BORDER SUBRANK OF TENSORS

A Thesis

by

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ABSTRACT

The subrank, border subrank, and asymptotic subrank play central roles in several areas including algebraic complexity theory and quantum information theory. The set of maximal subrank tensors is understood. However, less things were know about maximal border subrank tensors. We prove a lower bound on the dimension of the set of maximal border subrank tensors. This is the first such bound of its type.

DEDICATION

To my mother, father, and sisters.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

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TABLE OF CONTENTS

Page

ABSTRACT ii

DEDICATION iii

CONTRIBUTORS AND FUNDING SOURCES iv

TABLE OF CONTENTS v

1. INTRODUCTION 1

1.1 Motivations 4
1.1.1 Complexity Theory 4
1.1.2 Quantum Information Theory 5
1.2 Subrank and border subrank 6

2. PREPARATION FOR THE PROOF OF MAIN THEOREM 8

2.1 Symmetry Group 8
2.1.1 Symmetry group of the unit tensor $M_{\langle 1\rangle}^{\oplus n}$ 9
2.2 The nullcone $\mathcal{N}_{G_{M_{\langle 1\rangle}^{\oplus n}}}$ 11

3. PROOF OF THE MAIN THEOREM 15

3.1 A lower bound for the dimension of $\underline{\mathbf{Q}}_{\max}$ 15

4. SUMMARY AND CONCLUSIONS 19

4.1 Summary 19
4.2 Further Study 19
4.2.1 Find the nullcone $\mathcal{N}_{G_{M_{\langle 1\rangle}^{\oplus n}}}$ 19
4.2.2 The set of all maximal border subrank tensors 21
4.2.3 Border subrank for tensors with higher number of factors 21

REFERENCES 22

1. INTRODUCTION*

Let A, B, and C be n-dimensional vector spaces over $\mathbb{C}$, and let

$$A^* := \{f : A \to \mathbb{C} \mid f \text{ is linear } \}$$

denote the dual vector spaces of A. Similarly, B^* and C^* are the dual vector spaces of B and C, respectively. Let $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ respectively be bases of A, B, and C, and $\{\alpha_i\}$, $\{\beta_i\}$, and $\{\gamma_i\}$ the corresponding dual bases of the dual spaces A^*, B^*, and C^*, respectively.

The 2-way tensor space $A \otimes B$ is the linear space which consists of all bilinear maps from $A^* \times B^*$ to $\mathbb{C}$, or, equivalently, the space of all linear maps from A^* to B. If $a \in A$ and $b \in B$, one can define a linear map $a \otimes b : A^* \to B$ by $\alpha \mapsto \alpha(a)b$. Any element in the tensor space $A \otimes B$ can be written as a sum of such linear maps. A basis of $A \otimes B$ is $\{a_i \otimes b_j\}_{i,j=1,\dots,n}$.

The 3-way tensor space $A \otimes B \otimes C$ is the linear space which consists of all trilinear maps from $A^* \times B^* \times C^*$ to $\mathbb{C}$, or, equivalently, the space of all bilinear maps from $A^* \times B^*$ to C. For $a \in A$, $b \in B$, and $c \in C$, one can define the trilinear map

$$\begin{aligned} a \otimes b \otimes c : A^* \times B^* \times C^* &\to \mathbb{C} \\ (\alpha, \beta, \gamma) &\mapsto \alpha(a)\beta(b)\gamma(c). \end{aligned}$$

A trilinear map is said to be of *rank one* if it is in the form $a \otimes b \otimes c$ for some $a \in A$, $b \in B$, and $c \in C$. Any element in the tensor space $A \otimes B \otimes C$ can be written as a sum of rank one trilinear maps. The *rank* of a trilinear map $T \in A \otimes B \otimes C$ is the smallest integer r such that T can be written as the sum of r rank one tensors. A basis of $A \otimes B \otimes C$ is $\{a_i \otimes b_j \otimes c_k\}_{i,j,k=1,\dots,n}$, and any

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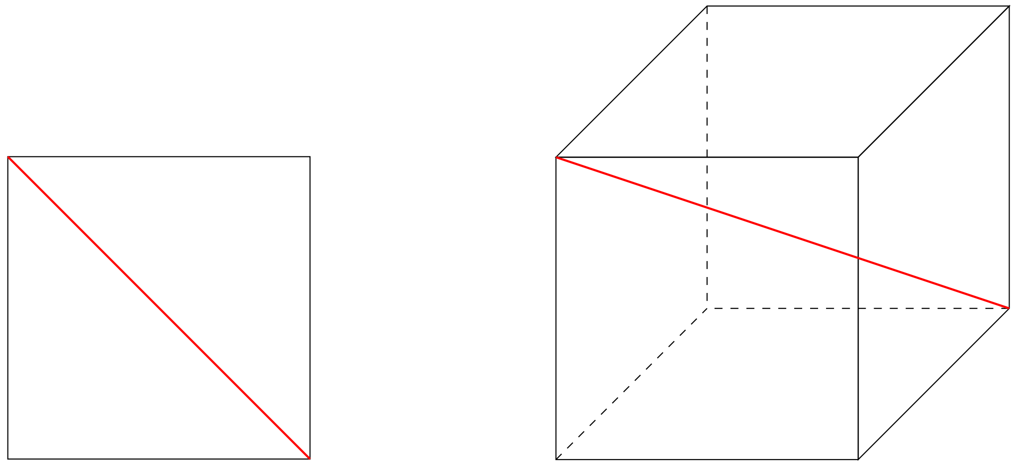

(a) A diagonal matrix (b) A diagonal 3-way tensor

Figure 1.1: Diagonal tensors

$T \in A \otimes B \otimes C$ can be written as

$$T = \sum_{i,j,k} T_{ijk} a_i \otimes b_j \otimes c_k$$

for some $T_{ijk} \in \mathbb{C}$.

For any positive integer r, define the unit tensor in $\mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$ (or size r diagonal tensor) with respect to the given bases

$$M_{\langle 1 \rangle}^{\oplus r} := \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i,$$

where $\{e_i\}$ is a basis for $\mathbb{C}^r$. One can also define the rank in the following way.

Definition 1.0.1. The *rank* of T, denoted $\mathbf{R}(T)$, is the minimal non-negative integer r such that

$$T \in (\mathrm{Hom}(\mathbb{C}^r, A) \times \mathrm{Hom}(\mathbb{C}^r, B) \times \mathrm{Hom}(\mathbb{C}^r, C)) \cdot M_{\langle 1 \rangle}^{\oplus r}.$$

That is $T = (\varphi_1 \otimes \varphi_2 \otimes \varphi_3) \cdot M_{\langle 1 \rangle}^{\oplus r} := \sum_{i=1}^{r} \varphi_1(e_i) \otimes \varphi_2(e_i) \otimes \varphi_3(e_i)$ for some $\varphi_1 \in \mathrm{Hom}(\mathbb{C}^r, A)$, $\varphi_2 \in \mathrm{Hom}(\mathbb{C}^r, B)$, and $\varphi_3 \in \mathrm{Hom}(\mathbb{C}^r, C)$. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the minimal non-negative integer r such that

$$T \in \overline{(\mathrm{Hom}(\mathbb{C}^r, A) \times \mathrm{Hom}(\mathbb{C}^r, B) \times \mathrm{Hom}(\mathbb{C}^r, C)) \cdot M_{\langle 1 \rangle}^{\oplus r}}.$$

Here and throughout, the overline denotes Zariski closure.

For a 2-way tensor $f \in A \otimes B$, as it is a linear map $f : A^* \to B$, by choosing bases, we can view f as an n by n matrix. One way to find the rank of a matrix is to count the number of 1's on the diagonal of its normal form, which is a diagonal matrix with only 0 and 1 obtained from its original matrix by undergoing transformations on the rows and columns. However, for a general 3-way tensor $T \in A \otimes B \otimes C$, there is no hope to get its normal form. Instead of applying general linear transformations, we may consider the maximal size of the possible diagonal tensors that can be obtained from the tensor T by applying linear transformations. We use *subrank* to measure this quantity.

Definition 1.0.2. The *subrank* of T, denoted $\mathbf{Q}(T)$, is the maximal non-negative integer s such that

$$M_{\langle 1 \rangle}^{\oplus s} \in (\mathrm{Hom}(A, \mathbb{C}^s) \times \mathrm{Hom}(B, \mathbb{C}^s) \times \mathrm{Hom}(C, \mathbb{C}^s)) \cdot T.$$

The *border subrank* of T, denoted $\underline{\mathbf{Q}}(T)$, is the maximal non-negative integer s such that

$$M_{\langle 1 \rangle}^{\oplus s} \in \overline{(\mathrm{Hom}(A, \mathbb{C}^s) \times \mathrm{Hom}(B, \mathbb{C}^s) \times \mathrm{Hom}(C, \mathbb{C}^s)) \cdot T}.$$

Remark 1.0.3. For any $T \in A \otimes B \otimes C$, we have inequalities of those quantities

$$\mathbf{Q}(T) \leq \underline{\mathbf{Q}}(T) \leq \underline{\mathbf{R}}(T) \leq \mathbf{R}(T) \text{ and } 0 \leq \mathbf{Q}(T) \leq \underline{\mathbf{Q}}(T) \leq n.$$

For $s \leq n$, we may view the size s diagonal tensor $M_{\langle 1 \rangle}^{\oplus s}$ as a tensor in $A \otimes B \otimes C$ by letting $M_{\langle 1 \rangle}^{\oplus s} = \sum_{i=1}^{s} a_i \otimes b_i \otimes c_i$. Then one can prove that the subrank of T is the maximal non-negative integer $s \leq n$ such that

$$M_{\langle 1 \rangle}^{\oplus s} \in (\mathrm{End}(A) \times \mathrm{End}(B) \times \mathrm{End}(C)) \cdot T$$

and the border subrank of T is the maximal non-negative integer $s \leq n$ such that

$$M_{\langle 1 \rangle}^{\oplus s} \in \overline{(\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)) \cdot T}.$$

Note that these definitions are independent of the choice of bases, and they are invatiant under the action of the general linear group $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$. Among the four quantities, border subrank is the least understood.

1.1 Motivations

1.1.1 Complexity Theory

The complexity of matrix multiplication is measured by the *exponent of matrix multiplication*, a constant that is denoted ω. It is defined as the smallest number such that for all $\epsilon > 0$, as a function in n, the rank of the $n \times n \times n$ matrix multiplication tensor is $O(n^{\omega+\epsilon})$. It is known that ω is less than 3 and at least 2. In 1969 [1], Strassen showed the first non-trivial upper bound $\omega \leq \log_2 7$. Several methods for finding an upper bound for ω or trying to prove $\omega = 2$ have been developed. A *barrier* [2] for a method is any lower bound for all upper bounds on ω that can be obtained by that method. When a tensor is of big subrank (as explained in the next two paragraphs), it is less likely that it will affected by the barriers, and when it is of maximal border subrank, it is not subject to any barrier. On the other hand, if the border rank is large, since it is not known how to compute the asymptotic rank (defined below), it is effectively useless. The tensor rank and border rank measure the “cost” of a tensor and the subrank and border subrank measure the “value” of a tensor.

The most effective (so far) method to find upper bounds on ω is the *Laser method*, which is based on the work of Strassen [3]. The idea is to study an intermediate tensor T which can be proven to have low border rank and is "close to" being a matrix multiplication tensor. For $N \in \mathbb{N}$ and $T \in A \otimes B \otimes C$, define the Kronecker power $T^{\boxtimes N}$ to be the N-th tensor power of T viewed as a three factor tensor, namely an element of $(A^{\otimes N}) \otimes (B^{\otimes N}) \otimes (C^{\otimes N})$. For $T \in A \otimes B \otimes C$ and $T' \in A' \otimes B' \otimes C'$ where A', B', and C' are vector spaces, we say T *restricts* to T' if

$$T' \in (\mathrm{Hom}(A, A') \times \mathrm{Hom}(B, B') \times \mathrm{Hom}(C, C')) \cdot T.$$

If some big Kronecker power $T^{\boxtimes N}$ restricts to a big direct sum $\oplus_i M_{\langle n_i, n_i, n_i \rangle}$ of matrix multiplication tensors, then by [4] we have an upper bound for ω. In [5], Strassen defined the asymptotic rank of T to be $\underset{\sim}{\mathbf{R}}(T) := \lim_{N\to\infty} \left(\mathbf{R}(T^{\boxtimes N})\right)^{1/N}$ and the asymptotic subrank of T to be $\underset{\sim}{\mathbf{Q}}(T) := \lim_{N\to\infty} \left(\mathbf{Q}(T^{\boxtimes N})\right)^{1/N}$, and he proved that they are well-defined. When $\underset{\sim}{\mathbf{R}}(T)/\underset{\sim}{\mathbf{Q}}(T) > 1$, the tensor T cannot be used to prove that $\omega = 2$. [2]

Asymptotic rank and asymptotic subrank may also be defined using border rank and border subrank. The tensors of maximal border subrank are not subject to barriers for their utility in the Laser method. More generally this is true of tensors of maximal asymptotic subrank, but that is more difficult to determine. Thus one motivation for this project is to expand the list of tensors that could potentially be used to prove $\omega = 2$ via the Laser method.

The rank and border rank of a generic tensor are the same and are equal to the maximal border rank. However, the behavior of the subrank and border subrank are different from the behavior of the rank and border rank. In 2022 [6], Derksen, Makam, and Zuiddam proved that the generic subrank of tensors have bounds $3(\lfloor\sqrt{n/3 + 1/4} - 1/2\rfloor) \leq \mathbf{Q}(n) \leq \lfloor\sqrt{3n-2}\rfloor$. In particular, the generic subrank is not maximal. The dimension of the set of maximal border subrank tensors is not even known. We give a lower bound of the dimension of the set of maximal border subrank tensors and a description of some members in that set.

1.1.2 Quantum Information Theory

There are connections between certain problems in algebraic complexity theory and questions in studying entanglement through stochastic local operations and classical communication

(SLOCC). Subrank measures the "rate" at which GHZ states can be distilled from a tripartite pure quantum state, via SLOCC. A GHZ state is a size two diagonal tensor $M_{\langle 1\rangle}^{\oplus 2} = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$. The asymptotic subrank of a tripartite pure quantum state T can be interpreted by using SLOCC (see [7] for definition of SLOCC). That is $\underset{\sim}{\mathbf{Q}}(T) = 2^s$ where

$$s = \max\{r : T^{\otimes k} \overset{\text{SLOCC}}{\dashrightarrow} (M_{\langle 1\rangle}^{\oplus 2})^{\otimes(rk - o(k))}\}.$$

The capacity of a quantum channel has the same form as the logs of the asymptotic ranks and asymptotic subranks. By the Holevo-Schumacher-Westmorland theorem (see [8], Theorem 8.27), the classical capacity of a quantum channel Λ is

$$\lim_{k\to\infty} \frac{\chi(\Lambda^{\otimes k})}{k},$$

where $\chi(\Lambda^{\otimes k})$ is the Holevo capacity of the channel $\Lambda^{\otimes k}$. By the quantum capacity theorem (see [8], Theorem 8.55), the quantum capacity of a quantum channel Λ is

$$\lim_{k\to\infty} \frac{I_C(\Lambda^{\otimes k})}{k},$$

where $I_C(\Lambda^{\otimes k})$ is the maximum coherent information of the channel $\Lambda^{\otimes k}$. These capacity theorems may be compared with the logs of the asymptotic ranks and subranks:

$$\log(\underset{\sim}{\mathbf{R}}(T)) = \lim_{k\to\infty} \frac{\log(\mathbf{R}(T^{\boxtimes k}))}{k} \text{ and } \log(\underset{\sim}{\mathbf{Q}}(T)) = \lim_{k\to\infty} \frac{\log(\mathbf{Q}(T^{\boxtimes k}))}{k}.$$

1.2 Subrank and border subrank

For any tensor T, it is clear that $\mathbf{Q}(T) \leq \underline{\mathbf{Q}}(T)$ By the definition of subrank and border subrank, the maximal possible subrank and border subrank for a tensor in $A \otimes B \otimes C$ is n and the unit tensor $M_{\langle 1\rangle}^{\oplus n}$ has the maximal subrank. The following proposition was "known to the experts" but we did not find it in the literature.

Proposition 1.2.1. *The orbit of the unit tensor* $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C) \cdot M_{\langle 1 \rangle}^{\oplus n}$ *is the set of all maximal subrank tensors.*

Proof. Let T be a tensor of maximal subrank. Then there exist $X \in \mathrm{End}(A)$, $Y \in \mathrm{End}(B)$, and $Z \in \mathrm{End}(C)$ such that $M_{\langle 1 \rangle}^{\oplus n} = (X \otimes Y \otimes Z) \cdot T \in \mathrm{im}(X) \otimes \mathrm{im}(Y) \otimes \mathrm{im}(Z)$. Since $M_{\langle 1 \rangle}^{\oplus n}$ cannot be put in a smaller tensor space, we get that X, Y, and Z are invertible. □

Since the subrank and border subrank of a tensor are invariant under multiplying by nonzero scalars, we can consider tensors in the projective space $\mathbb{P}(A \otimes B \otimes C)$. Let

$$\underline{\mathbf{Q}}_{\max} := \overline{\left\{[T] \in \mathbb{P}(A \otimes B \otimes C) : \underline{\mathbf{Q}}(T) = n\right\}}$$

be the closure of the set of the maximal border subrank tensors in projective space. Then

$$\overline{\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C) \cdot [M_{\langle 1 \rangle}^{\oplus n}]} \subset \underline{\mathbf{Q}}_{\max}$$

and we want to find and describe the set of $w \in A \otimes B \otimes C$ such that $M_{\langle 1 \rangle}^{\oplus n} + w$ is also of maximal border subrank. Our main theorem shows that the set of tensors not subject to barriers for the Laser method is large.

Theorem 1.2.2 (Main Theorem)**.** $\dim(\underline{\mathbf{Q}}_{\max}) \geq (2n^3 + 3n^2 - 2n - 3)/3.$

Proof sketch

We consider the nullcone determined by the symmetry group of the unit tensor. The Hilbert-Mumford criterion, see e.g., [9, Theorem 2.4.3], implies that the nullcone can be determined using a maximal torus of the symmetry group. The sum of the unit tensor and a general element in the nullcone is of maximal border subrank. By counting the dimension of the set of such tensors and its orbit closure, we find a lower bound of the dimension of the set of all maximal border subrank tensors.

2. PREPARATION FOR THE PROOF OF MAIN THEOREM*

2.1 Symmetry Group

Let $\tilde{G}$ be the group $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$, and let $\tilde{\mathfrak{g}} := \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ be the Lie algebra of $\tilde{G}$. For any tensor $T \in A \otimes B \otimes C$, the subgroup of $\tilde{G}$ preserving T is

$$\tilde{G}_T := \{g \in \mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C) : g \cdot T = T\}.$$

Then the Lie algebra of $\tilde{G}_T$ is

$$\tilde{\mathfrak{g}}_T = \{(x, y, z) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C) : (x, y, z).T = 0\}.$$

Since the action of $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ on $A \otimes B \otimes C$ is not faithful, we will consider the symmetry group of tensors, which is defined as follows.

Let $\Phi : \mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C) \longrightarrow \mathrm{GL}(A \otimes B \otimes C)$ be the natural group action of $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ on $A \otimes B \otimes C$. Then the kernel of Φ is

$$\ker(\Phi) = \{(\lambda \mathrm{Id}_A, \mu \mathrm{Id}_B, \nu \mathrm{Id}_C) : \lambda\mu\nu = 1\},$$

whose Lie algebra is the kernel of the differential of Φ

$$\ker(d\Phi) = \{(\lambda \mathrm{Id}_A, \mu \mathrm{Id}_B, \nu \mathrm{Id}_C) : \lambda + \mu + \nu = 0\},$$

which has dimension two. Let $G := \tilde{G}/\ker(\Phi)$, and let $\mathfrak{g}$ be its Lie algebra. Then there is a faithful action of G on $A \otimes B \otimes C$.

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For any $T \in A \otimes B \otimes C$, *the symmetry group of* T, denoted G_T, is the stabilizer of T in G

$$G_T := \{g \in G : g \cdot T = T\}.$$

The Lie algebra of the symmetry group of T is

$$\mathfrak{g}_T = \{X \in \mathfrak{g} : X.T = 0\}.$$

Note that for any tensor T, the subgroup $\tilde{G}_T$ of $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ preserving T contains the kernel of Φ and its Lie algebra $\tilde{\mathfrak{g}}_T$ contains the kernel of $d\Phi$. So the Lie algebra of the symmetry group of T is $\mathfrak{g}_T = \tilde{\mathfrak{g}}_T / \ker(d\Phi)$ and the symmetry group of T is $G_T = \tilde{G}_T / \ker(\Phi)$.

2.1.1 Symmetry group of the unit tensor $M_{\langle 1 \rangle}^{\oplus n}$

Note that the Lie algebra of the group preserving the unit tensor $M_{\langle 1 \rangle}^{\oplus n}$ is

$$\tilde{\mathfrak{g}}_{M_{\langle 1 \rangle}^{\oplus n}} = \{(x, y, z) \in \mathfrak{gl}(A) \times \mathfrak{gl}(B) \times \mathfrak{gl}(C) : (x, y, z).M_{\langle 1 \rangle}^{\oplus n} = 0\}.$$

Given $(x, y, z) \in \tilde{\mathfrak{g}}_{M_{\langle 1 \rangle}^{\oplus n}}$, we have that

$$\begin{aligned}
0 = (x, y, z).M_{\langle 1 \rangle}^{\oplus n} &= (x, y, z). \sum_{i=1}^{n} a_i \otimes b_i \otimes c_i \\
&= \sum_{i=1}^{n} [x(a_i) \otimes b_i \otimes c_i + a_i \otimes y(b_i) \otimes c_i + a_i \otimes b_i \otimes z(c_i)] \\
&= \sum_{i=1}^{n} \left(\sum_{s=1}^{n} x_{si} a_s \otimes b_i \otimes c_i + a_i \otimes \sum_{t=1}^{n} y_{ti} b_t \otimes c_i + a_i \otimes b_i \otimes \sum_{u=1}^{n} z_{ui} c_u \right),
\end{aligned}$$

which implies that $x_{si} = y_{ti} = z_{ui} = 0$ for $s, t, u \neq i$ and $x_{ii} + y_{ii} + z_{ii} = 0$ for all $i = 1, \ldots, n$. So the Lie algebra is

$$\tilde{\mathfrak{g}}_{M_{\langle 1 \rangle}^{\oplus n}} = \{(x, y, z) : x, y, z \text{ are diagonal } n \times n \text{ matrices and } x + y + z = 0\}.$$

Thus the dimension of the subgroup $\tilde{G}_{M_{\langle 1\rangle}^{\oplus n}}$ preserving the unit tensor is $2n$. Thus the dimension of the symmetry group of the unit tensor is

$$\dim G_{M_{\langle 1\rangle}^{\oplus n}} = \dim(\tilde{G}_{M_{\langle 1\rangle}^{\oplus n}}) - 2 = 2n - 2.$$

By Proposition 1.2.1, the dimension of the set of all maximal subrank tensors is $3n^2 - 2n$.

The symmetric group on n elements $\mathfrak{S}_n$ can be viewed as a subgroup of $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ by identifying $\sigma \in \mathfrak{S}_n$ with $(f_\sigma, g_\sigma, h_\sigma) \in \mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ where $f_\sigma(a_i) = a_{\sigma(i)}$, $g_\sigma(b_i) = b_{\sigma(i)}$, and $h_\sigma(c_i) = c_{\sigma(i)}$. It is clear that $\mathfrak{S}_n$ and the torus

$$\tilde{\mathbf{T}} = \{(\lambda, \mu, \nu) : \lambda, \mu, \nu \text{ are } n \times n \text{ diagonal matrices and } \lambda\mu\nu = \mathrm{Id}\}$$

are both subgroups of $\tilde{G}_{M_{\langle 1\rangle}^{\oplus n}}$. We claim that $\tilde{G}_{M_{\langle 1\rangle}^{\oplus n}} = \mathfrak{S}_n \ltimes \tilde{\mathbf{T}}$. Write the unit tensor $M_{\langle 1\rangle}^{\oplus n} = \sum_{i=1}^n a_i \otimes b_i \otimes c_i$ as a matrix whose entries are in C:

$$\begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$$

The determinant of this matrix is $c_1 c_2 \cdots c_n \in S^n C$. For a linear map $T : B^* \longrightarrow A$, we have $T^{\wedge n} : \Lambda^n B^* \longrightarrow \Lambda^n A$ and an element $(f, g) \in \mathrm{GL}(A) \times \mathrm{GL}(B)$ acts on $\Lambda^n T$ by

$$(f, g) \cdot T^{\wedge n} = \det(f)\det(g) T^{\wedge n}.$$

The determinant of the matrix presenting $(f, g, h) \cdot M_{\langle 1\rangle}^{\oplus n}$ for $(f, g, h) \in \mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ is

$$h \cdot (\det(f)\det(g) c_1 c_2 \cdots c_n) = \det(f)\det(g) h(c_1) h(c_2) \cdots h(c_n).$$

For $(f,g,h) \in \tilde{G}_{M_{\langle 1\rangle}^{\oplus n}}$, as $M_{\langle 1\rangle}^{\oplus n} = (f,g,h)\cdot M_{\langle 1\rangle}^{\oplus n}$, we have

$$c_1c_2\cdots c_n = \det(f)\det(g)h(c_1)h(c_2)\cdots h(c_n) \in S^nC.$$

By the unique factorization of polynomials, there exist $\sigma \in \mathfrak{S}_n$ and $\nu_1,\ldots,\nu_n \in \mathbb{C}$ such that $h(c_i) = \nu_i c_{\sigma(i)}$. Similarly, we have $\sigma',\sigma'' \in \mathfrak{S}_n$ and $\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n \in \mathbb{C}$ such that $f(a_i) = \lambda_i a_{\sigma'(i)}$ and $g(b_i) = \mu_i b_{\sigma''(i)}$. Since the unit tensor is fixed by (f,g,h), we see that $\sigma = \sigma' = \sigma''$ and $\lambda_i\mu_i\nu_i = 1$. So,

$$(f,g,h) = \sigma \circ \left(\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}, \begin{pmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_n \end{pmatrix}\right).$$

Given $\sigma \in \mathfrak{S}_n$ and $(\lambda,\mu,\nu) \in \tilde{\mathbf{T}}$, we have $\sigma(\lambda,\mu,\nu)\sigma^{-1} = (\lambda',\mu',\nu') \in \tilde{\mathbf{T}}$ where $\lambda'_{ii} = \lambda_{\sigma^{-1}(i)\sigma^{-1}(i)}$, $\mu'_{ii} = \mu_{\sigma^{-1}(i)\sigma^{-1}(i)}$, and $\nu'_{ii} = \nu_{\sigma^{-1}(i)\sigma^{-1}(i)}$. This gives the semidirect product $\tilde{G}_{M_{\langle 1\rangle}^{\oplus n}} = \mathfrak{S}_n \ltimes \tilde{\mathbf{T}}$.

Note that $\tilde{\mathbf{T}}$ contains the kernel of Φ and $\mathfrak{S}_n$ has trivial intersection with the kernel of Φ. We can consider the torus $\mathbf{T} := \tilde{\mathbf{T}}/\ker(\Phi)$ and see that the symmetry group of the unit tensor is $G_{M_{\langle 1\rangle}^{\oplus n}} = \mathfrak{S}_n \ltimes \mathbf{T}$.

2.2 The nullcone $\mathcal{N}_{G_{M_{\langle 1\rangle}^{\oplus n}}}$

Let

$$\mathcal{N}_{G_{M_{\langle 1\rangle}^{\oplus n}}} := \{v \in A\otimes B\otimes C : 0 \in \overline{G_{M_{\langle 1\rangle}^{\oplus n}}\cdot v}\},$$

denote the nullcone of the action of $G_{M_{\langle 1\rangle}^{\oplus n}}$ on $A\otimes B\otimes C$, and let

$$\mathcal{C} := \mathrm{Cone}(M_{\langle 1\rangle}^{\oplus n}, \mathcal{N}_{G_{M_{\langle 1\rangle}^{\oplus n}}}) = \left\{\left[M_{\langle 1\rangle}^{\oplus n} + w\right] \in \mathbb{P}(A\otimes B\otimes C) : w \in \mathcal{N}_{G_{M_{\langle 1\rangle}^{\oplus n}}}\right\}$$

be the projective cone over $\mathbb{P}\mathcal{N}_{G_{M_{\langle 1\rangle}^{\oplus n}}}$ with vertex $\left[M_{\langle 1\rangle}^{\oplus n}\right]$. Then we have $\overline{G\cdot\mathcal{C}} \subset \underline{\mathbf{Q}}_{\max}$.

Note that the nullcone defined by the maximal torus $\mathbf{T}$,

$$\mathcal{N}_{\mathbf{T}} := \{v \in A \otimes B \otimes C : 0 \in \overline{\mathbf{T} \cdot v}\}$$

is a subset of the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$. By the Hilbert-Mumford criterion, we have $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}} = G_{M_{\langle 1 \rangle}^{\oplus n}} \cdot \mathcal{N}_{\mathbf{T}}$. Thus we have that $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}} = \mathcal{N}_{\mathbf{T}}$ as $\mathbf{T}$ is normal in $G_{M_{\langle 1 \rangle}^{\oplus n}}$.

Let $x_{ijk} = \alpha_i \otimes \beta_j \otimes \gamma_k \in Sym(A \otimes B \otimes C)^*$. Then the coordinate ring of $A \otimes B \otimes C$ is the polynomial ring $\mathbb{C}[x_{ijk}]$. The nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}} = \mathcal{N}_{\mathbf{T}}$ is the zero set of all homogeneous polynomials of positive degree that are invariant under the action of $\mathbf{T}$, see e.g., [9, Lemma 2.4.2]. Note that $\mathbf{T}$ is a torus, and the monomials in $\mathbb{C}[x_{ijk}]$ are weight vectors of the action of $\mathbf{T}$:

$$\overline{(\lambda, \mu, \nu)} \cdot x_{ijk} = \frac{1}{\lambda_i \mu_j \nu_k} x_{ijk},$$

where $\overline{(\lambda, \mu, \nu)}$ is the image of (λ, μ, ν) in $\mathbf{T} = \tilde{\mathbf{T}} / \ker(\Phi)$ and λ_i, μ_j, ν_k are the i-th, j-th, and k-th diagonal term of λ, μ, ν, respectively. So the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$ is the zero set of monomials that are invariant under the action of $\mathbf{T}$, which implies that the nullcone is the union of linear spaces. Let I be the ideal generated by the monomials

$$x_{iii}, x_{iij}x_{jji}, x_{iji}x_{jij}, x_{ijj}x_{jii}, x_{ijk}x_{jki}x_{kij}$$

for distinct $i, j, k \in \{1, \ldots, n\}$. Since these monomials are invariant under the action of $\mathbf{T}$, the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$ is contained in the zero set of the ideal I. Since no variables in the generators of I are repeated and the number of the generators is $n + 3\binom{n}{2} + 2\binom{n}{3}$, the zero set of I is a union of linear spaces with dimension $n^3 - (n + 3\binom{n}{2} + 2\binom{n}{3})$.

Let W be a vector subspace of $A \otimes B \otimes C$ spanned by elements $a_i \otimes b_j \otimes c_k$ satisfying $1 \leq i, j, k \leq n$ and at least one of j and k less than i. For any $T = \sum_{i,j,k} T_{ijk} a_i \otimes b_j \otimes c_k \in W$, note that T_{ijk} is nonzero only when one of j and k is less than i, so we have that

$$x_{iii}(T) = T_{iii} = 0 \text{ for all } i = 1, \ldots, n$$

$$x_{iij}x_{jji}(T) = T_{iij}T_{jji} = 0 \cdot T_{jji} = 0 \text{ for all } 1 \leq i < j \leq n$$

$$x_{iji}x_{jij}(T) = T_{iji}T_{jij} = 0 \cdot T_{jij} = 0 \text{ for all } 1 \leq i < j \leq n$$

$$x_{ijj}x_{jii}(T) = T_{ijj}T_{jii} = 0 \cdot T_{jii} = 0 \text{ for all } 1 \leq i < j \leq n$$

$$x_{ijk}x_{jki}x_{kij}(T) = 0 \cdot T_{jki}T_{kij} = 0 \text{ for all } 1 \leq i < j < k \leq n \text{ or } 1 \leq i < k < j \leq n.$$

Then we have that W is contained in the zero set of I. Note that W has dimension

$$\dim(W) = n^3 - (n^2 + (n-1)^2 + \cdots + 2^2 + 1^2) = n^3 - \frac{n(n+1)(2n+1)}{6},$$

which is equal to the dimension of the zero set of I. Consider

$$c(t) = \left(\begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_n} \end{pmatrix}, \begin{pmatrix} t^{\mu_1} & & 0 \\ & \ddots & \\ 0 & & t^{\mu_n} \end{pmatrix}, \begin{pmatrix} t^{\nu_1} & & 0 \\ & \ddots & \\ 0 & & t^{\nu_n} \end{pmatrix} \right) \in G_{M_{\langle 1 \rangle}^{\oplus n}}$$

where $\lambda_k = 2^n - 2^{n-k+1}$ and $\mu_k = \nu_k = 2^{n-k} - 2^{n-1}$ for $k = 1, \ldots, n$. For $a_i \otimes b_j \otimes c_k \in W$, we have that

$$\begin{aligned} \lim_{t \to 0} c(t) \cdot (a_i \otimes b_j \otimes c_k) &= \lim_{t \to 0} t^{\lambda_i + \mu_j + \nu_k} (a_i \otimes b_j \otimes c_k) \\ &= \lim_{t \to 0} t^{2^{n-j} + 2^{n-k} - 2^{n-i+1}} (a_i \otimes b_j \otimes c_k) = 0 \end{aligned}$$

since $2^{n-j} + 2^{n-k} \geq 2^{n-i+1} + 1$. By the definition of the nullcone, we have that $W \subset \mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$. Since the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$ is in between two varieties having the same dimension, we can focus on the subset W, which has dimension

$$\dim(W) = n^3 - \frac{n(n+1)(2n+1)}{6} = \frac{4n^3 - 3n^2 - n}{6}.$$

Let $\mathcal{C}' := \mathrm{Cone}(M_{\langle 1 \rangle}^{\oplus n}, W)$ be the projective cone over $\mathbb{P}W$ with vertex $\left[M_{\langle 1 \rangle}^{\oplus n}\right]$. Then its dimension is

$$\dim(\mathcal{C}') = \dim \mathrm{Cone}(M_{\langle 1 \rangle}^{\oplus n}, W) = \frac{4n^3 - 3n^2 - n}{6}.$$

3. PROOF OF THE MAIN THEOREM*

3.1 A lower bound for the dimension of $\underline{\mathbf{Q}}_{\max}$

Note that $\mathcal{C}' \subset \mathcal{C} \subset \underline{\mathbf{Q}}_{\max}$ and $\underline{\mathbf{Q}}_{\max}$ is invariant under the action of G. The dimension of $\overline{G \cdot \mathcal{C}'}$ gives a lower bound of the dimension of $\underline{\mathbf{Q}}_{\max}$.

Proof of Theorem 1.2.2 (Main Theorem). Consider the map

$$\begin{aligned} \varphi : G \times \mathcal{C}' &\to \mathbb{P}(A \otimes B \otimes C) \\ (g, [v]) &\mapsto g \cdot [v] = [g \cdot v], \text{ where } g \in G \text{ and } [v] \in \mathcal{C}'. \end{aligned}$$

The orbit closure of $\mathcal{C}'$ is the closure of the image of the map φ. Then

$$\dim(\overline{G \cdot \mathcal{C}'}) = \dim(G) + \dim(\mathcal{C}') - \dim(\varphi^{-1}([u])),$$

where $[u]$ is a general point of the image of φ, and $\varphi^{-1}([u]) = \{(g, [v]) \in G \times \mathcal{C}' : g \cdot [v] = [u]\}$. Let $[u]$ be a general point of the image of φ. Then $u = g_0 \cdot v_0$ for some $g_0 \in G$ and $[v_0] \in \mathcal{C}'$. Define the transporter $\mathrm{Tran}_G(v_0, \mathcal{C}') := \{h \in G : h \cdot [v_0] \in \mathcal{C}'\}$, which is a subvariety of G. Note that the transporter $\mathrm{Tran}_G(v_0, \mathcal{C}')$ is isomorphic to the fiber $\varphi^{-1}([u])$ by

$$\begin{aligned} \mathrm{Tran}_G(v_0, \mathcal{C}') &\leftrightarrow \varphi^{-1}([u]) \\ h &\mapsto (g_0^{-1} h^{-1}, [h \cdot v_0]), \text{ where } h \in \mathrm{Tran}_G(v_0, \mathcal{C}') \\ g^{-1} g_0 &\leftarrow\!\shortmid (g, [v]), \text{ where } (g, [v]) \in \varphi^{-1}([u]). \end{aligned}$$

We may assume that $[v_0]$ is a general point in $\mathcal{C}'$. Then $[v_0] = [M_{\langle 1 \rangle}^{\oplus n} + w_0]$ for some $w_0 \in W$,

---

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and the transporter is

$$\begin{aligned}\operatorname{Tran}_G(v_0,\mathcal{C}') &:= \{g\in G : g\cdot[v_0]\in\mathcal{C}'\}\\ &= \{g\in G : g\cdot[M_{\langle 1\rangle}^{\oplus n}+w_0]\in\operatorname{Cone}(M_{\langle 1\rangle}^{\oplus n},W)\}\\ &= \{g\in G : g\cdot(M_{\langle 1\rangle}^{\oplus n}+w_0)=\lambda M_{\langle 1\rangle}^{\oplus n}+w' \text{ for some } \lambda\in\mathbb{C} \text{ and } w'\in W\}.\end{aligned}$$

The tangent space of the transporter $\operatorname{Tran}_G(v_0,\mathcal{C}')$ at the identity element of G is contained in the following subspace of the Lie algebra $\mathfrak{g}$ of G

$$\mathfrak{g}' := \{X\in\mathfrak{g} : X.(M_{\langle 1\rangle}^{\oplus n}+w_0)\in\langle M_{\langle 1\rangle}^{\oplus n},W\rangle\}\subset\mathfrak{g},$$

where $\langle M_{\langle 1\rangle}^{\oplus n},W\rangle$ is the linear subspace of $A\otimes B\otimes C$ spanned by $M_{\langle 1\rangle}^{\oplus n}$ and W.

Consider a Lie subalgebra of $\tilde{\mathfrak{g}}=\mathfrak{gl}(A)\oplus\mathfrak{gl}(B)\oplus\mathfrak{gl}(C)$:

$$\tilde{\mathfrak{g}}' := \{(x,y,z)\in\tilde{\mathfrak{g}} : (x,y,z).(M_{\langle 1\rangle}^{\oplus n}+w_0)\in\langle M_{\langle 1\rangle}^{\oplus n},W\rangle\}.$$

Then $\mathfrak{g}'=\tilde{\mathfrak{g}}'/\ker(d\Phi)$. Write $w_0=\sum_{i,j,k}w_{ijk}a_i\otimes b_j\otimes c_k\in W$. For $(x,y,z)\in\tilde{\mathfrak{g}}'$, we have the following equations

$$\begin{aligned}
&(\sigma,\sigma,\sigma) && x_{\sigma\sigma}+y_{\sigma\sigma}+z_{\sigma\sigma}+\sum_{i>\sigma}w_{i\sigma\sigma}x_{\sigma i}+\sum_{j<\sigma}w_{\sigma j\sigma}y_{\sigma j}+\sum_{k<\sigma}w_{\sigma\sigma k}z_{\sigma k}\\
&&&= x_{11}+y_{11}+z_{11}+\sum_{i>1}w_{i11}x_{1i} \text{ for all } 1\le\sigma\le n,\\
&(\sigma,\rho,\rho) && x_{\sigma\rho}+\sum_{i>\rho}w_{i\rho\rho}x_{\sigma i}+\sum_{j}w_{\sigma j\rho}y_{\rho j}+\sum_{k}w_{\sigma\rho k}z_{\rho k}=0 \text{ for all } 1\le\sigma<\rho\le n,\\
&(\sigma,\rho,\sigma) && y_{\rho\sigma}+\sum_{i}w_{i\rho\sigma}x_{\sigma i}+\sum_{j<\sigma}w_{\sigma j\sigma}y_{\rho j}+\sum_{k}w_{\sigma\rho k}z_{\sigma k}=0 \text{ for all } 1\le\sigma<\rho\le n,\\
&(\sigma,\sigma,\rho) && z_{\rho\sigma}+\sum_{i}w_{i\sigma\rho}x_{\sigma i}+\sum_{j}w_{\sigma j\rho}y_{\sigma j}+\sum_{k<\sigma}w_{\sigma\sigma k}z_{\rho k}=0 \text{ for all } 1\le\sigma<\rho\le n,\\
&(\sigma,\rho,\tau) && \sum_{i}w_{i\rho\tau}x_{\sigma i}+\sum_{j}w_{\sigma j\tau}y_{\rho j}+\sum_{k}w_{\sigma\rho k}z_{\tau k}=0 \text{ for all } 1\le\sigma<\rho,\tau\le n.
\end{aligned}$$

For $\sigma < \rho$, we use induction on σ to prove that $x_{\sigma\rho} = y_{\rho\sigma} = z_{\rho\sigma} = 0$. For $\sigma = 1$ and $\rho > 1$, equation (σ, ρ, ρ) implies that $x_{1\rho} = 0$ since $w_{1jk} = 0$ for any j, k. Then equations (σ, ρ, σ) and (σ, σ, ρ) imply that $y_{\rho 1} = z_{\rho 1} = 0$ for all $\rho > 1$. Now fix some $\sigma > 1$. Assume that $x_{\sigma'\rho} = y_{\rho\sigma'} = z_{\rho\sigma'} = 0$ for all $\sigma' < \sigma$ and $\rho > \sigma'$. For all $\rho > \sigma$, by the induction hypothesis, we have the equations

$$
\begin{aligned}
(\sigma, \rho, \rho) \quad & x_{\sigma\rho} = x_{\sigma\rho} + \sum_{j<\sigma} w_{\sigma j\rho} y_{\rho j} + \sum_{k<\sigma} w_{\sigma\rho k} z_{\rho k} = 0, \\
(\sigma, \rho, \sigma) \quad & y_{\rho\sigma} + \sum_{\rho>i>\sigma} w_{i\rho\sigma} x_{\sigma i} = y_{\rho\sigma} + \sum_{\rho>i>\sigma} w_{i\rho\sigma} x_{\sigma i} + \sum_{k<\sigma} w_{\sigma\rho k} z_{\sigma k} = 0, \text{ and} \\
(\sigma, \sigma, \rho) \quad & z_{\rho\sigma} + \sum_{\rho>i>\sigma} w_{i\sigma\rho} x_{\sigma i} = z_{\rho\sigma} + \sum_{\rho>i>\sigma} w_{i\sigma\rho} x_{\sigma i} + \sum_{j<\sigma} w_{\sigma j\rho} y_{\sigma j} = 0.
\end{aligned}
$$

Then we have that

$$
(x, y, z) = \left(\begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ x_{n1} & \cdots & \cdots & x_{nn} \end{pmatrix}, \begin{pmatrix} y_{11} & \cdots & \cdots & y_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & y_{nn} \end{pmatrix}, \begin{pmatrix} z_{11} & \cdots & \cdots & z_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & z_{nn} \end{pmatrix} \right)
$$

satisfy

$$
(\sigma, \sigma, \sigma) \quad x_{\sigma\sigma} + y_{\sigma\sigma} + z_{\sigma\sigma} = x_{11} + y_{11} + z_{11} \text{ for all } 1 < \sigma \leq n.
$$

Therefore we have $\mathfrak{g}'$ is of dimension

$$
\dim(\tilde{\mathfrak{g}}' / \ker(d\Phi)) = 3 \cdot \frac{(n+1)n}{2} - (n-1) - 2 = \frac{3n^2 + n - 2}{2}.
$$

Since the transporter $\mathrm{Tran}_G(v_0, \mathcal{C}')$ contains a Lie subgroup of G whose Lie algebra is $\mathfrak{g}'$, we have that

$$
\dim(\varphi^{-1}([u])) = \dim(\mathrm{Tran}_G(v_0, \mathcal{C}')) = \frac{3n^2 + n - 2}{2}.
$$

Then the dimension of $\overline{G \cdot \mathcal{C}'}$ is

$$\begin{aligned}
&\dim(G) + \dim(\mathcal{C}') - \dim(\varphi^{-1}([u])) \\
&\qquad = (3n^2 - 2) + \frac{4n^3 - 3n^2 - n}{6} - \frac{3n^2 + n - 2}{2} \\
&\qquad = \frac{2n^3 + 3n^2 - 2n - 3}{3},
\end{aligned}$$

which is a lower bound for the dimension of $\underline{\mathbf{Q}}_{\max}$.

$\square$

4. SUMMARY AND CONCLUSIONS

4.1 Summary

The set of all maximal border subrank tensors has dimension bounded above by the dimension of the whole space $\dim(A \otimes B \otimes C) = n^3$ and bounded below by the dimension of the set of all maximal subrank tensors which is $\dim\left(\overline{G \cdot M_{\langle 1 \rangle}^{\oplus n}}\right) = 3n^2 - 2n$. We showed that another nontrivial lower bound on the dimension of the set of all maximal border subrank tensors is around $\frac{2}{3}n^3$. The methods we used to prove this result include constructing certain tensors which are of maximal border subrank. We are hoping that such tensors can be utilized in the Laser method.

4.2 Further Study

In this section I list some future problems related to subrank and border subrank.

4.2.1 Find the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$*

Recall that

$$W = \langle a_i \otimes b_j \otimes c_k : \text{at least one of } j, k \text{ less than } i \rangle$$

is of dimension $(4n^3 - 3n^2 - n)/6$. Similarly, we can define

$$W' := \langle a_i \otimes b_j \otimes c_k : \text{at least one of } i, k \text{ less than } j \rangle$$

and

$$W'' := \langle a_i \otimes b_j \otimes c_k : \text{at least one of } i, j \text{ less than } k \rangle.$$

Remark that W' and W'' are both subsets of the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$. For $\sigma \in \mathfrak{S}_n$, define

$$\sigma(W) := \langle a_{\sigma(i)} \otimes b_{\sigma(j)} \otimes c_{\sigma(k)} : \text{at least one of } j, k \text{ less than } i \rangle.$$

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$\sigma(W')$ and $\sigma(W'')$ are defined similarly. Then we have

$$\bigcup_{\sigma \in \mathfrak{S}_n} (\sigma(W) \cup \sigma(W') \cup \sigma(W'')) \subset \mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}.$$

However this is not the whole nullcone and in fact that the nullcone is not equidimensional.

Example 4.2.1. Consider a one-parameter subgroup

$$c(t) = \left(\begin{pmatrix} t^5 & & 0 \\ & 1 & \\ 0 & & t^2 \end{pmatrix}, \begin{pmatrix} 1 & & 0 \\ & t & \\ 0 & & t^{-3} \end{pmatrix}, \begin{pmatrix} t^{-5} & & 0 \\ & t^{-1} & \\ 0 & & t \end{pmatrix} \right) \in G_{M_{\langle 1 \rangle}^{\oplus 3}}.$$

Let U be a subspace of $A \otimes B \otimes C$ generated by $a_i \otimes b_j \otimes c_k$ where $\lim_{t \to 0} c(t) \cdot a_i \otimes b_j \otimes c_k = 0$. Then U is a subset of the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus 3}}}$ and is of dimension $13 = \dim W$, but U is not any of $\sigma(W), \sigma(W'), \sigma(W'')$.

Example 4.2.2. Consider a one-parameter subgroup

$$c(t) = \left(\begin{pmatrix} t^{-2} & & 0 \\ & t^{-1} & \\ 0 & & 1 \end{pmatrix}, \begin{pmatrix} t^3 & & 0 \\ & t^{-2} & \\ 0 & & 1 \end{pmatrix}, \begin{pmatrix} t^{-1} & & 0 \\ & t^3 & \\ 0 & & 1 \end{pmatrix} \right) \in G_{M_{\langle 1 \rangle}^{\oplus 3}}.$$

Let U be a subspace of $A \otimes B \otimes C$ generated by $a_i \otimes b_j \otimes c_k$ where $\lim_{t \to 0} c(t) \cdot a_i \otimes b_j \otimes c_k = 0$. Then U is a subset of the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus 3}}}$ and is of dimension $12 < 13 = \dim W$. By considering the monomials listed in section 2.2 and these three monomials $x_{123}x_{211}x_{332}$, $x_{231}x_{322}x_{113}$, and $x_{132}x_{321}x_{213}$, we can see that U is a maximal component in the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus 3}}}$.

From the previous two examples, we see that the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$ is not the union of spaces obtained from W by applying permutations on bases or on order of A, B, and C, and it is even not equidimensional.

Problem. What are all components of the nullcone? How can we describe it?

4.2.2 The set of all maximal border subrank tensors

From the argument in section 2.2, we have that $M_{\langle 1 \rangle}^{\oplus n} + w$ is of maximal border subrank for some w in the nullcone $\mathcal{N}_{G_{M_{\langle 1 \rangle}^{\oplus n}}}$ of the action of $G_{M_{\langle 1 \rangle}^{\oplus n}}$. However we don't know if all tensors with maximal border subrank are in this form or lies in the orbit closure of the set of tensors in this form.

Problem. Is any maximal border subrank tensor in this form $M_{\langle 1 \rangle}^{\oplus n} + w$ or in the orbit closure of set of tensors in this form?

4.2.3 Border subrank for tensors with higher number of factors

In this paper, we only consider tensors in $(\mathbb{C}^n)^{\otimes 3}$. Note that we may define subrank and border subrank for d-way tensors, i.e., tensors in the tensor space with d factors $(\mathbb{C}^n)^{\otimes d}$. Similarly, we may define the unit tensor $\mathbf{u}_d(n) := \sum_{i=1}^{n} (e_i)^{\otimes d}$, its symmetry group $G_{\mathbf{u}_d(n)}$ in $\mathrm{GL}(n)^{\times d}$, and the nullcone $\mathcal{N}_{G_{\mathbf{u}_d(n)}}$ of the action of the symmetry group. Even for $d = 3$, we cannot find the whole nullcone. It may not be easy to find the dimension of the nullcone for arbitrary d.

Problem. How can we generalize this method to tensors with higher number of factors, i.e., tensors in $(\mathbb{C}^n)^{\otimes d}$? Or tensors in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_d}$, where n_j may not be all the same?

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