DIFFERENTIAL GEOMETRY 1
PROBLEM SET 3 SOLUTIONS

Lee: 3-1, 3-4, 3-5, 3-7, 3-8
CFB: A.2.3:1-3, omit the symplectic group for exercise 1

Lee 3-1: *If $\mathcal{M}$ and $\mathcal{N}$ are smooth connected manifolds and $F : \mathcal{M} \to \mathcal{N}$ is a smooth map such that $F_* : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is identically 0 for all p then show F is constant.*

Proof:

Let $p \in \mathcal{M}$ and fix a coordinate neighborhood of p say (U, φ) and let x^i denote the coordinates here. Similarly, let (V, ψ) be a coordinate neighborhood of $F(p)$ in $\mathcal{N}$ with coordinates y^a.
Then $\left\{\frac{\partial}{\partial x^i}|_p\right\}$ are a basis for $T_p\mathcal{M}$ and $\left\{\frac{\partial}{\partial y^a}|_{F(p)}\right\}$ form a basis for $T_{F(p)}\mathcal{N}$.
Letting $\hat{F}$ denote the coordinate representation of F and $\hat{p}$ the coordinate representation of p we compute

$$\begin{aligned}
0 = F_*\left(\frac{\partial}{\partial x^i}|_p\right) &= F_*\left(\left(\varphi^{-1}\right)_* \frac{\partial}{\partial x^i}|_{\varphi(p)}\right) \\
&= \left(\psi^{-1} \circ \psi \circ F \circ \varphi^{-1} \circ \varphi\right)_* \left(\frac{\partial}{\partial x^i}|_{\varphi(p)}\right) \\
&= \left(\psi^{-1}\right)_* \hat{F}_* \left(\frac{\partial}{\partial x^i}|_{\varphi(p)}\right) \\
&= \left(\psi^{-1}\right)_* \frac{\partial \hat{F}^a}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^a}|_{\hat{F}(\varphi(p))} \\
&= \frac{\partial \hat{F}^a}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^a}|_{F(p)}
\end{aligned}$$

Since $\left\{\frac{\partial}{\partial y^a}|_{F(p)}\right\}$ is a basis of $T_{F(p)}\mathcal{N}$ we can conclude that $\frac{\partial F^a(\hat{p})}{\partial x^i} = 0$ for all a, i and so $\hat{F}^a$ is constant on $\varphi\left(U \cap F^{-1}(V)\right)$.
Consequently F is constant on $U \cap F^{-1}(V)$.
Since this construction can be performed for any $p \in \mathcal{M}$ we can conclude that F is a continuous discrete map on $\mathcal{M}$.
However, $\mathcal{M}$ is connected and so F must be constant.

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Date: February 21, 2012.

Lee 3-4: *Let $C \subset \mathbb{R}^2$ be the unit circle and $S \subset \mathbb{R}^2$ the boundary of the square of side-length 2 centered at $(0,0)$ i.e.*

$$S = \{(x,y) \mid \max(|x|,|y|) = 1\}$$

Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(C) = S$ but that there is no diffeomorphism with the same property.

Proof:

Consider the ray from the origin passing through $e^{i\theta}$ for $\theta \in \left(-\frac{\pi}{4}, \frac{7\pi}{4}\right]$. Such a ray is unique and will intersect S in exactly one point given by

$$f\left(e^{i\theta}\right) = \begin{cases} (1, \tan\theta) & -\frac{\pi}{4} < \theta \leq \frac{\pi}{4} \\ (\cot\theta, 1) & \frac{\pi}{4} < \theta \leq \frac{3\pi}{4} \\ (-1, -\tan(\theta)) & \frac{3\pi}{4} < \theta \leq \frac{5\pi}{4} \\ (-\cot\theta, 1) & \frac{5\pi}{4} < \theta \leq \frac{7\pi}{4} \end{cases}$$

This can be extended to a map $f : \mathbb{R} \to \mathbb{R}$ by $f\left(re^{i\theta}\right) = rf\left(e^{i\theta}\right)$.

Next note that the trig functions are continuous from $\mathbb{R} \to \mathbb{R}$ and so f can be seen to be continuous.[1]
However, by construction f is bijective. Indeed, f maps origin centered circles of radius r bijectively onto origin centered squares of side-length $2r$ and sends the origin to the origin. Furthermore, one can construct the inverse of f and see that it is smooth. In particular when restricting f to a circle of radius r we see that f is continuous bijection from a compact set (a circle) to a Hausdorff space (a square) and hence is a homeomorphism.

Next, we note that there can be no such diffeomorphism. Indeed, if such a diffeomorphism existed then it would restrict to a diffeomorphism from C to S but S is not smooth at $(1,1)$ which gives a contradiction. A more rigorous argument is as follows: If F is a diffeomorphism from C to S then we can restrict to the subset of C that gets mapped onto a connected neighborhood of $(1,1)$ in S.
This subset of S is necessarily connected (since F is a homeomorphism) and hence can be parameterized by an arc $\gamma : (-1,1) \to S$ (you can take γ to be given by $t \mapsto e^{i\pi t + \varphi_0}$ for some fixed $\varphi_0 \in \mathbb{R}$ if you like). Then we necessarily have

$$F \circ \gamma(t) = \begin{cases} (1, y(t)) & -1 < t \leq t_0 \\ (x(t), 1) & -t_0 < t < 1 \end{cases}$$

By reparameterization of γ we may, without loss of generality, assume that $t_0 = 0$.
Then computing we find

$$F_* \gamma'(t) = \begin{cases} (0, y'(t)) & -1 < t \leq 0 \\ (x'(t), 0) & 0 < t < 1 \end{cases}$$

[1] One might worry about the point $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$ but these essentially resolve quadrant ambiguities inherent in tan and so one can show that f is continuous at these values of θ

In particular, we must have $x'(0) = y'(0) = 0$ and so $F_*\gamma'(0) = (0,0)$.
But F a diffeomorphism implies that F_* is a linear bijection and hence $\gamma'(0) = 0$, a contradiction.

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Lee 3-5 *Consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$ under the usual identification of $\mathbb{C}^2$ with $\mathbb{R}^4$. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \to \mathbb{S}^2$ by*

$$\gamma_z(t) = \left(e^{it}z^1, e^{it}z^2\right)$$

(a) *Compute the coordinate representation of $\gamma_z(t)$ in stereographic coordinates, and use this to show that γ_z is a smooth curve.*

(b) *Compute $\gamma_z'(t)$ in stereographic coordinates, and show that it is never zero.*

Proof:

Part a
Recall that stereographic coordinates are defined by:

$$\sigma\left(x^1, x^2, x^3, x^4\right) = \frac{1}{1-x^4}\left(x^1, x^2, x^3\right) : \mathbb{S}^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$$
$$\tilde{\sigma}(x) = -\sigma(-x) : \mathbb{S}^3 \setminus \{(0,0,0,-1)\} \to \mathbb{R}^3$$
$$\sigma^{-1}\left(x^1, x^2, x^3\right) = \frac{1}{1+|x|^2}\left(2x^1, 2x^2, 2x^3, |x|^2 - 1\right) : \mathbb{R}^3 \to \mathbb{S}^3 \setminus \{(0,0,0,1)\}$$
$$\tilde{\sigma}^{-1}(x) = -\sigma^{-1}(-x) : \mathbb{R}^3 \to \mathbb{S}^3 \setminus \{(0,0,0,-1)\}$$

So under the usual identification of $\mathbb{C}^2$ with $\mathbb{R}^4$ we have

$$\gamma_z(t) = \left(x^1\cos t - y^1\sin t, y^1\cos t + x^1\sin t, x^2\cos t - y^2\sin t, y^2\cos t + x^2\sin t\right)$$

where $z^j = x^j + iy^j$. Then direct computation gives

$$\sigma \circ \gamma_z(t) = \frac{1}{1 - y^2\cos t - x^2\sin(t)}\left(x^1\cos t - y^1\sin t, y^1\cos t + x^1\sin t, x^2\cos t - y^2\sin t\right)$$
$$\tilde{\sigma} \circ \gamma_z(t) = \frac{1}{1 + y^2\cos t + x^2\sin t}\left(x^1\cos t - y^1\sin t, y^1\cos t + x^1\sin t, x^1\cos t - y^1\sin t\right)$$

Which is smooth by the quotient rule.

Part b

Differentiating our result from part (a) we find

$$\begin{aligned}\sigma\circ\gamma_z'(t) = \frac{1}{\left(-1+y^2\cos t + x^2\sin t\right)^2}\Big(&x^1x^2 + y^1y^2 - x^1\sin t - y^1\cos t,\\ &x^2y^1 - x^1y^2 + x^1\cos t - y^1\sin t,\\ &\left(x^2\right)^2 + \left(y^2\right)^2 - y^2\cos t - x^2\sin t\Big)\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}\circ\gamma_z'(t) = \frac{1}{\left(1+y^2\cos t + x^2\sin t\right)^2}\Big(&-x^1x^2 - y^1y^2 - x^1\sin t - y^1\cos t,\\ &-x^2y^1 + x^1y^2 + x^1\cos t - y^1\sin t,\\ &-\left(x^2\right)^2 - \left(y^2\right)^2 - y^2\cos t - x^2\sin t\Big)\end{aligned}$$

Showing this never vanishes is an exercise in algebra.

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Lee 3-7 *Let $\mathcal{M}$ be a smooth manifold then for any $p\in\mathcal{M}$ let $\mathcal{C}_p^\infty$ denote the algebra of germs of smooth real valued functions at p. Then let D_p denote the vector space of derivations of $\mathcal{C}_p^\infty$ at p. Show that $T_p\mathcal{M}$ is naturally isomorphic to D_p.*

Proof:

Consider the map $\Phi : T_p\mathcal{M} \to D_p$ defined by $X \mapsto \tilde{X}$ where $\tilde{X}\in D_p$ is defined by $\tilde{X}\left[f\right]_p = Xf$ where $\left[f\right]_p$ is the germ of f at p.

First we need to show that $\tilde{X}$ is well defined. To do this we let $f\neq g$ be smooth real valued functions such that $\left[f\right]_p = \left[g\right]_p$.
Then on some neighborhood U of p we know that $f\mid_U = g\mid_U$ by definition of germs of functions. Then we have $Xf = Xg$ by Lee's Proposition 3.6.
Consequently, $\tilde{X}\left[f\right]_p = \tilde{X}\left[g\right]_p$ and thus $\tilde{X}$ is well defined.

Next we need to show that $\tilde{X}$ is a derivation of $\mathcal{C}_p^\infty$ at p i.e. that the map sends $T_p\mathcal{M}$ into D_p as claimed. To do this we let $\left[f\right]_p, \left[g\right]_p \in \mathcal{C}_p^\infty$ and let $c,d\in\mathbb{R}$ then we have

$$\begin{aligned}\tilde{X}\left(c\left[f\right]_p + d\left[g\right]_p\right) &= \tilde{X}\left(\left[cf+dg\right]_p\right) \quad \text{by definition of germs}\\ &= X\left(cf+dg\right) \quad \text{by definition of } \tilde{X}\\ &= c\left(Xf\right) + d\left(Xg\right) \quad \text{by linearity of } X\in T_p\mathcal{M}\\ &= c\tilde{X}\left[f\right]_p + d\tilde{X}\left[g\right]_p \quad \text{by definition of } \tilde{X}\end{aligned}$$

Furthermore,

$$\begin{aligned}\tilde{X}\left([f]_p\,[g]_p\right) &= \tilde{X}\,[fg]_p \quad \text{by definition of germs}\\ &= X\,(fg) \quad \text{by definition of } \tilde{X}\\ &= (Xf)\,g\,(p) + f\,(p)\,(Xg) \quad \text{since } X \in T_p\mathcal{M}\\ &= g\,(p)\,\tilde{X}\,[f]_p + f\,(p)\,\tilde{X}\,[g]_p \quad \text{by definition of } \tilde{X}\end{aligned}$$

Therefore, $\tilde{X} \in D_p$.

It now remains to show that this map is a linear isomorphism. To this end let $X, Y \in T_p\mathcal{M}$, $c, d \in \mathbb{R}$ and $[f]_p \in \mathcal{C}_p^\infty$. Then computing we find

$$\begin{aligned}\Phi\,(cX + dY)\,[f]_p &= (cX + dY)\,f\\ &= (cX)\,f + (dY)\,f\\ &= c\,(Xf) + d\,(Yf)\\ &= c\Phi\,(X)\,[f]_p + d\Phi\,(X)\,[f]_p\end{aligned}$$

Therefore Φ is linear.

Next, let $\delta \in D_p$. Then define $Xf = \delta\,[f]_p$. This uniquely defines a vector $X \in T_p\mathcal{M}$ by Lee's Lemma 3.11 and the definition of $T_p\mathcal{M}$ on page 65.

Finally, $X \in \ker\Phi$ if and only if $\tilde{X}\,[f]_p = 0$ for all $[f]_p\,\mathcal{C}_p^\infty$.
Applying the definition of $\tilde{X}$ we see that $X \in \ker\Phi$ if and only if $Xf = 0$ for all smooth real valued functions f defined in a neighborhood of p.
But this can only happen if $X \equiv 0$. This can be made transparent by working in local coordinates x^i in a neighborhood of p and expanding X in a basis $\left\{\frac{\partial}{\partial x^i}\,|_p\right\}$.

Therefore, $\Phi : T_p\mathcal{M} \to D_p$ is a linear bijection and hence $T_p\mathcal{M}$ and D_p are isomorphic as vector spaces.

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Lee 3-8*Let $\mathcal{M}$ be a smooth manifold. For any $p \in \mathcal{M}$ let $\mathcal{C}_p$ denote the set of all smooth curves $\gamma : J \to \mathcal{M}$ such that $0 \in J$ and $\gamma(0) = p$. Now define an equivalence relation on $\mathcal{C}_p$ by $\gamma_1 \tilde{\gamma}_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for all smooth f defined on some neighborhood of p. Let $\mathcal{V}_p$ denote the set of equivalence classes. Show that $\Phi : \mathcal{V}_p \to T_p\mathcal{M}$ defined by $\Phi[\gamma] = \gamma'(0)$ is well defined and gives a 1-1 correspondence.*

Proof:

First we need to show that Φ is well defined. To do this we fix $[\gamma_1] = [\gamma_2] \in \mathcal{V}_p$ such that $\gamma_1 \neq \gamma_2$.
Then we know $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for all smooth f defined on a neighborhood of p.
In particular, if we take f to be the map $f(x) = 1$ then f is smooth and the chain rule implies $(f \circ \gamma)'(0) = \gamma'(0)$ whence $\gamma_1'(0) = \gamma_2'(0)$.[2]
Therefore $\Phi[\gamma_1] = \Phi[\gamma_2]$ and hence Φ is well defined.

Next we want to show that Φ is injective. To do this we let $[\gamma_1], [\gamma_2] \in \mathcal{V}_p$ such that $\Phi[\gamma_1] = \Phi[\gamma_2]$.
Therefore $\gamma_1'(0) = \gamma_2'(0)$.
So if f is a smooth function defined on a neighborhood of p then we know $(f \circ \gamma_1)'(0) = f_*\gamma_1'(0) = f_*\gamma_2'(0) = (f \circ \gamma_1)'(0)$.
Since this was true for any smooth function defined on a neighborhood of p we can conclude that $[\gamma_1] = [\gamma_2]$ in $\mathcal{V}_p$ and hence Φ is injective.

Finally, we need to show that Φ is surjective. To do this we let $X \in T_p\mathcal{M}$.
Then by Lee's Lemma 3.11 we know that $\exists \gamma$ a smooth curve on $\mathcal{M}$ such that $\gamma(0) = p$ and $\gamma'(0) = X$.
Therefore $X = \gamma'(0) = \Phi[\gamma]$.
Since X was arbitrary we can conclude that Φ is surjective.

■

[2] alternatively one can take f to be the i^{th} coordinate function defined on some coordinate neighborhood of p. This then implies $\left(\gamma_1^i\right)'(0) = \left(\gamma_2^i\right)'(0)$ for each i.

CFB: A.2.3.1 *Show that*

$$SL(V) = \{g \in V \otimes V^* \mid \det(g) = 1\}$$
$$O(V,Q) = \{g \in V \otimes V^* \mid Q(v,w) = Q(gv,gw)\ \forall v,w \in V\}$$ *for a nondegenerate bilinear form $Q SO(V,Q)$*

are matrix Lie groups. i.e. that they are smooth submanifolds of $GL(V)$ that are closed under multiplication and inverses.

Proof:

First we'll show that each of these are subgroups of $GL(V)$.
For $SL(V)$ we let $g,h \in SL(V)$ then $\det(gh) = \det(g)\det(h) = 1$. Furthermore, $h \in SL(V)$ implies $h \in GL(V)$ and so let h^{-1} denote the inverse of h in $GL(V)$ then $\det(h^{-1}) = \frac{1}{\det h} = 1$. Thus $gh, h^{-1} \in SL(V)$ for all $g,h \in SL(V)$. Therefore $SL(V)$ is a subgroup of $GL(V)$.
For $O(V,Q)$ let $g,h \in O(V,Q)$, let h^{-1} be the inverse of h in $GL(V)$ and let $v,w \in V$ be arbitrary vectors. Then $h(v), h(w), h^{-1}(v), h^{-1}(w) \in V$ and so we have

$$\begin{aligned} Q((gh)v,(gh)w) &= Q(g(hv), g(hw)) \\ &= Q(hv,hw) \quad \text{since } g \in O(V,Q) \\ &= Q(v,w) \quad \text{since } h \in O(V,Q) \end{aligned}$$

and

$$\begin{aligned} Q\left(h^{-1}v, h^{-1}w\right) &= Q\left(h\left(h^{-1}v\right), h\left(h^{-1}w\right)\right) \quad \text{since } h \in O(V,Q) \\ &= Q(v,w) \quad \text{since } hh^{-1} = 1 \in GL(V) \end{aligned}$$

Therefore $gh, h^{-1} \in O(V,Q)$ for all $g,h \in O(V,Q)$ and thus $O(V,Q)$ is a subgroup of $GL(V)$.
Finally, note that the intersection of two subgroups is again a subgroup and so $SO(V,Q) = O(V,Q) \cap SL(V)$ is a subgroup of $GL(V)$.

So it remains to show that these subgroups are submanifolds. Again we proceed case by case beginning with $SL(V)$.
First note that for $\det : GL(V) \to \mathbb{R}^\times$ we have $SL(V) = \det^{-1}\{1\}$.[3]
Then computing the differential one finds that for any $A,B \in GL(V)$ we have

$$d(\det)_A(B) = \det(A)\, tr\left(A^{-1}B\right)$$

So if we take $A = B$ then $d(det)_A(A) = n\det(A)$ which cannot vanish since $A \in GL(V)$. Thus $d(det) \neq 0$ on $GL(V)$ and hence is a submersion. In particular 1 is a regular value of det and so $SL(V)$ is an embedded submanifold of dimension (n^2-1).

Next, if we fix a basis $v_1,\dots,v_n$ of V with dual basis v^i such that $Q(v^i,v_j) = \delta_i^j$ then as

[3]One can use this to establish that $SL(V)$ is a subgroup since $\det : GL(V) \to \mathbb{R}^\times$ is a homomorphism of multiplicative groups and $SL(V)$ is its kernel.

we'll see in this basis the elements of $O(V,Q)$ are exactly those elements $A \in GL(V)$ such that $AA^T = \mathbb{I}$ (see the next problem for details) that is we can work with $GL_n\mathbb{R}$ instead of $GL(V)$ and $O(n,\mathbb{R})$ instead of $O(V,Q)$. Furthermore, working in this basis, we define $S_n\mathbb{R} \subset GL(V)$ to be the set of symmetric matrices. Then we consider the map

$$\Phi : GL(n,\mathbb{R}) \to S_n\mathbb{R} = \varphi(A) = AA^T$$

Next, we note that $T_A GL(n,\mathbb{R}) \cong M_n\mathbb{R}$, the set of $n \times n$ matrices over $\mathbb{R}$ and $T_{\Phi(A)} S_n\mathbb{R} \cong S_n\mathbb{R}$.
So if we fix $A \in GL(n,\mathbb{R})$ and $B \in M_n\mathbb{R} \cong T_A GL(n,\mathbb{R})$ then we can form the curve $\gamma = A + Bt$ and use it to compute $\Phi_* : T_A GL(n,\mathbb{R}) \to T_{\Phi(A)} S_n\mathbb{R}$ as follows

$$\Phi_* B = \Phi_* \gamma'(0) = (\Phi \circ \gamma)'(0) = \frac{d}{dt}|_{t=0}\, \Phi(A + tB) = B^T A + A^T B$$

So for arbitrary $C \in S_n\mathbb{R}$ we have that $\Phi_*\left(\frac{1}{2}C\right) = \frac{1}{2}\left(C + C^T\right) = C$ and so $\mathbb{I}$ is a regular value of Φ.
Therefore $O(n,\mathbb{R}) = \Phi^{-1}(\mathbb{I})$ is an embedded submanifold.

Finally, one can show that $SO(V,Q)$ is simply a connected component of $O(V,Q)$ and hence is a submanifold.

■

CFB A.2.3.1*Suppose $v_1, \ldots, v_n$ is an orthonormal basis with respect to Q determine which matrices correspond to $SL(V)$, $O(V,Q)$ and $SO(V,Q)$.*

Proof:

Since the determinant function is independent of basis choice we can immediately deduce that $SL(V) \cong SL_n\mathbb{R}$.
Next, letting $\left[A_i^j\right]$ be the matrix representation of A with respect to this basis we have

$$A = A_i^j v_j \otimes v^i$$

where we're enforcing Einstein summation convention and we have v^j dual to v_i. Then we have

$$\begin{aligned}\delta_{ab} &= Q\left(v_a, v_b\right)\\ &= Q\left(Av_a, Av_b\right)\\ &= Q\left(A_i^j v_j \otimes v^i\left(v_a\right), A_k^\ell v_\ell \otimes v^k\left(v_b\right)\right)\\ &= A_i^j A_k^\ell Q\left(v_j \otimes v^i\left(v_a\right), v_\ell \otimes v^k\left(v_b\right)\right) \quad \text{by bilinearity of } Q\\ &= A_a^j A_b^\ell Q\left(v_j, v_k\right) \quad \text{since } v^\alpha\left(v_\beta\right) = \delta_\beta^\alpha \ \forall \alpha, \beta\\ &= A_a^j A_b^\ell \delta_{j,k}\\ &= A_a^j A_b^j\\ &= \left(AA^T\right)_a^b\end{aligned}$$

Thus $AA^T = \mathbb{I}$ in this basis. So we can conclude that $O\left(V, Q\right) \cong O\left(n, \mathbb{R}\right)$.
Finally, since $SO\left(V, Q\right) = O\left(V, Q\right) \cap SL\left(V\right)$ we have $SO\left(V, Q\right) \cong SO\left(n, \mathbb{R}\right)$.

■

CFB A.2.3.3 *Every matrix in $SO(3)$ represents a rotation fixing a line through the origin in $\mathbb{R}^3$. Is the same true for matrices in $SO(4)$?*

Proof:

Let $A \in SO\left(n, \mathbb{R}\right)$ then we have

$$\begin{aligned}\det\left(A - \mathbb{I}\right) &= \det\left(A^T A\left(A - \mathbb{I}\right)\right) \quad \text{Since } AA^T = \mathbb{I}\\ &= \det\left(A\right)\det\left(A^T A - A^T\right)\\ &= \det\left(-\left(A^T - \mathbb{I}\right)\right) \quad \text{since } \det\left(A\right) = 1 \text{ and } AA^T = \mathbb{I}\\ &= \left(-1\right)^3 \det\left(\left(A - \mathbb{I}\right)^T\right) = \det\left(A - \mathbb{I}\right)\end{aligned}$$

Since our field is not of characteristic two we can conclude that $\det\left(A - \mathbb{I}\right) = 0$ and consequently 1 is an eigenvalue of A.
Thus $\exists v \in V$ such that $Av = v$ and hence the eigenspace of v is fixed i.e. any $A \in SO\left(n, \mathbb{R}\right)$ fixes some linear subspace of $\mathbb{R}^3$ aka a line.

This is not true of $SO\left(4, \mathbb{R}\right)$ to see this consider the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

This matrix has eigenvalues $\pm i$ each with multiplicity 2 and so this matrix fixes no real linear subspace of $\mathbb{R}^4$ (if it did this would lead to an real eigenvalue).

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