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REMARKS ON "LIE ALGEBRA COHOMOLOGY AND THE GENERALIZED BOREL-WEIL THEOREM", BY B. KOSTANT

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In Kostant's paper, which appears in this issue of the *Annals*, that author studies the cohomology of certain subalgebras of a semi-simple Lie algebra. One of his results is Theorem 5.14 from which he is able to deduce the Borel-Weil theorem as generalized by R. Bott. I would like to point out that the whole reasoning by which Kostant makes this deduction effective can be inverted; therefore his result is *equivalent* to Borel-Weil-Bott theorem.

Next, following an idea of Bott, he proceeds to prove H. Weyl's well-known character formula. The point is to interpret each character as a ratio of two Euler-Poincaré characteristics which amount respectively for the numerator and the denominator of Weyl's formula. Unfortunately, as pointed out by Kostant himself, one needs a very particular case of Weyl's formula in the course of proof of Theorem 5.14. We would like to sketch a proof of Theorem 5.14 which does not use even this particular case of Weyl's formula. The point is to give a direct proof of Lemma 5.12 omitting all of §§ 5.8 to 5.11. Such a proof follows. (We do not repeat all notations).

Δ being the set of all roots, let Φ be any subset of Δ; by $\langle\Phi\rangle$ we mean the sum of all roots belonging to Φ and g is $1/2\langle\Delta_+\rangle$ (Δ_+ is the set of all positive roots). There is a one-to-one correspondence between the set of all subsets Φ of Δ_+ and the set of all subsets Ψ of Δ such that Δ be a disjoint union of Ψ and $-\Psi$; this correspondence is expressed by the formulas $\Phi = \Psi \cap \Delta_+$ and $\Psi = \Phi \cup -(\Delta_+ \cap C\Phi)$; furthermore, by an easy computation we get

$$(1) \qquad g - \langle\Phi\rangle = -1/2\langle\Psi\rangle$$

when Φ and Ψ are so related.

Let $\mathfrak{m}$ be the Lie subalgebra of the semi-simple Lie algebra $\mathfrak{g}$ generated by the Cartan subalgebra $\mathfrak{h}$ and the root vectors belonging to positive roots. If V^λ is the space of an irreducible representation π of $\mathfrak{g}$ with maximal weight λ, one considers the natural representation ζ of $\mathfrak{h}$ on the space $\Lambda\mathfrak{m}^* \otimes V^\lambda$. Using the well-known basis of an exterior algebra, one sees the weights of ζ are of the form:

$$(2) \qquad \xi = \mu - \langle\Phi\rangle$$

where μ is a weight of π and Φ is any subset of Δ_+; furthermore the multiplicity of ξ is the sum of the multiplicities m_μ (for μ considered as a weight of π) extended over all decompositions of ξ in the form (2).

I claim

$$(3) \qquad |g + \lambda| \geqq |g + \xi|$$

for any weight ξ of ζ. Using (1) and (2) we get

$$(4) \qquad g + \xi = \mu - 1/2\langle\Psi\rangle .$$

Since the set of weights of π is invariant under the Weyl group W, so it is for the set of linear forms $g + \xi$ on $\mathfrak{h}$:

$$(5) \qquad s(g + \xi) = s\cdot\mu - 1/2\langle s\cdot\Psi\rangle = g + s\cdot\mu - \langle\Phi(s)\rangle$$

where $\Phi(s) = s\cdot\Psi \cap \Delta_+$. We can therefore find an s in W so that $s(g + \xi)$ is dominant. As is well-known, $s\cdot\mu$ is equal to $\lambda - \sum_i m_i\cdot\alpha_i$ with positive roots α_i and non-negative integers m_i; it implies:

$$(6) \qquad s(g + \xi) = (g + \lambda) - \sum_i m_i'\cdot\alpha_i ,$$

and finally

$$\begin{aligned} |g + \lambda|^2 &= |s(g + \xi)|^2 + |\textstyle\sum_i m_i'\cdot\alpha_i|^2 + \sum_i m_i'\langle s(g + \xi), \alpha_i\rangle \\ &= |g + \xi|^2 + |\textstyle\sum_i m_i'\cdot\alpha_i|^2 + \sum_i m_i'\langle s(g + \xi), \alpha_i\rangle . \end{aligned}$$

Since $s(g + \xi)$ is dominant, the scalar product $\langle s(g + \xi), \alpha\rangle$ is non-negative for each positive root α. It follows formula (3) immediately.

From this deduction of formula (3), one sees sign "equal" can occur only for all m_i' equal to 0, that is $s(g + \xi) = g + \lambda$ or $\xi = \xi_s$ with

$$(7) \qquad \xi_s = s^{-1}(g + \lambda) - g .$$

Furthermore λ is dominant and $\langle g, \alpha\rangle > 0$ for any positive root α, so that $\langle g + \lambda, \alpha\rangle > 0$ under the same assumptions; as is well-known, this implies $s(g + \lambda) \neq g + \lambda$ for $s \neq 1$ and there is a unique s in W for which (7) holds. The map $s \to \xi_s$ is bijective from W to the set of all weights ξ of ζ such that $|g + \xi| = |g + \lambda|$.

It remains to show that the weight ξ_s given by (7) has multiplicity one. Since λ occurs with multiplicity one in π, it is sufficient to show ξ_s has a unique decomposition in the form (2) and $\lambda = \mu$ in this decomposition. But (2) implies (5) and using (7) one gets:

$$(8) \qquad \lambda = s\cdot\mu - \langle\Phi(s)\rangle .$$

Recalling $s\cdot\mu = \lambda - \sum_i m_i\cdot\alpha_i$ with non-negative integers m_i, this is possible only if all m_i are 0 and $\Phi(s)$ is empty, that is $\lambda = \mu$ and $s \cdot \Psi \cap \Delta_+$ empty which amounts to $\Psi = s^{-1} \cdot \Delta_-$, or finally $\Phi = \Delta_+ \cap s^{-1} \cdot \Delta_-$. This achieves the proof.

We have proved all of Lemma 5.12, the last assertion in it being trivial any way. This concludes our task.

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