Introduction to the Geometry of Tensors Part 1:

The fundamental theorem of linear algebra is a miracle + introduction to symmetry

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Linear algebra review

$\mathbf{a} \times \mathbf{a}$ matrix M

Could represent

$L_M : \mathbb{C}^{\mathbf{a}} \to \mathbb{C}^{\mathbf{a}}$ linear map (or $\mathbb{R}^{\mathbf{a}} \to \mathbb{R}^{\mathbf{a}}$...)

$v \mapsto Mv$.

Write $L_M : A \to A$.

Or could represent bilinear form

$B_M : A \times A \to \mathbb{C}$

$(v, w) \mapsto v^t M v$

Both cases: group action $GL(A)$: group of invertible linear maps $A \to A$, invertible $\mathbf{a} \times \mathbf{a}$ matrices

linear map $\rightsquigarrow$ Jordan normal form (eigenvalues, Jordan blocks....)

bilinear maps ???

Notation

$A = \mathbb{C}^{\mathbf{a}}$: column vectors,

A^*: row vectors = space of linear maps $A \to \mathbb{C}$, where $\alpha \in A^*$, $v \in A$, $\alpha(v) = \alpha v$, row-column mult.

$A^* \otimes A$: linear maps $A \to A$

$A^* \otimes A^*$: bilinear forms $A \times A \to \mathbb{C}$.

$GL(A)$ acts on $A^* \otimes A$. $g \in GL(A)$, $M \in A^* \otimes A$, $g \cdot M = gMg^{-1}$. Jordan normal form: infinite number of orbits (open subset described by **a** parameters) "tame" orbit structure.

Bilinear forms: $GL(A)$ acts on $A^* \otimes A^*$ $g \in GL(A)$, $M \in A^* \otimes A^*$, $g \cdot M = gMg^t$. Normal form?

Easier case

$\mathbf{a} \times \mathbf{b}$ matrix M

Could represent

$M : B \to A$ linear map

$w \mapsto Mw$.

Or bilinear form

$M : B \times A^* \to \mathbb{C}$

$(\alpha, w) \mapsto \alpha M w$

Both cases: Same group action $GL(A) \times GL(B)$

Normal forms $\begin{pmatrix} \mathrm{Id}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$

$0 \leq k \leq \min\{\mathbf{a}, \mathbf{b}\}$: finite

Group actions

Bilinear forms: $GL(A) \times GL(B)$ acts on $A \otimes B$, finite number of orbits, simple normal form for each.

Use: efficient algorithm to solve system of linear equations (ancient China, rediscovered by Gauss) Exploit (part of) group action to put system in easy form.

$GL(A)$ action on $A^*{\otimes}A^*$

If $B \in A^*{\otimes}A^*$ symmetric, i.e., $B(v,w) = B(w,v)\ \forall v, w \in A$, $\Rightarrow$ $g \cdot B$ is too

same for skew.

$\rightsquigarrow$

$A^*{\otimes}A^* = S^2 A^* \oplus \Lambda^2 A^*$

as $GL(A)$-module.

Exercise: Show orbit structure on $A^*{\otimes}A^*$ is "tame", analog of Jordan normal form.

Symmetry groups

Given $T \in A^* \otimes A$, let $G_T := \{g \in GL(A) \mid g \cdot T = T\}$, symmetry group of T

Let $\mathbb{T}_A \subset GL(A)$ diagonal matrices.

Exercise: $T \in A^* \otimes A$ "generic" $G_T \cong g\mathbb{T}_A g^{-1}$, some fixed $g \in G$. In particular **a**-dimensional subgroup of $GL(A)$.

Exercise: Let $M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, G_M?

Question: Before doing calc, what do we expect in general?

Given $T \in A \otimes B$, let $G_T := \{g \in GL(A) \times GL(B) \mid g \cdot T = T\}$

Exercise: Say $\mathbf{a} = \mathbf{b}$ and T: generic, what is G_T?

Open Q: What are possible $G_T \subset GL(A)$ for $T \in A^* \otimes A^*$?

Fundamental Theorem of linear algebra

Fix bases $\{a_i\}$, $\{b_j\}$ of A, B and for $r \leq \min\{\mathbf{a}, \mathbf{b}\}$, set $I_r = \sum_{k=1}^{r} a_k \otimes b_k$. Let $\mathrm{End}(A) = A^* \otimes A$. The following quantities all equal the **rank** of $T \in A \otimes B$:

(Q) The largest r such that $I_r \in \mathrm{End}(A) \times \mathrm{End}(B) \cdot T$.

($\underline{\mathbf{Q}}$) The largest r such that $I_r \in \overline{GL(A) \times GL(B) \cdot T}$.

($\mathbf{ml}_A$) $\dim A - \dim \ker(T_A : A^* \to B)$

($\mathbf{ml}_B$) $\dim B - \dim \ker(T_B : B^* \to A)$

($\underline{\mathbf{R}}$) The smallest r such that T is a limit of a sum of r rank one elements, i.e., such that $T \in \overline{GL(A) \times GL(B) \cdot I_r}$

(R) The smallest r such that T is a sum of r rank one elements. i.e., such that $T \in \mathrm{End}(A) \times \mathrm{End}(B) \cdot I_r$

Tensors

Now consider $T \in A \otimes B \otimes C$. (or $T \in A_1 \otimes \cdots \otimes A_k$)

Trilinear form $A^* \times B^* \times C^* \to \mathbb{C}$.

Bilinear map $A^* \times B^* \to C$.

Linear map $T_A : A^* \to B \otimes C$

Example: $A^*, B^*, C = \mathcal{A}$ algebra, $T = T_{\mathcal{A}}$ structure tensor. i.e., $T_{\mathcal{A}}(a_1, a_2) := a_1 a_2$.

In particular, A, B, C space of $n \times n$ matrices $T = M_{\langle n \rangle}$ structure tensor of matrix multiplication.

$T \in A \otimes B \otimes C$ has *rank one* if $\exists\ a \in A$, $b \in B$, $c \in C$ such that $T = a \otimes b \otimes c$.

Tensors

For $r \leq \min\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, write $I_r = \sum_{\ell=1}^{r} a_\ell \otimes b_\ell \otimes c_\ell$.

Definitions:

$\mathbf{Q}(T)$ *subrank*: largest r such that $I_r \in \operatorname{End}(A) \times \operatorname{End}(B) \times \operatorname{End}(C) \cdot T$

$\underline{\mathbf{Q}}(T)$ *border subrank*: largest r such that $I_r \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$

$\mathbf{ml}$ *multi-linear ranks*: $(\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)) := (\operatorname{rank} T_A, \operatorname{rank} T_B, \operatorname{rank} T_C)$

$\underline{\mathbf{R}}(T)$ *border rank*: The smallest r such that T is a limit of rank r tensors i.e. such that $T \in \overline{GL(A) \times GL(B) \times GL(C) \cdot I_r}$, allowing re-embeddings

$\mathbf{R}(T)$ *rank*: smallest r such that T is a sum of r rank one tensors i.e., such that $T \in \operatorname{End}(A) \times \operatorname{End}(B) \times \operatorname{End}(C) \cdot I_r$, allowing re-embeddings of T to $\mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$

Inequalities and first open problems

$$\mathbf{Q}(T) \leq \underline{\mathbf{Q}}(T) \leq \min\{\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)\}$$
$$\leq \max\{\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)\} \leq \underline{\mathbf{R}}(T) \leq \mathbf{R}(T)$$

all may be strict, even when $\mathbf{a} = \mathbf{b} = \mathbf{c}$.

Say $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$, then T: generic $\Rightarrow \underline{\mathbf{R}}(T) = \mathbf{R}(T) \simeq \frac{m^2}{3}$ and this is largest possible $\underline{\mathbf{R}}$. (Lickteig 1980's, symmetric case Terracini 1916, higher order symmetric mostly Terracini 1916, finished Alexander-Hirschowitz 1990's)

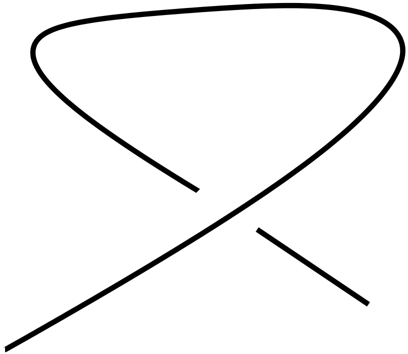
Open Q: Exact largest possible in general 3-factor (see Abo-Ottaviani-Peterson for state of art).

Open Q: Largest possible $\mathbf{R}(T)$? (state of art, see Buczynski-Han-Mella-Teitler)

If multilinear ranks maximal $= m$, call T *concise* $\Rightarrow \underline{\mathbf{R}}(T) \geq m$, say *minimal border rank* if $= m$.

Open Problem:Classify concise tensors of minimal border rank.

Geometry of rank

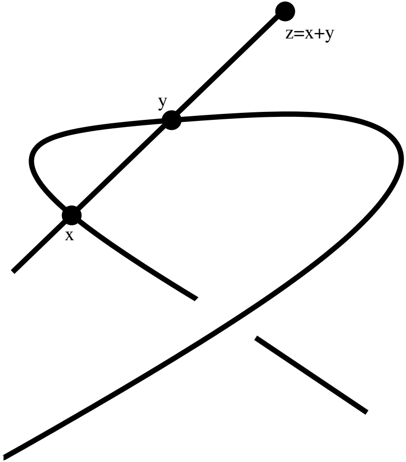


Imagine curve represents the set of tensors of rank one sitting in the N^3 dimensional space of tensors.

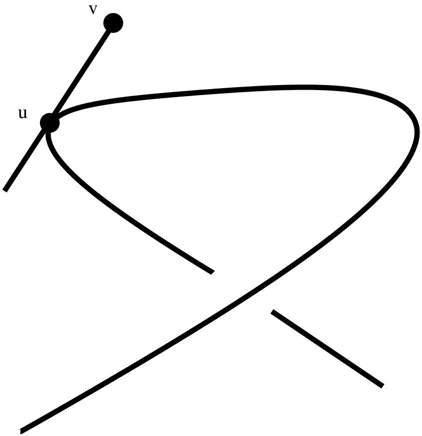
Geometry of rank

{ tensors of rank two} =

{ points on a secant line to set of tensors of rank one}



Geometry of border rank



The limit of secant lines is a tangent line!

Note: most points on just one secant line.

Most points: if on secant line, usually not on tangent line

Plane curve: both. Rank one matrices like curves in the plane

Polynomials and limits

Clear: P: poly, $P(T_t) = 0$ for $t > 0 \Rightarrow P(T_0) = 0$.

$\Rightarrow$ Cannot describe rank via zero sets of polynomials.

Matrices: Matrix border rank given by polynomials.

Tensor border rank?

Tensors of border rank $\leq r$ Euclidean closed

$S \subset V$ set, define Zariski closure by first
$I_S := \{\text{polys } P \mid P(s) = 0 \forall s \in S\}$.

$\overline{S}^{zar} := \{v \in V \mid P(v) = 0 \forall P \in I_S\}$.

Theorem: In our situation $\overline{S} = \overline{S}^{zar}$ (whenever $\overline{S}^{zar}$ is irreducible and S contains a Zariski-open subset of $\overline{S}^{zar}$).

$\Rightarrow$ can determine border rank with polynomials!

Border rank via Polynomials

Matrices: easy, just minors (efficient to compute thanks to Gaussian elimination)

Tensors??

Open

State of the art: border rank ≤ 4 (Friedland)

Next time: some known equations.

Normal forms?

Bilinear forms: finite number of orbits

Endomorphisms: finite number of cases, each with finite number of parameters "tame"

Tensors?

Kronecker $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$: yes! tame

$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$: yes! tame

In general: NO "wild"

Aside for those familiar with Dynkin diags.

Write marked Dynkin diag. for space with group action. Cases $A \otimes B$, $A \otimes A^*$, $\mathbb{C}^2 \otimes A \otimes B$. Add new node and adjoin edges from new node to marked nodes.Finite if get Dynkin diag. of finite dimensional simple Lie alg. Tame (not finite) if get Dynkin diag. of affine simple Lie alg. Otherwise wild.

Tensor Rank Decomposition

Linear algebra: determine rank of matrix easy. finding a rank decomposition easy. $r > 1$, never unique.

Tensors: determine rank of tensor hard. No general technique. (methods for T low rank and with nice combinatorial properties)
But: often unique!

If can decompose, extremely useful for applications.

e.g. blind source separation (P. Comon)

Classical algebraic geometry

$V = \mathbb{C}^N$, $X \subset \mathbb{P}V$ variety,

$$\sigma_r(X) := \overline{\{z \in \mathbb{P}V \mid \exists x_1, \ldots, x_r \in X, \text{ such that } z \in \mathrm{span}(x_1, \ldots, x_r)\}}$$

r-th secant variety

Palantini, Terracini, Fulton-Hansen, Alexander-Hirschowitz, Zak, ...

$$\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) = \{[T] \mid \underline{\mathbf{R}}(T) \leq r\}$$

Classical algebraic geometry

Consider rank at most r matrices:
$\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B)) = \{[T] \mid \underline{\mathbf{R}}(T) \leq r\}$

Invariant under changes of bases $\Rightarrow$ its ideal
$I_{\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B))} \subset Sym(A^*{\otimes}B^*)$ invariant under changes of bases

Special case: rank one - saw matrix has rank one iff size two minors zero. Degree two polynomials.

Consider all homogeneous degree two polynomials on matrices:

$S^2(A^*{\otimes}B^*) = S^2A^*{\otimes}S^2B^* \oplus \Lambda^2A^*{\otimes}\Lambda^2B^*$

Size two minors ??

What about $S^2(A^*{\otimes}B^*{\otimes}C^*)$? any subspace in
$I_{\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))}$?

More Open Problems

• Tensors of minimal border rank revisited: find defining eqns. for $\sigma_m(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}))$ State of art: $m \leq 4$ (Friedland, 2010)

• Even more ambitious: generators of ideal. State of art: $m \leq 3$ (L-Weyman, 2007)

• Hay in a haystack: A random tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ has border rank $\sim \frac{m^2}{3}$. Find an *explicit* sequence of tensors of border rank $m^{1+\epsilon}$. State of art: $(2.03)m$. (L-Michalek, 2020).

Open Problems cont'd

• Cost v. Value in quantum information: Approximate Cost of T $\sim \underline{\mathbf{R}}(T)$, Approximate Value $\sim \underline{\mathbf{Q}}(T)$,

True cost/value $T^{\boxtimes N} := T^{\otimes N} \in (A^{\otimes N}) \otimes (B^{\otimes N}) \otimes (C^{\otimes N})$

$\underset{\sim}{\mathbf{R}}(T) := \lim_{N\to\infty} (\underline{\mathbf{R}}(T^{\otimes N}))^{\frac{1}{N}}$, $\underset{\sim}{\mathbf{Q}}(T) := \lim_{N\to\infty} (\underline{\mathbf{Q}}(T^{\otimes N}))^{\frac{1}{N}}$

Find low cost high value tensors. (see work of Christandl-Vrana-Zuiddam)

Approaches to value

$\mathbf{Q}$, $\underline{\mathbf{Q}}$ not related to classically studied objects.

Idea: define easier to compute quantities bounding $\underline{\mathbf{Q}}$

$\rightsquigarrow$ slice rank (Tao, 2016) and Strength/product rank (for higher order tensors)

Variant over finite fields inspired by random tensors: analytic rank (Gowers) "low (product) rank implies bias" Very recent: Cohen-Moshkovitz: bias implies low (product) rank.

over $\mathbb{C}$ $\rightsquigarrow$ geometric rank (Kopparty-Moshkovitz-Zuiddam, 2020)

$\rightsquigarrow$ classical linear algebra *and* classical algebraic geometry:

spaces of matrices of bounded rank, linear $\mathbb{P}^{m-1}$'s $\subset \mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^m)$ having non-transverse interections with $\sigma_r(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}))$

Next time : one running example to illustrate the utility of geometry in study of tensors - complexity of matrix multiplication.

Workshop lecture: geometry associated to tensor network states.

Thank you for your attention

For more on **tensors**, their geometry and applications, resp. **geometry and complexity**, resp. **recent developments**: