 + \frac{k_{2,1}}{k_1 - k_2} X_2. \circledcirc$$
4. Find all surfaces in $\mathbb{E}^3$ with $k_1 \equiv k_2$, i.e., surfaces where each point is an umbilic point.
5. Derive the *Codazzi equation* for Darboux frames, i.e., show that k_1, k_2 satisfy the differential equation

(2.3.1)
$$-k_1 k_2 = \frac{1}{(k_1 - k_2)^2}\left((k_1 - k_2)(k_{2,11} - k_{1,22}) + k_{1,2}k_{2,2} + k_{2,1}k_{1,1}\right). \circledcirc$$

This to some extent addresses the existence question for principal curvature functions. Namely, two functions $k_1(u, v)$, $k_2(u, v)$ that are never equal cannot be the principal curvature functions of some embedding of $U \to \mathbb{E}^3$ unless they satisfy the Codazzi equation. In particular, surfaces with both H and K constant must be either flat or totally umbilic.
6. Using the Codazzi equation, show that if $k_1 > k_2$ everywhere, and if there exists a point p at which k_1 has a local maximum and k_2 a local minimum, then $K(p) \leq 0$.

Even among surfaces of revolution, there are an infinite number of non-congruent surfaces with $K \equiv 1$. We will see in Example **??** and again in §**??** that surfaces with constant $K > 0$ are even more flexible in general. Thus, the following theorem might come as a surprise:

Theorem 2.3.0.64. *If $M^2 \subset \mathbb{E}^3$ is compact, without boundary, and has constant Gauss curvature $K > 0$, then M is the round sphere.*

Exercise 2.3.0.65: Prove the theorem. $\circledcirc$

2.4. What do H and K tell us?

Since Darboux frames provide a unique lift for M and well-defined differential invariants, it is natural to pose the question:

> **Question:** Are surfaces $M^2 \subset \mathbb{E}^3$ with no umbilic points locally determined, up to a Euclidean motion, by the functions H and K?

The answer is **NO!** Consider the following example:

The Catenoid. Let $\mathbb{R}^2$ have coordinates (u, v), let $a > 0$ be a constant, and consider the following mapping $f : \mathbb{R}^2 \to \mathbb{R}^3$:
$$f(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$$
The image is called the *catenoid*.

Exercise 2.4.0.66: Draw the catenoid. Calculate the mean and Gauss curvature functions by choosing an adapted orthonormal frame and differentiating as we did in §2.2. ⊚

Now consider the map $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $s = a \sinh v$, $t = u$.

Exercise 2.4.0.67: Show that $g^*(K_{cat}) = K_{helicoid}$, and of course the mean curvature functions match up as well.

Although we have the same Gauss and mean curvature functions, the helicoid is ruled, and it is not hard to check that the catenoid contains no line segments. Since a Euclidean transformation takes lines to lines, we see it is impossible for the helicoid to be equivalent to the catenoid via a Euclidean motion.

We will see in Example **??** that, given a non-umbilic surface with constant mean curvature, there are a circle's worth of non-congruent surfaces with the same Gauss curvature function and mean curvature as the given surface. On the other hand, the functions H and K are usually sufficient to determine M up to congruence. Those surfaces for which this is not the case either have constant mean curvature, or belong to a finite-dimensional family called *Bonnet surfaces*, after Ossian Bonnet, who first investigated them; see §**??** for more discussion.

We will also see, in §2.6, that using slightly more information, namely vector bundle valued differential invariants, one can always determine local equivalence of surfaces from second-order information.

Flat surfaces. Recall that a surface is *flat* if $K \equiv 0$. The name is justified by Theorem 2.6.0.78, which implies that the intrinsic geometry of such surfaces is the same as that of a plane, and also by the following exercise:

Exercise 2.4.0.68: Show that if M is flat, there exist local coordinates x^1, x^2 on M and an orthonormal frame (e_1, e_2, e_3) such that $\omega^1 = dx^1, \omega^2 = dx^2$. ⊚

Here are some examples of flat surfaces:

Cylinders. Let $C \subset \mathbb{E}^2 \subset \mathbb{E}^3$ be a plane curve parametrized by $c(u)$, and assume X is a unit normal to $\mathbb{E}^2$. Let $f(u,v) = c(u) + vX$.

Exercise 2.4.0.69: Find a Darboux framing for the cylinder and calculate its Gauss and mean curvature functions.

Cones. Let $C \subset \mathbb{E}^3$ be a curve parametrized by $c(u)$, and let $p \in \mathbb{E}^3 \backslash C$. Let $f(u,v) = c(u) + v(p - c(u))$. The resulting surface is called the *cone* over c with vertex p.

Exercise 2.4.0.70: Show that cones are indeed flat. ⊚

Remark 2.4.0.71. It turns out the property of being flat is invariant under a larger group than $ASO(3)$, and flat surfaces are best studied by exploiting this larger group. This topic will be taken up in , where we classify all flat surfaces: in the projective complex analytic category they are either cones, cylinders, or tangential surfaces to a curve. Even in the C^∞ category, the only complete flat surfaces are cylinders; see [**?**], where there are also extensive comments about the local characterization of flat C^∞-surfaces.

2.5. Invariants for n-dimensional submanifolds of $\mathbb{E}^{n+s}$

We already saw that for surfaces, the functions H, K alone were not sufficient to determine equivalence. We now begin the study of vector bundle valued functions as differential invariants for submanifolds of (oriented) Euclidean space $\mathbb{E}^{n+s} = ASO(n+s)/SO(n+s)$.

Let $(x, e_1, \dots, e_{n+s})$ denote an element of $ASO(n+s)$. Define the projection

$$\begin{aligned} \pi : ASO(n+s) &\to \mathbb{E}^{n+s}, \\ (x, e_1, \dots, e_{n+s}) &\mapsto x. \end{aligned}$$

Given an n-dimensional submanifold $M \subset \mathbb{E}^{n+s}$, let $\pi : \mathcal{F}^1 \to M$ denote the subbundle of $ASO(n+s)|_M$ of oriented *first-order adapted frames* for M, whose fiber over a point $x \in M$ is the set of oriented orthonormal bases such that $e_1, \dots, e_n$ are tangent to M (equivalently, $e_{n+1}, \dots, e_{n+s}$ are normal to M).

Using index ranges $1 \leq i, j, k \leq n$ and $n+1 \leq a, b \leq s$, we write the Maurer-Cartan form on $ASO(n+s)$ as

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ \omega^i & \omega^i_j & \omega^i_b \\ \omega^a & \omega^a_j & \omega^a_b \end{pmatrix}.$$

On $\mathcal{F}^1$, continuing our standard abuse in omitting pullbacks from the notation, $\omega^a = 0$ and thus $d\omega^a = -\omega^a_j \wedge \omega^j = 0$, which implies

$$\omega^a_j = h^a_{ij}\omega^j,$$

for some functions $h^a_{ij} = h^a_{ji} : \mathcal{F}^1 \to \mathbb{R}$.

We seek quantities that are invariant under motions in the fiber $(\mathcal{F}^1)_x$. The motions in the fiber of $\mathcal{F}^1$ are given by left-multiplication by

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g^i_j & 0 \\ 0 & 0 & u^a_b \end{pmatrix}, \tag{2.5.1}$$

where $(g^i_k) \in SO(n)$ and $(u^a_b) \in SO(s)$. If $\tilde{f} = Rf$, then

$$\tilde{h}_i^a j = (u^{-1})_b^a g_i^k g_j^l h_{kl}^b. \tag{2.5.2}$$

In this situation, if we were to look for scalar functions that are constant on the fibers, we would get a mess, but there are simple vector bundle valued functions that are constant on the fibers. Recall our general formula (1.6.1) for how the Maurer-Cartan form changes under a change of lift. Under a motion (2.5.1) we have

$$\omega^i \mapsto (g^{-1})_j^i \omega^j, \qquad e_a \mapsto u_a^b e_b, \qquad \omega_i^a \mapsto g_j^i (u^{-1})_b^a \omega_j^b. \tag{2.5.3}$$

Let NM denote the *normal bundle* of M, the bundle with fiber $N_xM = (T_xM)^\perp \subset T_x\mathbb{E}^{n+s}$. Define

$$\begin{aligned} \widetilde{II} :&= \omega_j^a \omega^j \otimes e_a \\ &= h_{ij}^a \omega^i \omega^j \otimes e_a \in \Gamma(\mathcal{F}^1, \pi^*(S^2T^*M \otimes NM)), \end{aligned}$$

where we write $\omega^i\omega^j$ for the symmetric product $\omega^i \circ \omega^j$. Then (2.5.3) shows that $\widetilde{II}$ is constant on fibers and thus is basic, i.e., if $s_1, s_2 : M \to \mathcal{F}^1$ are any two sections, then $s_1^*(\widetilde{II}) = s_2^*(\widetilde{II})$.

Proposition/Definition 2.5.0.72. *$\widetilde{II}$ descends to a well-defined differential invariant*

$$II \in \Gamma(M, S^2T^*M \otimes NM)$$

called the (Euclidean) second fundamental form of M.

When studying surfaces, we failed to mention the vector bundle valued first-order invariants of submanifolds described in the following exercises.

Exercises 2.5.0.73:

1. Consider

   $$\tilde{I} := \sum_i \omega^i \omega^i \in \Gamma(\mathcal{F}^1, \pi^*(S^2T^*M)).$$

   Verify that $\tilde{I}$ descends to a well-defined differential invariant

   $$I \in \Gamma(M, S^2T^*M),$$

   which is called the *first fundamental form* or *Riemannian metric* of M.
2. Show that $\mathrm{dvol} := \omega^1 \wedge \ldots \wedge \omega^n$ is invariant under motions in the fiber and descends to a well-defined invariant, called the *volume form* of M. Show that it is indeed the volume form induced by the Riemannian metric I. (Recall that an inner product on a vector space V induces an inner product on $\Lambda^n V$, and thus a volume form up to a sign.)

Interpretations of II and K. We now give a more geometrical definition of the second fundamental form for surfaces in $\mathbb{E}^3$. This definition will be extended to all dimensions and codimensions in .

The Gauss map. Let $M^2 \subset \mathbb{E}^3$ be oriented and let S^2 denote the unit sphere. Since e_3 is invariant under changes of first-order adapted frame, we obtain a well-defined mapping

$$\begin{aligned} M^2 &\to S^2, \\ x &\mapsto e_3(x), \end{aligned}$$

called the *Gauss map*.

Proposition 2.5.0.74. $II(v,w) = -\langle de_3(v), w\rangle$.

Thus, II admits the interpretation as the derivative of the Gauss map.

Exercises 2.5.0.75:

1. Prove Proposition 2.5.0.74.
2. Let $M^n \subset \mathbb{E}^{n+1}$ be an oriented hypersurface. Define the Gauss map of M and the analogous notions of principal curvatures, mean curvature and Gauss curvature.
3. Show that the generic fibers of the Gauss map are (open subsets of) linear spaces, and thus flat surfaces are ruled by lines. ⊚

A geometric interpretation of Gauss curvature. The round sphere S^2 may be considered as the homogeneous space $ASO(3)/ASO(2)$ via the projection $(x, e_1, e_2, e_3) \mapsto e_3$. As such, the form $\omega^3_1 \wedge \omega^3_2$ is the pullback of the area from on S^2 because $de_3 = -(\omega^3_1 e_1 + \omega^3_2 e_2)$. Since $\omega^3_1 \wedge \omega^3_2 = K\omega^1 \wedge \omega^2$, we may interpret K as a measure of how much the area of M is (infinitesimally) distorted under the Gauss map. (This is because, for a linear map $A : V \to V$, the determinant of A gives, up to sign, the factor by which volume is distorted under A. More precisely, if P is a parallelepiped with one vertex at the origin, $\mathrm{vol}(A(P)) = |\det A|\,\mathrm{vol}(P)$.)

The Gauss equation. There is another way to calculate the Gauss curvature of a surface, namely by differentiating ω^2_1. Using the Maurer-Cartan equation, we obtain

$$d\omega^2_1 = -K\omega^1 \wedge \omega^2, \tag{2.5.4}$$

which is called the *Gauss equation*.

Exercises 2.5.0.76:

1. Let $c(t) \subset M^2 \subset \mathbb{E}^3$ be a curve on a surface such that $|c'(t)| = 1$ and $c''(t) \perp T_{c(t)}M$ for all t. Show that $\langle II_{M,c(t)}(c'(t), c'(t)), e_3\rangle = \kappa_{c(t)}$, where κ denotes the curvature of the curve.
2. Let $M^2 \subset \mathbb{E}^3$ be a surface. Let $x \in M$ be a point such that there exists $\epsilon > 0$ such that $M \cap T_xM \cap B_\epsilon(x) = x$, where $B_\epsilon(x)$ denotes

a ball in $\mathbb{E}^3$ of radius ϵ around x. Show that $K_x \geq 0$. What can one say if $K_x < 0$?

2.6. Intrinsic and extrinsic geometry

Definition 2.6.0.77. A *Riemannian manifold* is a differentiable manifold M endowed with a smooth section $g \in \Gamma(M, S^2T^*M)$, called a *Riemannian metric*, that is positive definite at every point.

If $M^n \subset \mathbb{E}^{n+s}$, then the first fundamental form I is a Riemannian metric on M. Which of our differential invariants for M depend only on the induced Riemannian metric I? Such invariants are often called *intrinsic*, depending only on the Riemannian structure of M, as opposed to *extrinsic* invariants, which depend on how M sits in Euclidean space.

Intrinsic geometry for surfaces. By definition, I is intrinsic, while II is necessarily extrinsic, since it takes values in a bundle that is defined only by virtue of the embedding of M. However, one can obtain intrinsic invariants from II. Given a surface $M^2 \subset \mathbb{E}^3$, we have the "great theorem" of Gauss:

Theorem 2.6.0.78 (Gauss' theorema egregium)**.** *The Gauss curvature of a surface $M^2 \subset \mathbb{E}^3$ depends only on the induced Riemannian metric.*

Proof. Let $f : U \to M$ be a local parametrization, with $U \subset \mathbb{R}^2$, and let $F : U \to \mathcal{F}^1$ be a first-order adapted lift. Let X_1, X_2 be vector fields on U such that $f_*X_i = e_i$. Then X_1, X_2 are orthonormal for the metric $g = f^*I$ on U. Let η^1, η^2 be the dual 1-forms on U (i.e., $\eta^j(X_i) = \delta^j_i$). Since $\eta^1 \wedge \eta^2 \neq 0$, there exist functions a, b such that

$$d\eta^1 = a\eta^1 \wedge \eta^2,$$
$$d\eta^2 = b\eta^1 \wedge \eta^2.$$

The proof is completed by the following exercises:

Exercises 2.6.0.79:

1. Show that there exists a 1-form α such that

$$d\eta^1 = -\alpha \wedge \eta^2,$$
$$d\eta^2 = \alpha \wedge \eta^1.$$

2. If $\tilde{X}_1, \tilde{X}_2$ is another g-orthonormal framing on U, show that $d\tilde{\alpha} = d\alpha$. Show that the function κ defined by $d\alpha = \kappa\eta^1 \wedge \eta^2$ is also unchanged, and thus depends only on g.
3. Show that $(\eta^1, \eta^2, \alpha) = F^*(\omega^1, \omega^2, \omega^1_2)$, and thus $\kappa = f^*K$ by (2.5.4). □

The Codazzi equation. Given two functions H, and K on an open subset $U \subset \mathbb{R}^2$, does there exist (locally) a map $f : U \to \mathbb{E}^3$ such that H and K are the mean and Gauss curvature functions of $M = f(U)$? There are inequalities on admissible pairs of functions because H, K are supposed to be symmetric functions of the principal curvatures (so, e.g., $H = 0$ implies $K \leq 0$). However, as we have already seen in (2.3.1), stronger restrictions exist and are uncovered when one differentiates and checks that mixed partials commute.

To see these restrictions, set up an EDS on $ASO(3) \times \mathbb{R}^3$, where $\mathbb{R}^3$ has coordinates $h_{ij} = h_{ji}$, for lifts of surfaces equipped with second fundamental forms, namely

$$\mathcal{I} = \{\omega^3, \omega^3_1 - h_{11}\omega^1 - h_{12}\omega^3, \omega^3_2 - h_{21}\omega^1 - h_{22}\omega^3\}$$

with independence condition $\Omega = \omega^1 \wedge \omega^2$. We calculate $d\omega^3 \equiv 0 \mod \mathcal{I}$, but

$$\begin{aligned}
0 &= d\left\{\begin{pmatrix}\omega^3_1\\ \omega^3_2\end{pmatrix} - h\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}\right\}\\
&= -\begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^3_1\\ \omega^3_2\end{pmatrix} - dh\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} + h\begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}\\
&\equiv -\begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\wedge h\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} - dh\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} + h\begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \mod \mathcal{I}\\
&\equiv -\left\{dh - \left[h, \begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\right]\right\}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = 0,
\end{aligned}$$

(where $[,]$ denotes the commutator of matrices). Thus h must satisfy the matrix differential equation (2.6), which is called the *Codazzi equation.*

Given a Riemannian metric g on M and an orthonormal framing, we saw in the proof of Theorem 2.6.0.78 that $\omega^1, \omega^2, \omega^2_1$ are uniquely determined. In this situation, we may interpret (2.6) as a system of equations for the possible second fundamental forms $II = h_{ij}\omega^i\omega^j$ for embeddings of M into $\mathbb{E}^3$ that induce the metric g. These restrictions are well-defined, since (2.6) is invariant under changes of orthonormal framing.

Intrinsic geometry in higher dimensions.

Frames for any manifold. Given an n-dimensional differentiable manifold M^n, consider the bundle of all *coframings* of M. More precisely, write V for $\mathbb{R}^n$ and let $\pi : \mathcal{F}(M) \to M$ denote the bundle whose fiber over $x \in M$ is the set of all linear maps $f_x : T_xM \to V$. Once we fix a basis of V, we may write a local section of $\mathcal{F}(M)$ as $s(x) = (x, f_x) = (x, \eta^1_x, \dots, \eta^n_x)$ with $\eta^i \in \Omega^1(M)$ and the η^i_x a basis of T^*_xM. The η^i determine a dual framing $(e_1, \dots, e_n)$ of TM, so equivalently we may consider $\mathcal{F}(M)$ as the bundle of all *framings* of M.

Although $\mathcal{F}(M)$ is not a Lie group, $GL(V)$ acts simply transitively on the fibers by $g.f_x = g^{-1} \circ f_x$. We now try to obtain an analogue of the Maurer-Cartan form for $\mathcal{F}(M)$:

Define the *tautological* V-valued 1-form η on $\mathcal{F}(M)$ by, for $f = (x, f_x) \in \mathcal{F}(M)$,

$$\eta_f(w) := f_x(\pi_* w), \qquad w \in T_f\mathcal{F}(M).$$

With a basis for V fixed as above, the forms $\pi_f^*(\eta^i)$ (which, by our usual abuse of notation, we write as η^i) furnish a basis of the semi-basic forms on $\mathcal{F}(M)$.

The η^i generalize the semi-basic forms ω^i on the frame bundle of Euclidean space. We would also like to find analogues of the forms ω^i_j, i.e., additional forms α^i_j that fill out a coframing of $\mathcal{F}(M)$ and satisfy

$$d\eta^i = -\alpha^i_j \wedge \eta^j. \tag{2.6.1}$$

Such forms always exist; however, without some additional restrictions, the α^i_j will not be uniquely defined.

Exercise 2.6.0.80: Suppose $\overline{\eta}^i$ is a local coframing defined on $U \subset M$. Then we may define a local trivialization

$$t : GL(V) \times U \simeq \mathcal{F}(M)|_U$$

by $(g, x) \mapsto g^{-1}\overline{\eta}_x$. Show the existence of a desired coframing on $\mathcal{F}(M)|_U$ as follows:

(a) Show that $t^*\eta = g^{-1}\overline{\eta}$.

(b) Show that there exist $\alpha^i_j \in \Omega^1(\mathcal{F}(M)|_U)$ satisfying (2.6.1) such that $t^*\alpha^i_j \equiv (g^{-1}dg)^i_j$ modulo $\{\overline{\eta}^i\}$. ⊚

(c) Show that (η^i, α^i_j) is a coframing of $\mathcal{F}(M)|_U$.

(d) Show that any other coframing satisfying (2.6.1) must be of the form $\tilde{\alpha}^i_j = \alpha^i_j + C^i_{jk}\eta^k$ for some functions $C^i_{jk} = C^i_{kj}$. ⊚

Frames for Riemannian manifolds. Over a Riemannian manifold M, consider the bundle of *orthonormal coframes* which we denote $\pi : (M) \to M$. Namely, endow V with the standard inner product $\langle , \rangle$ and define the fiber of $(M) \subset \mathcal{F}(M)$ to be the linear maps $(T_xM, g_x) \to (V, \langle , \rangle)$ that are isometries. (We will sometimes denote the general frame bundle by (M) to distinguish it from (M).) The orthogonal group $O(V)$ acts simply transitively on the fibers of (M).

On (M) we still have the tautological forms η^i (which are pullbacks of those on $\mathcal{F}(M)$ under the obvious inclusion). If $s : M \to (M)$ is a smooth (local) section, then $\overline{\eta}^j = s^*(\eta^j)$ provides a local coframing such that

$$g = (\overline{\eta}^1)^2 + \cdots + (\overline{\eta}^n)^2.$$

Thanks to the additional structure of the Riemannian metric, we will uniquely define an $\mathfrak{so}(V)$-valued 1-form to obtain a canonical framing of (M) as follows:

Lemma 2.6.0.81 (The fundamental lemma of Riemannian geometry). *Let (M^n, g) be a Riemannian manifold and let η^i denote the tautological forms on (M). Let $s : M \to (M)$ be a smooth section. Then there exist unique forms $\eta^i_j \in \Omega^1(M)$ such that*

i. $s^*(d\eta^i) = -\eta^i_j \wedge s^*(\eta^j)$

and

ii. $\eta^i_j + \eta^j_i = 0.$

Proof. Write $\overline{\eta}^j = s^*(\eta^j)$. Since the $\overline{\eta}^i$ furnish a basis of T^*M at each point, we may write

$$d\overline{\eta}^i = -\alpha^i_j \wedge \overline{\eta}^j \tag{2.6.2}$$

for some $\alpha^i_j \in \Omega^1(M)$. Let $\alpha^i_j = \beta^i_j + \gamma^i_j$, where $\beta^i_j = -\beta^j_i$ and $\gamma^i_j = \gamma^j_i$.

We first prove existence by showing it is possible to choose forms $\tilde{\alpha}^i_j$ such that $\tilde{\gamma}^i_j = \frac{1}{2}(\tilde{\alpha}^i_j + \tilde{\alpha}^j_i) = 0$.

Exercise 2.6.0.82: Write $\gamma^i_j = T^i_{jk}\eta^k$ for some functions $T^i_{jk} = T^j_{ik} : M \to \mathbb{R}$, and let

$$\tilde{\alpha}^i_j = \alpha^i_j - (T^i_{jk} + T^i_{kj} - T^j_{ki})\eta^k.$$

Verify that $d\overline{\eta}^i = -\tilde{\alpha}^i_j \wedge \overline{\eta}^j$, and show that $\tilde{\gamma}^i_j = 0$.

To prove uniqueness, assume that α^i_j, $\tilde{\alpha}^i_j$ both satisfy i. and ii. Since $(\tilde{\alpha}^i_j - \alpha^i_j) \wedge \overline{\eta}^j = 0$, by the Cartan Lemma we have $\tilde{\alpha}^i_j - \alpha^i_j = C^i_{jk}\overline{\eta}^k$ for some functions $C^i_{jk} = C^i_{kj}$. Moreover, we also have $C^i_{jk} = -C^j_{ik}$.

Exercise 2.6.0.83: Show that $C^i_{jk} = 0$.

□

In case M is a submanifold of $\mathbb{E}^{n+s}$, our uniqueness argument implies that $\eta^i_j = F^*(\omega^i_j)$, where F is any extension of s to a first-order adapted framing $F : M \to \mathcal{F}_{\mathbb{E}^{n+s}}$.

In we will prove the following "upstairs" version of the fundamental lemma:

Lemma 2.6.0.84 (The fundamental lemma of Riemannian geometry). *Let M be an n-dimensional Riemannian manifold and let η^i denote the tautological forms on (M). Then there exist unique forms $\eta^i_j \in \Omega^1((M))$ such that $d\eta^i = -\eta^i_j \wedge \eta^j$ and $\eta^i_j + \eta^j_i = 0$.*

The η^i_j (in either the upstairs or downstairs version of the fundamental lemma) are referred to as *connection forms*. The upstairs version provides a coframing of (M), which we now differentiate to obtain differential invariants. (If you are unhappy that we haven't yet proven the upstairs version, you may use the downstairs version and note that all quantities we are about to define are independent of the choice of section s.) While $d\eta^i$ is given by Lemma 2.6.0.84, we calculate $d\eta^i_j$ by using

$$0 = d^2\eta^i = -(d\eta^i_j + \eta^i_k \wedge \eta^k_j) \wedge \eta^j.$$

Let $\tilde{\Theta}^i_j := d\eta^i_j + \eta^i_k \wedge \eta^k_j \in \Omega^2((M))$. The forms $\tilde{\Theta}^i_j$ are semi-basic (because 0 and the η^j are). Define

$$\tilde{\Theta} = \tilde{\Theta}^i_j e^j \otimes e_i \in \Omega^2\left((M), \pi^*(\mathrm{End}(TM))\right),$$

where $\mathrm{End}(TM) = T^*M \otimes TM$.

Exercise 2.6.0.85: Show that $\tilde{\Theta}$ is basic, i.e., show that there exists $\Theta \in \Omega^2(M, \mathrm{End}(TM))$ such that $\tilde{\Theta} = \pi^*(\Theta)$. ⊚

The differential invariant Θ is called the *Riemann curvature tensor*.

Let $\mathfrak{so}(TM) \subset \mathrm{End}(TM)$ denote the subbundle of endomorphisms of TM that are compatible with the Riemannian metric g, in the sense that if $A \in \mathfrak{so}(T_xM)$, then $\rho(A)g_x = 0$, where $\rho : \mathrm{End}(T_xM) \to \mathrm{End}(S^2T^*_xM)$ is the induced action. In other words, if $A \in \mathfrak{so}(T_xM)$, then

$$g_x(A\,v, w) = -g_x(v, A\,w) \quad \forall v, w \in T_xM.$$

Exercise 2.6.0.86: Show that $\Theta \in \Omega^2(M, \mathfrak{so}(TM))$.

Definition 2.6.0.87. A Riemannian manifold (M^n, g) is *flat* if there exist local coordinates $x^1, \dots, x^n$ such that $g = (dx^1)^2 + \dots + (dx^n)^2$. For example, the Riemannian metric on $\mathbb{E}^n$ is flat.

Theorem 2.6.0.88. *Let (M^n, g) be a Riemannian manifold such that $\Theta \equiv 0$. Then M is flat.*

Proof. By hypothesis, $d\eta^i_j + \eta^i_k \wedge \eta^k_j = 0$. By Cartan's Theorem 1.6.0.29, around any point $x \in M$ there exist an open set U and a map $g : U \to SO(n)$ such that $\eta^i_j = (g^{-1}dg)^i_j$ on U. We have

$$d\eta^j = -(g^{-1}dg)^j_k \wedge \eta^k.$$

Taking a new frame $\widetilde{\eta}^i = g^i_j\eta^j$, we obtain

$$\begin{aligned} d\widetilde{\eta}^i &= dg^i_j \wedge \eta^j - g^i_j \wedge d\eta^j \\ &= dg^i_j \wedge \eta^j - g^i_j \wedge ((g^{-1}dg)^j_k \wedge \eta^k = 0. \end{aligned}$$

Thus $\widetilde{\eta}^i = dx^i$ for some functions x^i defined on a possibly smaller open set U'. □

The following exercises show how some standard notions from Riemannian geometry are natural consequences of the structure equations described above. If you have not already seen these notions, you may wish to skip these exercises.

Exercises 2.6.0.89: Write $\Theta^i_j = \frac{1}{2}R^i_{jkl}\eta^k \wedge \eta^l = \sum_{k>l} R^i_{jkl}\eta^k \wedge \eta^l$, so that

$$\Theta = \tfrac{1}{2}R^i_{jkl}(\eta^j \otimes e_i)\otimes\eta^k \wedge \eta^l \in \Gamma(\mathfrak{so}(TM)\otimes\Lambda^2 T^*M) = \Omega^2(M, \mathfrak{so}(TM)).$$

(Here we should really be pulling everything back from (M), but we continue to abuse notation and omit the s^*'s.) We can use the Riemannian metric g to define a bundle isomorphism $\sharp : TM \to T^*M$; tensoring with T^*M, we obtain a map $\sharp \otimes \mathrm{Id}_{T^*M} : \mathfrak{so}(TM) \to \Lambda^2(T^*M)$. Applying $\sharp \otimes \mathrm{Id}_{T^*M}$ to the first two factors in Θ, we define

$$R = \tfrac{1}{2}R_{ijkl}(\eta^i \wedge \eta^j)\otimes(\eta^k \wedge \eta^l) \in \Gamma(M, \Lambda^2 T^*M \otimes \Lambda^2 T^*M).$$

1. (a) Show that since we are using orthonormal frames, $R_{ijkl} = R^i_{jkl}$. (If we were using frames that were not orthonormal, then $R_{ijkl} \neq R^i_{jkl}$; see §2.10.)
   (b) Show that $R \in \Gamma(S^2(\Lambda^2 T^*M))$, i.e., $R_{ijkl} = R_{klij}$.
   (c) Show that $R \in \Gamma(T^*M \otimes S_{21}T^*M))$, i.e., $R_{ijkl} + R_{iklj} + R_{iljk} = 0$. (See for the definition of the tensorial construct $S_{21}V$.) This is called the *first Bianchi identity*.
   (d) Let $R_{ik} = \sum_j R_{ijkj}$ and $\mathrm{Ric} = R_{ik}\eta^i\eta^k \in \Gamma(S^2 T^*M)$. Show that Ric is well-defined. It is called the *Ricci curvature* of M.
   (e) Show that $S := \sum_i R_{ii} \in C^\infty(M)$ is well-defined. It is called the *scalar curvature* of M.
   (f) Show that when $n = 3$ one can recover R from Ric, but this is not the case for $n > 3$.
   (g) Show that when M is a surface, $R_{1212} = K$, the Gauss curvature.
   (h) Let $E \in G(2, T_xM)$, the Grassmannian of two-planes in T_xM (see or), and let v_1, v_2 be an orthonormal basis of E. Then we define $K(E) := R(v_1, v_2, v_1, v_2)$ as the *sectional curvature* of E. Show that $K(E)$ is well-defined. (Remark: S, the scalar curvature, satisfies $S(x) = \int_{Gr(2,T_xM)} K(E)\,\mathrm{dvol}$ where dvol is the natural volume form on $Gr(2, T_xM) = SO(T_xM)/(S(O(2)\times S(n-2)))$.
   (i) Calculate the sectional curvature function on $G(2, T_x(S^2 \times S^2))$, where $S^2 \times S^2$ has the product metric. What are the maximum and minimum values for $K(E)$?
   (j) More generally, given Riemannian manifolds (M_1, g_1) and (M_2, g_2), we can form the product Riemannian manifold $(M_1 \times M_2, g_1 + g_2)$. Express the Riemann curvature tensor of $M_1 \times M_2$ in terms of R_1, R_2, the curvature tensors on M_1, M_2.

   For a more invariant description of the various curvatures one can extract from the Riemann curvature tensor, see Exercise **??.??**.

2. *(Covariant differential operators)* Let $X \in \Gamma(TM)$ be a vector field and $s : M \to (M)$ a (local) orthonormal coframing. We have $X = X^i e_i$ for some functions X^i. Define the *covariant derivative* of X to be

$$\nabla X = (dX^i + X^j \eta^i_j) \otimes e_i \in \Omega^1(M, TM) = \Gamma(TM \otimes T^*M), \tag{2.6.3}$$

with η^i and η^i_j being pulled back via s.

(a) Show that ∇X is well-defined, i.e., independent of the choice of section s.

(b) For $Y \in \Gamma(TM)$, we define $\nabla_Y X := Y \lrcorner \nabla X = (dX^i + X^j \eta^i_j)(Y) e_i$. Show that $\nabla_Y(fX) = f \nabla_Y X + X(f) Y$ for $f \in C^\infty(M)$. In other words, the differential operator ∇_Y obeys the Leibniz rule.

(c) Show that

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{2.6.4}$$

Note that the left-hand side, which is the Lie bracket of X and Y, is independent of the Riemannian metric.

(d) Show that ∇ is compatible with g in the sense that

$$Y(g(X_1, X_2)) = g(\nabla_Y X_1, X_2) + g(X_1, \nabla_Y X_2)$$

for all $Y, X_1, X_2 \in \Gamma(TM)$.

(e) If $\alpha = a_i \eta^i \in \Omega^1(M)$, we may define

$$\nabla \alpha = (da_i + a_j \eta^j_i) \otimes \eta^i \Gamma(T^*M \otimes T^*M).$$

Alternatively, define $\nabla \alpha$ by requiring, for all $X, Y \in \Gamma(TM)$, that

$$Y(X \lrcorner \alpha) = (\nabla_Y X) \lrcorner \alpha + X \lrcorner (\nabla_Y \alpha).$$

Since ∇_Y is linear in Y, this defines a tensor $\nabla \alpha \in \Gamma(T^*M \otimes T^*M)$ by

$$\nabla \alpha(X, Y) := X \lrcorner (\nabla_Y \alpha).$$

Show that these two definitions of $\nabla \alpha$ agree. Similarly, one can extend ∇ to act on sections of $T^{*\otimes a} M \otimes T^{\otimes b} M$ and its natural subbundles.

Let M be a C^∞ manifold. A *covariant differential operator* or *connection* on TM is an operator $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ that is $C^\infty(M)$-linear in the first factor and obeys the Leibniz rule. If the connection satisfies (2.6.4), it is called *torsion-free*. The fundamental Lemma 2.6.0.81 implies that there exists a unique connection that is torsion-free and compatible with the Riemannian metric. This connection is called the *Levi-Civita connection*. Connections are discussed in more detail in §**??**.

3. Let $\nabla R = R_{ijkl,m}(\eta^i \wedge \eta^j) \otimes (\eta^k \wedge \eta^l) \otimes \eta^m$. Show that ∇R satisfies the *second Bianchi identity* $R_{ijkl,m} + R_{ijmk,l} + R_{ijlm,k} = 0$. ⊚
4. For $X, Y, Z \in \Gamma(TM)$, define

$$\overline{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Show that $\overline{R}$ is a tensor, i.e., it is $C^\infty(M)$-linear in all three factors, and that $\overline{R} = -\Theta$. ⊚

The fundamental theorem for hypersurfaces. Suppose $M^n \subset \mathbb{E}^{n+s}$ is a submanifold. The relationship between the curvature tensor of the induced Riemannian metric on M and the second fundamental form of M is given by the *algebraic Gauss map* G, defined as follows:

Let V, W be vector spaces, where W has an inner product. Given a basis e^j of V^* and an orthonormal basis f_a of W, let

$$\begin{aligned} G : S^2 V^* \otimes W &\to S^2(\Lambda^2 V^*), \\ h^a_{ij} e^i e^j \otimes f_a &\mapsto \sum_a (h^a_{ij} h^a_{kl} - h^a_{il} h^a_{jk})(e^i \wedge e^k) \circ (e^j \wedge e^l), \end{aligned}$$

where $1 \le i, j, k \le n = \dim V$ and $1 \le a, b \le s = \dim W$. (If we think of each h^a as a matrix, we are taking the 2×2 minors.)

Exercises 2.6.0.90:

1. Show that G is independent of the choices of bases.
2. Taking $V = T_x M$, $W = N_x M$ (the normal space at x), show that $G(II) = R$.

We are now finally in a position to answer our original question regarding the equivalence of surfaces in $\mathbb{E}^3$, and at the same time see the generalization to hypersurfaces.

Theorem 2.6.0.91 (The fundamental theorem for hypersurfaces in $\mathbb{E}^{n+1}$)**.** *Let (M, g) be a Riemannian manifold with curvature tensor R, and let $h \in \Gamma(S^2 T^* M)$. Assume that*

i. (Gauss) $R = G(h)$

and

ii. (Codazzi) $\nabla h \in \Gamma(S^3 T^* M)$

hold. Then for every $x \in M$ there exist an open neighborhood U containing x, and an embedding $f : U \to \mathbb{E}^{n+1}$ as oriented hypersurface, such that $f^*(I) = g$ and $f^*(\langle II, e_{n+1} \rangle) = h$, where e_{n+1} is a unit vector in the direction of the orientation. Moreover, f is unique up to a Euclidean motion.

Corollary 2.6.0.92. *Let $M^n, \overline{M}^n \subset \mathbb{E}^{n+1}$ be two orientable hypersurfaces with fundamental forms I, II and $\overline{I}, \overline{II}$. Suppose there exist a diffeomorphism $\phi : M \to \overline{M}$ and unit vector fields $e_{n+1}, \overline{e}_{n+1}$ such that* *i.* $\phi^*(\overline{I}) = I$,

and

ii. $\phi^*\left(\langle \overline{II}, \bar{e}_{n+1}\rangle\right) = \langle II, e_{n+1}\rangle$.

Then there exists $g \in AO(n+1)$ *(the group* $ASO(n+s)$ *plus reflections) such that* $\phi = g\,|_M$.

Proof of 2.6.0.91. Let $\Sigma = (M) \times (\mathbb{E}^{n+1})$, where we use η's to denote forms on the first space and ω's for forms on the second. We omit pullback notation in the proof. Let Ω denote a volume form on (M) (e.g., wedge together all the entries of the Maurer-Cartan form) and let

$$\mathcal{I} = \{\eta^i - \omega^i, \eta^i_j - \omega^i_j, \omega^{n+1}, \omega^{n+1}_j - h_{jk}\omega^k\}.$$

Integral manifolds of $(\mathcal{I}, \Omega)$ are graphs of immersions $i : (M)|_U \to (\mathbb{E}^{n+1})$ that are lifts of immersions $j : U \to \mathbb{E}^{n+1}$, for $U \subset M$, satisfying $j^*(I) = g$ and $j^*(\langle II, e_{n+1}\rangle) = h$. We obtain existence by:

Exercise 2.6.0.93: Show that $(\mathcal{I}, \Omega)$ is Frobenius. ⊚

To prove uniqueness, fix an orthonormal frame $(e_1, \ldots, e_n)$ at x. If two such immersions $j, \tilde{j}$ exist, we can arrange (by composing $\tilde{j}$ with an element of $AO(n+1)$) that $j(x) = \tilde{j}(x)$, $j_* e_i = \tilde{j}_* e_i$ at x and the orientations at $j(x)$ match up. Thus, j and $\tilde{j}$ have lifts to Σ which are both integral manifolds of $(\mathcal{I}, \Omega)$ and pass through the same point $(p, q) \in \Sigma$. Thus, $j = \tilde{j}$ by the uniqueness part of the Frobenius Theorem. □

The Laplacian. Let (M^n, g) be a Riemannian manifold with a volume form. Recall the star operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ defined in . Define a differential operator of order two, the *Laplacian*, by

$$\Delta_g \alpha = (d * d * + * d * d)\alpha, \qquad \alpha \in \Omega^k(M).$$

If $(\omega^1, \ldots, \omega^n)$ is a coframing of M, recall the notation $df = f_j \omega^j$.

Exercise 2.6.0.94: (a) If $M^2 \subset \mathbb{E}^3$ is a surface and (e_1, e_2) is a Darboux framing, with principal curvature functions k_1, k_2, show that

$$\Delta_g f = -(f_{11} + f_{22}) - \frac{f_1 k_{2,1} - f_2 k_{1,2}}{k_1 - k_2}.$$

(b) If g is a flat metric and $(x^1, \ldots, x^n)$ are coordinates such that dx^j gives an orthonormal coframing, show that

$$\Delta_g f = -(f_{11} + \ldots + f_{nn}).$$

(c) If $x : M^2 \to \mathbb{E}^n$ is an isometric immersion (i.e., the metric g on M agrees with the pullback from $\mathbb{E}^n$), then calculating the Laplacian of each component of x as a vector-valued function gives

$$\Delta_g x = 2\vec{H},$$

where $\vec{H} = \mathrm{trace}_g\, II$ is the mean curvature vector.

Isothermal coordinates. Let (M^2, g) be a Riemannian manifold with coordinates (x, y). Write $g = a(x,y)dx^2 + b(x,y)dxdy + c(x,y)dy^2$; then one can calculate $K(x, y)$ by differentiating the functions a, b, c. In general one gets a mess (although this was the classical way of calculating K).

Let (M^n, g) be a Riemannian manifold. Coordinates $(x^1, \ldots, x^n)$ such that

$$g = e^{2u}(dx^1 \circ dx^1 + \ldots + dx^n \circ dx^n),$$

where $u = u(x, y)$ is a given function, are called *isothermal coordinates.* Note that a Riemannian manifold admits isothermal coordinates iff g is conformally equivalent to the flat metric. In we will show that every surface with an analytic Riemannian metric admits isothermal coordinates. In fact, this is true for C^∞ metrics as well—see ([**?**], vol. IV).

Specializing to surfaces with isothermal coordinates (x, y), the framing $e_1 = e^{-u}\partial x$, $e_2 = e^{-u}\partial y$ is orthonormal.

Exercise 2.6.0.95: (a) Show that the Gauss curvature is given by

$$K = -e^{-2u}\Delta u,$$

where Δ is the Laplacian. In particular, if $K = \pm 1$, then of course $\Delta u = \mp e^{2u}$. Writing $z = x + iy$, solutions to this are given by

$$u(z) = \log \frac{2|f'(z)|}{1 \pm |f(z)|^2},$$

where f is a holomorphic function on some $D \subset \mathbb{C}$ with $f' \neq 0$ on D and $1 \pm |f|^2 > 0$.

(b) Show that, in isothermal coordinates, $\Delta f = 0$ iff $f_{xx} + f_{yy} = 0$.

2.7. Space forms: the sphere and hyperbolic space

We have seen that $\mathbb{E}^n \cong ASO(n)/SO(n)$ as a homogeneous space. Expressing $\mathbb{E}^n$ in this way facilitated a study of the geometry of its submanifolds.

Let $S^n \subset \mathbb{E}^{n+1}$ be the sphere of radius one, with its inherited metric g. We may similarly express S^n as the quotient $SO(n+1)/SO(n)$. In this manner, $(S^n) = SO(n+1)$ with the basepoint projection given by $(e_0, e_1, \ldots, e_n) \mapsto e_0 \in S^n$.

Let $\mathbb{L}^{n+1}$ be $(n+1)$-dimensional Minkowski space, i.e., $\mathbb{R}^{n+1}$ equipped with a quadratic form

$$Q(x, y) = -x^0y^0 + x^1y^1 + \ldots + x^ny^n$$

of signature $(1,n)$. Let $O(V,Q)=O(1,n)$ denote the group of linear transformations preserving Q (see for details). Then $\mathbb{L}^{n+1}\cong ASO(1,n)/SO(1,n)$.

We define *hyperbolic space* to be

$$H^n=\{x\in\mathbb{L}^{n+1}\mid Q(x,x)=-1,x^0>0\}.$$

(The reasons for this name will become clear below.) Thus, H^n may be considered as (one half of) the "sphere of radius -1" in $\mathbb{L}^{n+1}$.

Exercise 2.7.0.96: Show that Q restricts to be positive definite on vectors tangent to H^n.

Thus, H^n inherits a Riemannian metric from $\mathbb{L}^{n+1}$. Moreover, H^n can be expressed as the quotient $SO(1,n)/SO(n)$. In this manner, $(H^n)=SO(1,n)$, with the basepoint projection given by $(e_0,e_1,\dots,e_n)\mapsto e_0\in H^n$.

Let $\epsilon=0,1,-1$ respectively for $X=\mathbb{E}^n,S^n,H^n$. This handy notation will enable us to study all three spaces and their submanifold geometry at the same time. Then $X=G/SO(n)$, where G is respectively $ASO(n)$, $SO(n+1)$, $SO(1,n)$, with Lie algebra $\mathfrak{g}=\mathfrak{aso}(n)$, $\mathfrak{so}(n+1)$, $\mathfrak{so}(1,n)$. The Maurer-Cartan form of G may be written as

$$\omega=\begin{pmatrix}0&-\epsilon\omega^j\\ \omega^i&\omega^i_j\end{pmatrix}$$

where $1\le i,j\le n$ and $\omega^i_j+\omega^j_i=0$.

As explained above, we identify G with (X), so that the ω^i are the tautological forms for the projection to X and $g:=\Sigma(\omega^\alpha)^2$ gives the Riemannian metric on X. The Maurer-Cartan equation for $d\omega^i$ implies that the ω^i_j are the (upstairs) Levi-Civita connection forms for g. We also use the Maurer-Cartan equation to compute the curvature of X:

$$d\omega^i_j=-\omega^i_k\wedge\omega^k_j-\omega^i\wedge(-\epsilon\omega^j).$$

Therefore, $\Theta^i_j=d\omega^i_j+\omega^i_k\wedge\omega^k_j=\epsilon\omega^i\wedge\omega^j$ and

$$R_{ijkl}=\epsilon(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}).$$

Exercises 2.7.0.97:

1. Show that the sectional curvature of X is constant for all 2-planes. (In particular, it is zero for $\mathbb{E}^n$, positive for S^n, and negative for H^n.)
2. Let $M^{n-1}\subset X^n$ be a hypersurface. Define its second fundamental form and describe the hypersurfaces with $II\equiv 0$.
3. Consider the surface $S^1\times S^1\subset S^3$ defined by $(x^0)^2+(x^1)^2=\cos^2\theta$ and $(x^2)^2+(x^3)^2=\sin^2\theta$ for some constant $\theta\in(0,\pi/2)$. Show that this surface, which is known as a *Clifford torus*, is flat, and calculate its mean curvature.

2.8. Curves on surfaces

The interaction between the geometry of surfaces in $\mathbb{E}^3$ and the geometry of curves lying on them was much studied by early differential geometers such as Dupin, Gauss, Minding and Monge (see [**?**] for more information). We will use curves on surfaces to prove the Gauss-Bonnet theorem and to study Cauchy-type problems associated to the exterior differential systems for surfaces in .

Let $c(s)$ be a regular curve in $\mathbb{E}^3$ parametrized by arclength. Recall from §1.8 that we can adapt frames so that

$$d(x,T,N,B) = (x,T,N,B)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{pmatrix} ds.$$

Now say that c lies on a surface $M \subset \mathbb{E}^3$. Let (e_1, e_2, e_3) be a first-order adapted lift of M (so $e_3 \perp TM$). Let θ denote the angle from e_1 to T, and let ϵ be T rotated counterclockwise by $\frac{\pi}{2}$ in T_pM, so that

$$\begin{pmatrix} T \\ \epsilon \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Then the $\{e_3, \epsilon\}$-plane is orthogonal to T. The angle between N and e_3 is traditionally denoted by ϖ,[1] so that

$$\begin{pmatrix} N \\ B \end{pmatrix} = \begin{pmatrix} \cos\varpi & \sin\varpi \\ -\sin\varpi & \cos\varpi \end{pmatrix} \begin{pmatrix} e_3 \\ \epsilon \end{pmatrix}.$$

Since (T, ϵ, e_3) gives an orthonormal frame of $\mathbb{E}^3$, when we restrict this frame to c we have

$$d(x,T,\epsilon,e_3) = (x,T,\epsilon,e_3)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \kappa_g & \kappa_n \\ 0 & -\kappa_g & 0 & \tau_g \\ 0 & -\kappa_n & -\tau_g & 0 \end{pmatrix} ds \tag{2.8.1}$$

for some functions $\kappa_g(s), \kappa_n(s), \tau_g(s)$. (This notation will become less mysterious in a moment.)

To interpret these functions, notice that

$\kappa_g = \kappa \sin\varpi =$ component of the orthogonal projection of κN onto ϵ;

$\kappa_n = \kappa \cos\varpi =$ component of the orthogonal projection of κN onto e_3.

The first, κ_g, is called the *geodesic curvature*. *Geodesics* are defined to be the constant speed curves with $\kappa_g \equiv 0$. Exercise 2.8.0.98.3 below shows

[1] This letter, pronounced "var-pi" by M. Spivak in our favorite introduction to differential geometry [**?**], is not a sickly omega, but an alternate way of writing π.

that $\kappa_g(c) = \nabla_{c'}c'$, and this shows how geodesic curvature is defined intrinsically for Riemannian manifolds. In Exercise 2.10.0.113, you will show that geodesics are locally the curves that are the shortest distance between two points on a Riemannian manifold. Thus the notion of a small geodesic disk about a point (which we will use in the proof of the Gauss-Bonnet theorem) makes sense.

Next, (2.8.1) shows that κ_n measures the curving of the surface in the direction of T (by means of measuring how the surface normal e_3 is bending); it is called the *normal curvature* of the surface along c. We say $c \subset M$ is an *asymptotic line* on M if $\kappa_n \equiv 0$.

Finally, notice that if $\kappa_g \equiv 0$, then all of the curvature of the curve lies in the normal direction, and ϵ is parallel to the binormal B of the curve. Thus τ_g measures what the torsion (as a curve in $\mathbb{R}^3$) of a geodesic having tangent vector T would be; it is called the *geodesic torsion* of c.

Exercises 2.8.0.98:

1. Show that

$$\begin{aligned} \kappa_n &= -\langle II(T,T), e_3\rangle, \\ \tau_g &= \langle II(\epsilon,\epsilon), e_3\rangle, \\ \kappa_g &= (d\theta + \omega^2_1)(T). \end{aligned} \tag{2.8.2}$$

   (Hint: Use Proposition 2.5.0.74.)

   (2.8.2) shows that κ_g is intrinsic to the induced Riemannian metric on the surface and depends on how the curve is situated on the surface (in particular, how T is turning as we move along c). By contrast, the values of τ_g, κ_n depend only on the pointwise value of T, and are really measuring properties of the immersion of the surface into $\mathbb{E}^3$.
2. Show that $\kappa_g \equiv 0$ if and only if the osculating plane to c is perpendicular to the surface at each point.
3. Show that $\kappa_g(c) = \nabla_{c'}c'$.
4. Find formulas for κ_n, τ_g in terms of the principal curvatures k_1, k_2 when our surface is given a principal (Darboux) framing.
5. Find formulas for κ_n, τ_g when our surface is given a framing such that $e_1 = T$.
6. Calculate τ_g of a curve c such that c' is a principal direction (i.e., a direction where κ_n is a principal curvature) at each point along c. Such curves are called *lines of curvature*.

Exercise 2.8.0.99: Prove the *local Gauss-Bonnet theorem*: Let (M^2, g) be a Riemannian manifold and let $U \subset M$ be a connected, simply-connected oriented open subset with smooth boundary ∂U. Show that

$$\int_U K dA = 2\pi - \int_{\partial U} \kappa_g ds,$$

where dA is the area form on U, ds is the arclength measure on ∂U, and ∂U is oriented so that if T points along ∂U and N points into the interior of U, then $T \wedge N$ agrees with the orientation on U. ⊚

2.9. The Gauss-Bonnet and Poincaré-Hopf theorems

Given a compact oriented surface, there are lots of Riemannian metrics we can put on it. With different metrics, the Gauss curvature can have wildly different behavior. However, as we will see, the integral of the Gauss curvature is independent of the metric and only relies on the underlying topology of M.

First, we review some concepts from simplicial and differential topology.

A *triangulation* of the plane is the plane together with a set of triangular tiles that fill up the plane. A triangulation of a neighborhood U in a surface M is the pullback of some triangulation of a plane under a diffeomorphism $f : U \to \mathbb{R}^2$, and a triangulation of a surface is a covering by triangulated open sets such that the triangulations agree on the overlaps.

Let T be a triangulation of M with V vertices, E edges and F faces. Define

$$\chi_\Delta(M, T) = V - E + F.$$

Exercise 2.9.0.100: Show that if T, T' are two triangulations of M, then $\chi_\Delta(M, T) = \chi_\Delta(M, T')$. ⊚

Since $\chi_\Delta(M, T)$ is independent of T, we will denote it by $\chi_\Delta(M)$.

Let $X \in \Gamma(TM)$ be a vector field with isolated zeros. Around such a zero $p \in M$ define the *index* of X at p as follows: Pick a closed embedded curve retractible within M to p such that no other zero of X lies in the region U enclosed by the curve. Choose a diffeomorphism $f : U \to D$, where D is the unit disc in $\mathbb{E}^2$. Let θ be the counterclockwise angle between $f_*(X)$ and some fixed vector $v \in \mathbb{E}^2$. Define the integer

$$\operatorname{ind}_X(p) = \frac{1}{2\pi} \int_{\partial D} d\theta,$$

where the circle $\partial D \subset \mathbb{E}^2$ is oriented counterclockwise. The index is well-defined by Stokes' theorem.

Intuitively, to obtain the index we draw a small circle around an isolated zero. Travel around the circle once and count how many times the vector field spins counterclockwise (counting clockwise spin negatively) when going around the circle once. See [**?**] or [**?**] for more on indices of vector fields.

Suppose $p_1, \dots, p_r \in M$ are the isolated zeros of X. Define

$$\chi_{\mathsf{vf}}(M, X) = \sum_i \mathrm{ind}_X(p_i).$$

Below, we will indirectly prove that $\chi_{\mathsf{vf}}(M, X)$ is independent of X. For a direct proof, again see [**?**] or [**?**].

Let M^2 be a compact oriented manifold without boundary. Let g be a Riemannian metric on M and define

$$\chi_{\mathsf{metric}}(M, g) = \frac{1}{2\pi}\int_M K dA.$$

Theorem 2.9.0.101 (Guass-Bonnet and Poincaré-Hopf)**.** *Let (M^2, g) be a compact orientable Riemannian manifold, let T be a triangulation of M and let X be a vector field on M with isolated zeros. Then*

$$\chi_{\mathsf{metric}}(M, g) = \chi_\Delta(M) = \chi_{\mathsf{vf}}(M, X).$$

The equality $\chi_{\mathsf{metric}}(M, g) = \chi_\Delta(M)$ is the *Gauss-Bonnet Theorem*, and $\chi_{\mathsf{vf}}(M, X) = \chi_\Delta(M)$ is the *Poincaré-Hopf Theorem*. The common value of these invariants is called the *Euler characteristic* of M, and is denoted by $\chi(M)$.

We first prove, for certain vector fields X, that $\chi_{\mathsf{vf}}(M, X) = \chi_\Delta(M)$. Next we show that $\chi_{\mathsf{metric}}(M, g) = \chi_{\mathsf{vf}}(M, X)$, which, since the left hand side is independent of X and the right hand side independent of g, shows that both are well-defined. Combined with Poincaré-Hopf for certain vector fields, this proves both theorems.

Just for fun, we afterwards give a direct proof that $\chi_{\mathsf{metric}}(M, g) = \chi_\Delta(M)$. (Actually, "we" here is a bit of a euphemism, as you, the reader, will do much of the work in Exercise 2.9.0.102.)

Proof. Given a triangulation T, one can associate a vector field X to it such that $\chi_{\mathsf{vf}}(M, X) = \chi_\Delta(M)$. Consider the following picture:

Notice that each vertex of the triangulation becomes a zero of index $+1$, and each edge and face contains a zero of index -1 or $+1$, respectively.

To prove that $\chi_{\mathsf{metric}}(M, g) = \chi_\Delta(M)$, we follow ([**?**], vol. III) (who probably followed someone else): We will divide M into two pieces, a subset $U \subset M$ where X is complicated but the topology of U is trivial, and $M \backslash U$ where X is simple but we know nothing about the topology.

Let $p_1, \dots, p_r$ be the zeros of X. Let $D_i(\epsilon)$ be an open geodesic disc of radius ϵ about p_i, where ϵ is small enough so that the discs are contractible and don't intersect each other. Let $N(\epsilon) = M \backslash (\cup_i D_i(\epsilon))$, a manifold with boundary. On $N(\epsilon)$, X is nonvanishing, so we may define a *global* oriented orthonormal framing on $N(\epsilon)$ by taking $e_1 = \frac{X}{|X|}$ with e_2 determined by the orientation. We let η^1, η^2 denote the dual coframing. Now we calculate

$$\int_{N(\epsilon)} KdA = \int_{N(\epsilon)} d\eta^2_1 = \int_{\partial N(\epsilon)} \eta^2_1 = \sum_i \int_{\partial D_i(\epsilon)} \eta^2_1.$$

Let $\tilde{e}^j_1, \tilde{e}^j_2$ be an orthonormal framing in $D_j(\epsilon)$, and from now on we suppress the j index. Let θ denote the angle between e_1 and $\tilde{e}_1$. We have $\tilde{\eta}^2_1 = \eta^2_1 - d\theta$ wherever both framings are defined. We calculate

$$\begin{aligned} \int_M KdA &= \lim_{\epsilon\to 0} \int_{N(\epsilon)} KdA \\ &= \lim_{\epsilon\to 0} \sum_i \int_{\partial D_i(\epsilon)} \tilde{\eta}^2_1 + d\theta \\ &= \lim_{\epsilon\to 0} \sum_i \int_{D_i(\epsilon)} KdA + \sum_i \int_{\partial D_i(\epsilon)} d\theta. \end{aligned} \tag{2.9.1}$$

Since K is bounded, as $\epsilon \to 0$ the first expression in (2.9.1) tends to zero. As ϵ tends to zero the vector $\tilde{e}_1$ tends to a constant vector, so the second term tends towards 2π times index of X at p_i. □

Exercise 2.9.0.102 (The Gauss-Bonnet Formula)**:** Let (M^2, g) be an oriented Riemannian manifold and let $R \subset M$ be an open subset that is contractible with ∂R the union of a finite number of smooth curves $C_1, \dots, C_p$, oriented so that if T points along C_j and N points into R, then $T \wedge N$ agrees with the orientation on R. Let δ_i denote the angle between the terminal position of the tangent vector to C_i and the initial position of the tangent vector to C_{i+1} (with the convention that $C_{p+1} = C_1$):

Prove the *Gauss-Bonnet formula*:

$$\int_R KdA = \int_{\partial R} \kappa_g ds + \sum_i \delta_i - 2\pi.$$

Then, use this formula to obtain a second proof of the Gauss-Bonnet theorem using the triangulation definition of the Euler characteristic.

Why are these theorems so wonderful? Take a plane in $\mathbb{E}^3$, and draw a circle in the plane. Now perturb the disk inside the circle—by stretching or squashing, whatever you like—so that the boundary of the disk stays flat (see Figure 1). What is the average curvature of the wildly curving surface you've made inside the circle? Zero! Next, take a round sphere, sit on it, twist it, fold it so it gets lots of negative curvature regions. What's the average curvature of your distorted sphere? 4π, no matter how strong you are!

Exercises 2.9.0.103:

1. Let $M^2 \subset \mathbb{E}^3$. Show that the degree of the Gauss map is $\chi(M)/2$.
2. What is $\int_{\partial R} \kappa_g ds$, where R is the region enclosed by the lower dashed curve (only half of which is pictured) in Figure 2?
3. Prove the Gauss-Bonnet theorem for compact even-dimensional hypersurfaces $M^n \subset \mathbb{E}^{n+1}$. Namely, let K_n denote the product of the principal curvatures $k_1, \ldots, k_n$ and dV the volume element. Then

$$\int_M K_n dV = \frac{1}{2}\,\mathrm{vol}(S^n)\chi(M),$$

where $\mathrm{vol}(S^n) = 2^{n+1}\pi^{n/2}(\frac{n}{2})!/n!$ is the volume of the n-dimensional unit sphere for n even.

The generalization by Chern [?] to Riemannian manifolds is:

Theorem 2.9.0.104 (Gauss-Bonnet-Chern). *Let M^n be a compact, oriented Riemannian manifold of even dimension. Then*

$$\int_M \mathrm{Pfaff}(\Theta)dV = 2^{n-1}\operatorname{vol}(S^n)\chi(M),$$

where $\Theta_{ij} = \frac{1}{2}R_{ijkl}\eta^k \wedge \eta^l$.

The Pfaffian is defined in Exercise **??.??**. Note that, since the entries of Θ are 2-forms, wedge products are commutative, and so the Pfaffian makes sense with the multiplications being wedge products.

The proof of the Gauss-Bonnet-Chern theorem is not too difficult; see, e.g., [?]. The essential point is that if one puts two metrics $g, \tilde{g}$ on M, the forms $\mathrm{Pfaff}(\Theta)dV$ and $\mathrm{Pfaff}(\widetilde{\Theta})dV$ differ by an exact form and therefore $[\mathrm{Pfaff}(\Theta)dV]$ is well-defined as a cohomology class, i.e., $\int_M \mathrm{Pfaff}(\Theta)dV$ is independent of the Riemannian metric. The same proof works for Riemannian metrics on arbitrary vector bundles over M, giving rise to curvature representations of the Euler class of a vector bundle (see [?] for the definition of the Euler class). More generally, any elementary symmetric combination of the eigenvalues of a skew-symmetric matrix with, e.g., positive imaginary part leads to a *characteristic class*; that is, the corresponding cohomology class obtained from Θ is independent of the Riemannian metric used. For an excellent introduction to representing characteristic classes via curvature, see the appendix to [?].

There are further generalizations of Gauss-Bonnet-Chern (e.g., the Atiyah-Singer index theorem), but discussion of them would take us too far afield at this point; for further reading, see [?].

2.10. Non-orthonormal frames

Non-orthonormal frames for Riemannian manifolds. Let M^n be a differentiable manifold and let $\mathcal{F} = (M)$ be the bundle of all framings of M, as in §2.6. Suppose M happens to have a Riemannian metric g, but we continue to use $\mathcal{F}(M)$. (This will be desirable if, for example, we wish to vary the metric on M.) We define the functions $g_{ij} = g_{ij}(f) := g(e_i, e_j)$ on $\mathcal{F}$, where $f = (x, e_1, \dots, e_n) \in \mathcal{F}$. The fundamental lemma of Riemannian geometry now takes the form:

Lemma 2.10.0.105 (Fundamental Lemma). *There exist unique forms $\eta^i_j \in \Omega^1((M))$ such that*

i. $d\eta^i = -\eta^i_j \wedge \eta^j$

and

ii. $dg_{ij} = g_{ik}\eta^k_j + g_{kj}\eta^k_i$, *where η^i are the tautological forms on (M).*

Note that the second condition replaces $0 = \eta^i_j + \eta^j_i$, which no longer holds on (M).

Exercise 2.10.0.106: Let (M, g) be oriented and let dvol_M denote the induced volume form. Show that

$$\mathrm{dvol}_M = \sqrt{\det(g_{ij})}\eta^1 \wedge \ldots \wedge \eta^n. \circledcirc$$

Submanifolds of $\mathbb{E}^{n+s}$. Consider the general frame bundle $(\mathbb{E}^{n+s})$, which we may identify with $AGL(n+s)$ in the usual way. Let $M^n \subset \mathbb{E}^{n+s}$, and let $\mathcal{F}^1 \subset (\mathbb{E}^{n+s})$ be the bundle of first-order adapted frames, i.e., frames such that T_xM is spanned by $e_1, \ldots, e_n$. As a prelude to , define the *quotient normal bundle* $\tilde{N}M$ as the bundle whose fiber at $x \in M$ is $T_x\mathbb{E}^{n+s}/T_xM$. Then $e_{n+1}, \ldots, e_{n+s}$ span $\tilde{N}_xM$ modulo T_xM, but these vectors are not necessarily perpendicular to T_xM.

Using index ranges as in §2.5, on $\mathcal{F}^1$ we have

$$d(x, e_j, e_a) = (x, e_k, e_b)\begin{pmatrix} 0 & 0 & 0 \\ \omega^j & \omega^j_k & \omega^k_b \\ 0 & \omega^a_k & \omega^a_b \end{pmatrix}.$$

Then the Maurer-Cartan equation $0 = d\omega^a = -\omega^a_i\omega^i$ again implies that $\omega^a_i = h^a_{ij}\omega^j$ for some functions $h^a_{ij} = h^a_{ji}$ on $\mathcal{F}^1$, so we have

$$I = g_{ij}\omega^i\omega^j \in \Gamma(M, S^2T^*M),$$
$$II = \omega^a_j\omega^j \otimes e_a \in \Gamma(M, S^2T^*M \otimes \tilde{N}M).$$

Now, for simplicity, assume M is a hypersurface. Fix an orientation on M (say, upward). Let N be an unit vector field perpendicular to the surface, and let $Q = \langle II, N\rangle$. *Then the eigenvalues of $g^{-1}Q$ are well-defined.* This can be explained as follows.

Given a vector space V with a quadratic form $Q \in S^2V^*$, we may think of Q as a map $V \to V^*$. Given a linear map between two different vector spaces, it does not make sense to talk of eigenvalues (and therefore traces and determinants). But now say we have a second, nondegenerate, quadratic form $g \in S^2V^*$. We may think of g^{-1} as a map $g^{-1} : V^* \to V$ and consider the composition $g^{-1} \circ Q : V \to V$. We can calculate the trace and determinant of $g^{-1} \circ Q$.

Exercise 2.10.0.107: Say M is a surface. Show that

$$K = \det(g^{-1} \circ Q)$$
$$H = \mathrm{trace}(g^{-1} \circ Q). \circledcirc$$

Coordinate formulas for H, K. Now we will finally prove the formulas (1.1.3). Say $M \subset \mathbb{E}^3$ is given locally by a graph $z = f(x, y)$, with $f(0, 0) = 0$ and $f_x(0, 0) = f_y(0, 0) = 0$.

A simple coframing of $T\mathbb{R}^3$ along M is

$$\begin{aligned} \omega^1 &= dx, \\ \omega^2 &= dy, \\ \omega^3 &= dz - f_x dx - f_y dy. \end{aligned}$$

Note that this coframing is first-order adapted in the sense that $TM = \{\omega^3\}^\perp$. The dual framing is

$$\begin{aligned} e_1 &= \partial_x + f_x \partial_z, \\ e_2 &= \partial_y + f_y \partial_z, \\ e_3 &= \partial_z. \end{aligned}$$

Then

$$(g_{ij}) = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_y f_x & 1 + f_y^2 \end{pmatrix},$$

and therefore

$$\begin{aligned} \text{dvol}_M &= (\det g)^{\frac{1}{2}} \omega^1 \wedge \omega^2 \\ &= [(1 + f_x^2)(1 + f_y)^2) - (f_x f_y)^2]^{\frac{1}{2}} dx \wedge dy \\ &= (1 + f_x^2 + f_y^2)^{\frac{1}{2}} dx \wedge dy. \end{aligned} \tag{2.10.1}$$

Computing de_1 and de_2 gives $\omega_1^3 = d(f_x)$ and $\omega_2^3 = d(f_x)$, so that $h = (h_{ij})$ is just the Hessian of f.

Exercise 2.10.0.108: Show that, relative to our framing for the graph $z = f(x, y)$,

$$\begin{aligned} Q &= (1 + f_x^2 + f_y^2)^{-\frac{1}{2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \\ (g_{ij})^{-1} &= (1 + f_x^2 + f_y^2)^{-1} \begin{pmatrix} 1 + f_x^2 & -f_x f_y \\ -f_x f_y & 1 + f_y^2 \end{pmatrix}. \end{aligned}$$

Then confirm that H, K are given by (1.1.3). ⊚

Geometric interpretation of $H \equiv 0$. Non-orthonormal frames are particularly useful if one wants to deform a submanifold.

Definition 2.10.0.109. $M^2 \subset \mathbb{E}^3$ is said to be *minimal* if for all $x \in M$ there exists a closed neighborhood U, with $x \in U \subset M$, such that for any $V \subset \mathbb{E}^3$ that is a small deformation of U with $\partial V = \partial U$, we have $\text{area}(V) \geq \text{area}(U)$.

Theorem 2.10.0.110. *$M^2 \subset \mathbb{E}^3$ is minimal iff $H \equiv 0$.*

Proof. We show that minimal implies $H \equiv 0$. We use our area formula (2.10.1) and deform the metric. We work locally, so we have $U \subset \mathbb{R}^2$ and $x : U \to \mathbb{E}^3$ giving the surface. Let u, v be coordinates on U.

Fix an orthonormal framing (e_1, e_2, e_3) along $x(U)$ and let $x^t(u, v)$ be a nontrivial deformation of x. For t sufficiently small, we may write

$$x^t(u,v) = x(u,v) + t\,s(u,v)e_3(u,v) + O(t^2),$$

where $s : U \to \mathbb{R}$ is some function. For fixed t, we calculate

$$\begin{aligned} dx^t &= dx + t\,e_3\,ds + t\,s\,de_3 + O(t^2) \\ &= e_1\omega^1 + e_2\omega^2 + t\,e_3(s_1\omega^1 + s_2\omega^2) - t\,s(e_1\omega^3_1 + e_2\omega^3_2) + O(t^2). \end{aligned}$$

Using the same ω^1, ω^2, we may write $dx^t = e^t_1\omega^1 + e^t_2\omega^2$ for

$$\begin{aligned} e^t_1 &= (1 - t\,s\,h_{11})e_1 - t(s\,h_{21}e_2 + s_1e_3) + O(t^2), \\ e^t_2 &= (1 - t\,s\,h_{22})e_2 - t(s\,h_{12}e_1 + s_2e_3) + O(t^2). \end{aligned}$$

Of course, this framing is no longer orthonormal, and in fact the metric now looks like

$$g^t = \begin{pmatrix} 1 - t\,s\,h_{11} & 2t\,s\,h_{21} \\ 2t\,s\,h_{12} & 1 - t\,s\,h_{22} \end{pmatrix} + O(t^2).$$

Exercise 2.10.0.111: Calculate $\frac{d}{dt}|_{t=0}(\det g^t)^{\frac{1}{2}}$ and show $\frac{d}{dt}|_{t=0}\int_U \mathrm{dvol}(g^t) = 0$ iff $H \equiv 0$.

Now we wave our hands a little and ask you to trust that calculus in infinite dimensions behaves the same way as in finite dimensions. That is, a function has a critical point at a point where the derivative vanishes, and in our case its easy to see we are at a minimum. (If you don't believe us, consult a rigorous book on the calculus of variations, such as [**?**].) □

Exercise 2.10.0.112: More generally, for $M^n \subset \mathbb{E}^{n+s}$, define $\vec{H} \in \Gamma(NM)$, the *mean curvature vector*, to be $\mathrm{trace}_g(II)$. Show that M is minimal iff $\vec{H} \equiv 0$. In particular, show that straight lines are locally the shortest curves between two points in the plane

Exercise 2.10.0.113: More generally, let X^{n+s} be a Riemannian manifold and $M^n \subset X$ a submanifold, and define $\vec{H} \in \Gamma(NM)$, the *mean curvature vector*, to be $\mathrm{trace}_g(II)$. Show that M is minimal iff $\vec{H} \equiv 0$. In particular, show that geodesics are locally the shortest curves between two points of X.

Figure 1. A distorted disk

Figure 2. What's the average Gauss curvature?