

**Cartan for Beginners:
Differential Geometry via Moving Frames
and Exterior Differential Systems**

Thomas A. Ivey and J.M. Landsberg

Contents

| | |
|---|----|
| Preface | ix |
| Chapter 1. Moving Frames and Exterior Differential Systems | 1 |
| §1.1. Geometry of surfaces in \mathbb{E}^3 in coordinates | 2 |
| §1.2. Differential equations in coordinates | 5 |
| §1.3. Introduction to differential equations without coordinates | 8 |
| §1.4. Introduction to geometry without coordinates: curves in \mathbb{E}^2 | 12 |
| §1.5. Submanifolds of homogeneous spaces | 15 |
| §1.6. The Maurer-Cartan form | 16 |
| §1.7. Plane curves in other geometries | 20 |
| §1.8. Curves in \mathbb{E}^3 | 23 |
| §1.9. Exterior differential systems and jet spaces | 26 |
| Chapter 2. Euclidean Geometry and Riemannian Geometry | 35 |
| §2.1. Gauss and mean curvature via frames | 36 |
| §2.2. Calculation of H and K for some examples | 39 |
| §2.3. Darboux frames and applications | 42 |
| §2.4. What do H and K tell us? | 43 |
| §2.5. Invariants for n -dimensional submanifolds of \mathbb{E}^{n+s} | 45 |
| §2.6. Intrinsic and extrinsic geometry | 47 |
| §2.7. Space forms: the sphere and hyperbolic space | 57 |
| §2.8. Curves on surfaces | 58 |
| §2.9. The Gauss-Bonnet and Poincaré-Hopf theorems | 61 |

| | |
|---|-----|
| §2.10. Non-orthonormal frames | 66 |
| Chapter 3. Projective Geometry | 71 |
| §3.1. Grassmannians | 72 |
| §3.2. Frames and the projective second fundamental form | 76 |
| §3.3. Algebraic varieties | 81 |
| §3.4. Varieties with degenerate Gauss mappings | 89 |
| §3.5. Higher-order differential invariants | 94 |
| §3.6. Fundamental forms of some homogeneous varieties | 98 |
| §3.7. Higher-order Fubini forms | 107 |
| §3.8. Ruled and uniruled varieties | 113 |
| §3.9. Varieties with vanishing Fubini cubic | 115 |
| §3.10. Dual varieties | 118 |
| §3.11. Associated varieties | 123 |
| §3.12. More on varieties with degenerate Gauss maps | 125 |
| §3.13. Secant and tangential varieties | 128 |
| §3.14. Rank restriction theorems | 132 |
| §3.15. Local study of smooth varieties with degenerate tangential varieties | 134 |
| §3.16. Generalized Monge systems | 137 |
| §3.17. Complete intersections | 139 |
| Chapter 4. Cartan-Kähler I: Linear Algebra and Constant-Coefficient Homogeneous Systems | 143 |
| §4.1. Tableaux | 144 |
| §4.2. First example | 148 |
| §4.3. Second example | 150 |
| §4.4. Third example | 153 |
| §4.5. The general case | 154 |
| §4.6. The characteristic variety of a tableau | 157 |
| Chapter 5. Cartan-Kähler II: The Cartan Algorithm for Linear Pfaffian Systems | 163 |
| §5.1. Linear Pfaffian systems | 163 |
| §5.2. First example | 165 |
| §5.3. Second example: constant coefficient homogeneous systems | 166 |
| §5.4. The local isometric embedding problem | 169 |

| | |
|---|-----|
| §5.5. The Cartan algorithm formalized: tableau, torsion and prolongation | 173 |
| §5.6. Summary of Cartan's algorithm for linear Pfaffian systems | 177 |
| §5.7. Additional remarks on the theory | 179 |
| §5.8. Examples | 182 |
| §5.9. Functions whose Hessians commute, with remarks on singular solutions | 189 |
| §5.10. The Cartan-Janet Isometric Embedding Theorem | 191 |
| §5.11. Isometric embeddings of space forms (mostly flat ones) | 194 |
| §5.12. Calibrated submanifolds | 197 |
| Chapter 6. Applications to PDE | 203 |
| §6.1. Symmetries and Cauchy characteristics | 204 |
| §6.2. Second-order PDE and Monge characteristics | 212 |
| §6.3. Derived systems and the method of Darboux | 215 |
| §6.4. Monge-Ampère systems and Weingarten surfaces | 222 |
| §6.5. Integrable extensions and Bäcklund transformations | 231 |
| Chapter 7. Cartan-Kähler III: The General Case | 243 |
| §7.1. Integral elements and polar spaces | 244 |
| §7.2. Example: Triply orthogonal systems | 251 |
| §7.3. Statement and proof of Cartan-Kähler | 254 |
| §7.4. Cartan's Test | 256 |
| §7.5. More examples of Cartan's Test | 259 |
| Chapter 8. Geometric Structures and Connections | 267 |
| §8.1. G -structures | 267 |
| §8.2. How to differentiate sections of vector bundles | 275 |
| §8.3. Connections on \mathcal{F}_G and differential invariants of G -structures | 278 |
| §8.4. Induced vector bundles and connections on induced bundles | 283 |
| §8.5. Holonomy | 286 |
| §8.6. Extended example: Path geometry | 295 |
| §8.7. Frobenius and generalized conformal structures | 308 |
| Appendix A. Linear Algebra and Representation Theory | 311 |
| §A.1. Dual spaces and tensor products | 311 |
| §A.2. Matrix Lie groups | 316 |
| §A.3. Complex vector spaces and complex structures | 318 |

| | |
|--|-----|
| §A.4. Lie algebras | 320 |
| §A.5. Division algebras and the simple group G_2 | 323 |
| §A.6. A smidgen of representation theory | 326 |
| §A.7. Clifford algebras and spin groups | 330 |
| Appendix B. Differential Forms | 335 |
| §B.1. Differential forms and vector fields | 335 |
| §B.2. Three definitions of the exterior derivative | 337 |
| §B.3. Basic and semi-basic forms | 339 |
| §B.4. Differential ideals | 340 |
| Appendix C. Complex Structures and Complex Manifolds | 343 |
| §C.1. Complex manifolds | 343 |
| §C.2. The Cauchy-Riemann equations | 347 |
| Appendix D. Initial Value Problems | 349 |
| Hints and Answers to Selected Exercises | 355 |
| Bibliography | 363 |
| Index | 371 |

Preface

In this book, we use moving frames and exterior differential systems to study geometry and partial differential equations. These ideas originated about a century ago in the works of several mathematicians, including Gaston Darboux, Edouard Goursat and, most importantly, Elie Cartan. Over the years these techniques have been refined and extended; major contributors to the subject are mentioned below, under “Further Reading”.

The book has the following features: It concisely covers the classical geometry of surfaces and basic Riemannian geometry in the language of moving frames. It includes results from projective differential geometry that update and expand the classic paper [69] of Griffiths and Harris. It provides an elementary introduction to the machinery of exterior differential systems (EDS), and an introduction to the basics of G -structures and the general theory of connections. Classical and recent geometric applications of these techniques are discussed throughout the text.

This book is intended to be used as a textbook for a graduate-level course; there are numerous exercises throughout. It is suitable for a one-year course, although it has more material than can be covered in a year, and parts of it are suitable for one-semester course (see the end of this preface for some suggestions). The intended audience is both graduate students who have some familiarity with classical differential geometry and differentiable manifolds, and experts in areas such as PDE and algebraic geometry who want to learn how moving frame and EDS techniques apply to their fields.

In addition to the geometric applications presented here, EDS techniques are also applied in CR geometry (see, e.g., [98]), robotics, and control theory (see [55, 56, 129]). This book prepares the reader for such areas, as well as

for more advanced texts on exterior differential systems, such as [20], and papers on recent advances in the theory, such as [58, 117].

Overview. Each section begins with geometric examples and problems. Techniques and definitions are introduced when they become useful to help solve the geometric questions under discussion. We generally keep the presentation elementary, although advanced topics are interspersed throughout the text.

In Chapter 1, we introduce moving frames via the geometry of curves in the Euclidean plane \mathbb{E}^2 . We define the Maurer-Cartan form of a Lie group G and explain its use in the study of submanifolds of G -homogeneous spaces. We give additional examples, including the equivalence of holomorphic mappings up to fractional linear transformation, where the machinery leads one naturally to the Schwarzian derivative.

We define exterior differential systems and jet spaces, and explain how to rephrase any system of partial differential equations as an EDS using jets. We state and prove the Frobenius system, leading up to it via an elementary example of an overdetermined system of PDE.

In Chapter 2, we cover traditional material—the geometry of surfaces in three-dimensional Euclidean space, submanifolds of higher-dimensional Euclidean space, and the rudiments of Riemannian geometry—all using moving frames. Our emphasis is on local geometry, although we include standard global theorems such as the rigidity of the sphere and the Gauss-Bonnet Theorem. Our presentation emphasizes finding and interpreting differential invariants to enable the reader to use the same techniques in other settings.

We begin Chapter 3 with a discussion of Grassmannians and the Plücker embedding. We present some well-known material (e.g., Fubini's theorem on the rigidity of the quadric) which is not readily available in other textbooks. We present several recent results, including the Zak and Landman theorems on the dual defect, and results of the second author on complete intersections, osculating hypersurfaces, uniruled varieties and varieties covered by lines. We keep the use of terminology and results from algebraic geometry to a minimum, but we believe we have included enough so that algebraic geometers will find this chapter useful.

Chapter 4 begins our multi-chapter discussion of the Cartan algorithm and Cartan-Kähler Theorem. In this chapter we study constant coefficient homogeneous systems of PDE and the linear algebra associated to the corresponding exterior differential systems. We define tableaux and involutivity of tableaux. One way to understand the Cartan-Kähler Theorem is as follows: given a system of PDE, if the linear algebra at the infinitesimal level

“works out right” (in a way explained precisely in the chapter), then existence of solutions follows.

In Chapter 5 we present the Cartan algorithm for linear Pfaffian systems, a very large class of exterior differential systems that includes systems of PDE rephrased as exterior differential systems. We give numerous examples, including many from Cartan’s classic treatise [31], as well as the isometric immersion problem, problems related to calibrated submanifolds, and an example motivated by variation of Hodge structure.

In Chapter 6 we take a detour to discuss the classical theory of characteristics, Darboux’s method for solving PDE, and Monge-Ampère equations in modern language. By studying the exterior differential systems associated to such equations, we recover the sine-Gordon representation of pseudospherical surfaces, the Weierstrass representation of minimal surfaces, and the one-parameter family of non-congruent isometric deformations of a surface of constant mean curvature. We also discuss integrable extensions and Bäcklund transformations of exterior differential systems, and the relationship between such transformations and Darboux integrability.

In Chapter 7, we present the general version of the Cartan-Kähler Theorem. Doing so involves a detailed study of the integral elements of an EDS. In particular, we arrive at the notion of a Kähler-regular flag of integral elements, which may be understood as the analogue of a sequence of well-posed Cauchy problems. After proving both the Cartan-Kähler Theorem and Cartan’s test for regularity, we apply them to several examples of non-Pfaffian systems arising in submanifold geometry.

Finally, in Chapter 8 we give an introduction to geometric structures (G -structures) and connections. We arrive at these notions at a leisurely pace, in order to develop the intuition as to why one needs them. Rather than attempt to describe the theory in complete generality, we present one extended example, path geometry in the plane, to give the reader an idea of the general theory. We conclude with a discussion of some recent generalizations of G -structures and their applications.

There are four appendices, covering background material for the main part of the book: linear algebra and rudiments of representation theory, differential forms and vector fields, complex and almost complex manifolds, and a brief discussion of initial value problems and the Cauchy-Kowalevski Theorem, of which the Cartan-Kähler Theorem is a generalization.

Layout. All theorems, propositions, remarks, examples, etc., are numbered together within each section; for example, Theorem 1.3.2 is the second numbered item in section 1.3. Equations are numbered sequentially within each chapter. We have included hints for selected exercises, those marked with the symbol © at the end, which is meant to be suggestive of a life preserver.

Further Reading on EDS. To our knowledge, there are only a small number of textbooks on exterior differential systems. The first is Cartan's classic text [31], which has an extraordinarily beautiful collection of examples, some of which are reproduced here. We learned the subject from our teacher Bryant and the book by Bryant, Chern, Griffiths, Gardner and Goldschmidt [20], which is an elaboration of an earlier monograph [19], and is at a more advanced level than this book. One text at a comparable level to this book, but more formal in approach, is [156]. The monograph [70], which is centered around the isometric embedding problem, is similar in spirit but covers less material. The memoir [155] is dedicated to extending the Cartan-Kähler Theorem to the C^∞ setting for hyperbolic systems, but contains an exposition of the general theory. There is also a monograph by Kähler [89] and lectures by Kuranishi [97], as well the survey articles [66, 90]. Some discussion of the theory may be found in the differential geometry texts [142] and [145].

We give references for other topics discussed in the book in the text.

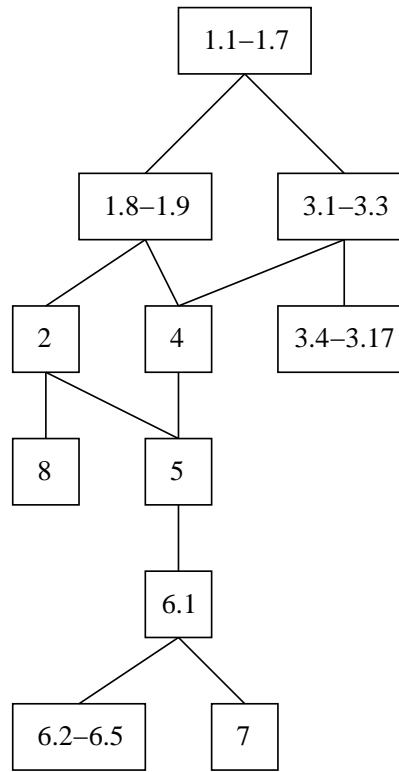
History and Acknowledgements. This book started out about a decade ago. We thought we would write up notes from Robert Bryant's Tuesday night seminar, held in 1988–89 while we were graduate students, as well as some notes on exterior differential systems which would be more introductory than [20]. The seminar material is contained in §8.6 and parts of Chapter 6. Chapter 2 is influenced by the many standard texts on the subject, especially [43] and [142], while Chapter 3 is influenced by the paper [69]. Several examples in Chapter 5 and Chapter 7 are from [31], and the examples of Darboux's method in Chapter 6 are from [63]. In each case, specific attributions are given in the text. Chapter 7 follows Chapter III of [20] with some variations. In particular, to our knowledge, Lemmas 7.1.10 and 7.1.13 are original. The presentation in §8.5 is influenced by [11], [94] and unpublished lectures of Bryant.

The first author has given graduate courses based on the material in Chapters 6 and 7 at the University of California, San Diego and at Case Western Reserve University. The second author has given year-long graduate courses using Chapters 1, 2, 4, 5, and 8 at the University of Pennsylvania and Université de Toulouse III, and a one-semester course based on Chapters 1, 2, 4 and 5 at Columbia University. He has also taught one-semester

undergraduate courses using Chapters 1 and 2 and the discussion of connections in Chapter 8 (supplemented by [141] and [142] for background material) at Toulouse and at Georgia Institute of Technology, as well as one-semester graduate courses on projective geometry from Chapters 1 and 3 (supplemented by some material from algebraic geometry), at Toulouse, Georgia Tech. and the University of Trieste. He also gave more advanced lectures based on Chapter 3 at Seoul National University, which were published as [107] and became a precursor to Chapter 3. Preliminary versions of Chapters 5 and 8 respectively appeared in [104, 103].

We would like to thank the students in the above classes for their feedback. We also thank Megan Dillon, Phillipe Eyssidieux, Daniel Fox, Sung-Eun Koh, Emilia Mezzetti, Joseph Montgomery, Giorgio Ottaviani, Jens Piontkowski, Margaret Symington, Magdalena Toda, Sung-Ho Wang and Peter Vassiliou for comments on the earlier drafts of this book, and Annette Rohrs for help with the figures. The staff of the publications division of the AMS—in particular, Ralph Sizer, Tom Kacvinsky, and our editor, Ed Dunne—were of tremendous help in pulling the book together. We are grateful to our teacher Robert Bryant for introducing us to the subject. Lastly, this project would not have been possible without the support and patience of our families.

Dependence of Chapters



Suggested uses of this book:

- a year-long graduate course covering moving frames and exterior differential systems (chapters 1–8);
- a one-semester course on exterior differential systems and applications to partial differential equations (chapters 1 and 4–7);
- a one-semester course on the use of moving frames in algebraic geometry (chapter 3, preceded by part of chapter 1);
- a one-semester beginning graduate course on differential geometry (chapters 1, 2 and 8).

Moving Frames and Exterior Differential Systems

In this chapter we motivate the use of differential forms to study problems in geometry and partial differential equations. We begin with familiar material: the Gauss and mean curvature of surfaces in \mathbb{E}^3 in §1.1, and Picard's Theorem for local existence of solutions of ordinary differential equations in §1.2. We continue in §1.2 with a discussion of a simple system of partial differential equations, and then in §1.3 rephrase it in terms of differential forms, which facilitates interpreting it geometrically. We also state the Frobenius Theorem.

In §1.4, we review curves in \mathbb{E}^2 in the language of moving frames. We generalize this example in §§1.5–1.6, describing how one studies submanifolds of homogeneous spaces using moving frames, and introducing the Maurer-Cartan form. We give two examples of the geometry of curves in homogeneous spaces: classifying holomorphic mappings of the complex plane under fractional linear transformations in §1.7, and classifying curves in \mathbb{E}^3 under Euclidean motions (i.e., rotations and translations) in §1.8. We also include exercises on plane curves in other geometries.

In §1.9, we define exterior differential systems and integral manifolds. We prove the Frobenius Theorem, give a few basic examples of exterior differential systems, and explain how to express a system of partial differential equations as an exterior differential system using jet bundles.

Throughout this book we use the summation convention: unless otherwise indicated, summation is implied whenever repeated indices occur up and down in an expression.

1.1. Geometry of surfaces in \mathbb{E}^3 in coordinates

Let \mathbb{E}^3 denote Euclidean three-space, i.e., the affine space \mathbb{R}^3 equipped with its standard inner product.

Given two smooth surfaces $S, S' \subset \mathbb{E}^3$, when are they “equivalent”? For the moment, we will say that two surfaces are (locally) equivalent if there exist a rotation and translation taking (an open subset of) S onto (an open subset of) S' .

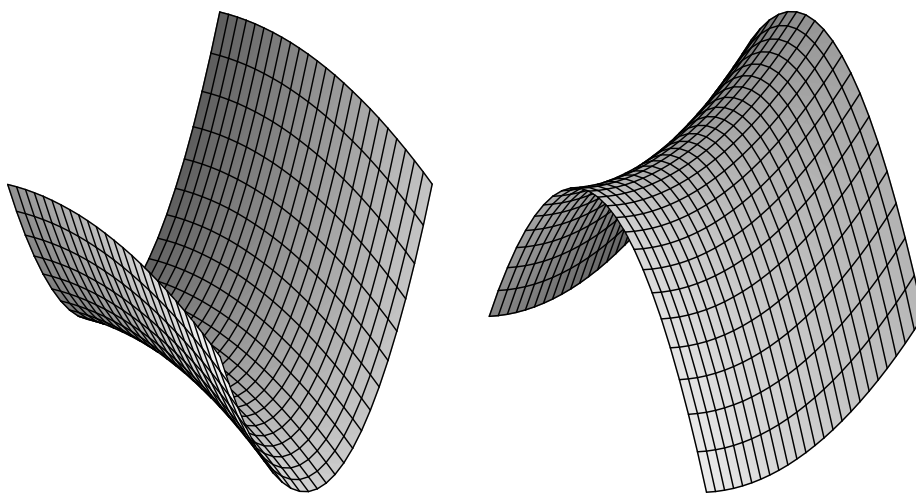


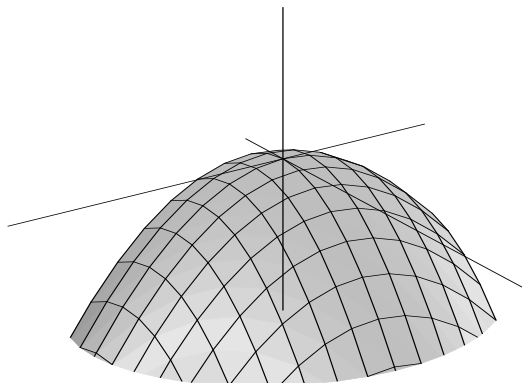
Figure 1. Are these two surfaces equivalent?

It would be impractical and not illuminating to try to test all possible motions to see if one of them maps S onto S' . Instead, we will work as follows:

Fix one surface S and a point $p \in S$. We will use the Euclidean motions to put S into a normalized position in space with respect to p . Then any other surface S' will be locally equivalent to S at p if there is a point $p' \in S'$ such that the pair (S', p') can be put into the same normalized position as (S, p) .

The implicit function theorem implies that there always exist coordinates such that S is given locally by a graph $z = f(x, y)$. To obtain a normalized position for our surface S , first translate so that $p = (0, 0, 0)$, then use a rotation to make $T_p S$ the xy -plane, i.e., so that $z_x(0, 0) = z_y(0, 0) = 0$. We

will call such coordinates *adapted to p* . At this point we have used up all our freedom of motion except for a rotation in the xy -plane.



If coordinates are adapted to p and we expand $f(x, y)$ in a Taylor series centered at the origin, then functions of the coefficients of the series that are invariant under this rotation are differential invariants.

In this context, a (Euclidean) *differential invariant* of S at p is a function I of the coefficients of the Taylor series for f at p , with the property that, if we perform a Euclidean change of coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where A is a rotation matrix and a, b, c are arbitrary constants, after which S is expressed as a graph $\tilde{z} = \tilde{f}(\tilde{x}, \tilde{y})$ near p , then I has the same value when computed using the Taylor coefficients of \tilde{f} at p . Clearly a necessary condition for (S, p) to be locally equivalent to (S', p') is that the values of differential invariants of S at p match the values of the corresponding invariants of S' at p' .

For example, consider the Hessian of $z = z(x, y)$ at p :

$$(1.1) \quad \text{Hess}_p = \begin{pmatrix} z_{xx} & z_{yx} \\ z_{xy} & z_{yy} \end{pmatrix} \Big|_p.$$

Assume we have adapted coordinates to p . If we rotate in the xy plane, the Hessian gets conjugated by the rotation matrix. The quantities

$$(1.2) \quad \begin{aligned} K_0 &= \det(\text{Hess}_p) = (z_{xx}z_{yy} - z_{xy}^2) \Big|_p, \\ H_0 &= \frac{1}{2}\text{trace}(\text{Hess}_p) = \frac{1}{2}(z_{xx} + z_{yy}) \Big|_p. \end{aligned}$$

are differential invariants because the determinant and trace of a matrix are unchanged by conjugation by a rotation matrix. Thus, if we are given two surfaces S, S' and we normalize them both at respective points p and p' as

above, a necessary condition for there to be a rigid motion taking p' to p such that the Taylor expansions for the two surfaces at the point p coincide is that $K_0(S) = K_0(S')$ and $H_0(S) = H_0(S')$.

The formulas (1.2) are only valid at one point, and only after the surface has been put in normalized position relative to that point. To calculate K and H as functions on S it would be too much work to move each point to the origin and arrange its tangent plane to be horizontal. But it is possible to adjust the formulas to account for tilted tangent planes (see §2.10). One then obtains the following functions, which are differential invariants under Euclidean motions of surfaces that are locally described as graphs $z = z(x, y)$:

$$(1.3) \quad \begin{aligned} K(x, y) &= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}, \\ H(x, y) &= \frac{1}{2} \frac{(1 + z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1 + z_x^2)z_{yy}}{(1 + z_x^2 + z_y^2)^{\frac{3}{2}}}, \end{aligned}$$

respectively giving the *Gauss* and *mean* curvature of S at $p = (x, y, z(x, y))$.

Exercise 1.1.1: By locally describing each surface as a graph, calculate the Gauss and mean curvature functions for a sphere of radius R , a cylinder of radius r (e.g., $x^2 + y^2 = r^2$) and the smooth points of the cone $x^2 + y^2 = z^2$.

Once one has found invariants for a given submanifold geometry, one may ask questions about submanifolds with special invariants. For surfaces in \mathbb{E}^3 , one might ask which surfaces have K constant or H constant. These can be treated as questions about solutions to certain partial differential equations (PDE). For example, from (1.3) we see that surfaces with $K \equiv 1$ are locally given by solutions to the PDE

$$(1.4) \quad z_{xx}z_{yy} - z_{xy}^2 = (1 + z_x^2 + z_y^2)^2.$$

We will soon free ourselves of coordinates and use moving frames and differential forms. As a provisional definition, a *moving frame* is a smoothly varying basis of the tangent space to \mathbb{E}^3 defined at each point of our surface. In general, using moving frames one can obtain formulas valid at every point analogous to coordinate formulas valid at just one preferred point. In the present context, the Gauss and mean curvatures will be described at all points by expressions like (1.2) rather than (1.3); see §2.1.

Another reason to use moving frames is that the method gives a uniform procedure for dealing with diverse geometric settings. Even if one is originally only interested in Euclidean geometry, other geometries arise naturally. For example, consider the *warp* of a surface, which is defined to be $(k_1 - k_2)^2$, where the k_j are the eigenvalues of (1.1). It turns out that this

quantity is invariant under a larger change of coordinates than the Euclidean group, namely *conformal* changes of coordinates, and thus it is easier to study the warp in the context of conformal geometry.

Regardless of how unfamiliar a geometry initially appears, the method of moving frames provides an algorithm to find differential invariants. Thus we will have a single method for dealing with conformal, Hermitian, projective and other geometries. Because it is familiar, we will often use the geometry of surfaces in \mathbb{E}^3 as an example, but the reader should keep in mind that the beauty of the method is its wide range of applicability. As for the use of differential forms, we shall see that when we express a system of PDE as an exterior differential system, the geometric features of the system—i.e., those which are independent of coordinates—will become transparent.

1.2. Differential equations in coordinates

The first questions one might ask when confronted with a system of differential equations are: Are there any solutions? If so, how many?

In the case of a single ordinary differential equation (ODE), here is the answer:

Theorem 1.2.1 (Picard¹). *Let $f(x, u) : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function with f and f_u continuous. Then for all $(x_0, u_0) \in \mathbb{R}^2$, there exist an open interval $I \ni x_0$ and a function $u(x)$ defined on I , satisfying $u(x_0) = u_0$ and the differential equation*

$$(1.5) \quad \frac{du}{dx} = f(x, u).$$

Moreover, any other solution of this initial value problem must coincide with this solution on I .

In other words, for a given ODE there exists a solution defined near x_0 and this solution is unique given the choice of a constant u_0 . Thus for an ODE for one function of one variable, we say that *solutions depend on one constant*. More generally, Picard's Theorem applies to systems of n first-order ODE's involving n unknowns, where solutions depend on n constants.

The graph in \mathbb{R}^2 of any solution to (1.5) is tangent at each point to the vector field $X = \frac{\partial}{\partial x} + f(x, u) \frac{\partial}{\partial u}$. This indicates how determined ODE systems generalize to the setting of differentiable manifolds (see Appendix B). If M is a manifold and X is a vector field on M , then a solution to the *system defined by X* is an immersed curve $c : I \rightarrow M$ such that $c'(t) = X_{c(t)}$ for all $t \in I$. (This is also referred to as an *integral curve* of X .) Away from singular points, one is guaranteed existence of local solutions to such systems and can even take the solution curves as coordinate curves:

¹See, e.g., [140], p.423

Theorem 1.2.2 (Flowbox coordinates²). *Let M be an m -dimensional C^∞ manifold, let $p \in M$, and let $X \in \Gamma(TM)$ be a smooth vector field which is nonzero at p . Then there exists a local coordinate system (x^1, \dots, x^m) , defined in a neighborhood U of p , such that $\frac{\partial}{\partial x^1} = X$.*

Consequently, there exists an open set $V \subset U \times \mathbb{R}$ on which we may define the *flow of X* , $\phi : V \rightarrow M$, by requiring that for any point $q \in U$, $\frac{\partial}{\partial t}\phi(q, t) = X|_{\phi(q, t)}$. The flow is given in flowbox coordinates by

$$(x^1, \dots, x^m, t) \mapsto (x^1 + t, x^2, \dots, x^m).$$

With systems of PDE, it becomes difficult to determine the appropriate initial data for a given system (see Appendix D for examples). We now examine a simple PDE system, first in coordinates, and then later (in §5.2) using differential forms.

Example 1.2.3. Consider the system for $u(x, y)$ given by

$$(1.6) \quad \begin{aligned} u_x &= A(x, y, u), \\ u_y &= B(x, y, u), \end{aligned}$$

where A, B are given smooth functions. Since (1.6) specifies both partial derivatives of u , at any given point $p = (x, y, u) \in \mathbb{R}^3$ the tangent plane to the graph of a solution passing through p is uniquely determined.

In this way, (1.6) defines a smoothly-varying field of two-planes on \mathbb{R}^3 , just as the ODE (1.5) defines a field of one-planes (i.e., a line field) on \mathbb{R}^2 . For (1.5), Picard's Theorem guarantees that the one-planes "fit together" to form a solution curve through any given point. For (1.6), existence of solutions amounts to whether or not the two-planes "fit together".

We can attempt to solve (1.6) in a neighborhood of $(0, 0)$ by solving a succession of ODE's. Namely, if we set $y = 0$ and $u(0, 0) = u_0$, Picard's Theorem implies that there exists a unique function $\tilde{u}(x)$ satisfying

$$(1.7) \quad \frac{d\tilde{u}}{dx} = A(x, 0, \tilde{u}), \quad \tilde{u}(0) = u_0.$$

After solving (1.7), hold x fixed and use Picard's Theorem again on the initial value problem

$$(1.8) \quad \frac{du}{dy} = B(x, y, u), \quad u(x, 0) = \tilde{u}(x).$$

This determines a function $u(x, y)$ on some neighborhood of $(0, 0)$. The problem is that this function may not satisfy our original equation.

Whether or not (1.8) actually gives a solution to (1.6) depends on whether or not the equations (1.6) are "compatible" as differential equations. For smooth solutions to a system of PDE, compatibility conditions

²See, e.g., [142] vol. I, p.205

arise because mixed partials must commute, i.e., $(u_x)_y = (u_y)_x$. In our example,

$$\begin{aligned}(u_x)_y &= \frac{\partial}{\partial y} A(x, y, u) = A_y(x, y, u) + A_u(x, y, u) \frac{\partial u}{\partial y} = A_y + BA_u, \\ (u_y)_x &= B_x + AB_u,\end{aligned}$$

so setting $(u_x)_y = (u_y)_x$ reveals a “hidden equation”, the compatibility condition

$$(1.9) \quad A_y + BA_u = B_x + AB_u.$$

We will prove in §1.9 that the commuting of second-order partials in this case implies that all higher-order mixed partials commute as well, so that there are no further hidden equations. In other words, if (1.9) is an identity in x, y, u , then solving the ODE’s (1.7) and (1.8) in succession gives a solution to (1.6), and solutions depend on one constant.

Exercise 1.2.4: Show that, if (1.9) is an identity, then one gets the same solution by first solving for $\tilde{u}(y) = u(0, y)$.

If (1.9) is not an identity, there are several possibilities. If u appears in (1.9), then it gives an equation which every solution to (1.6) must satisfy. Given a point $p = (0, 0, u_0)$ at which (1.9) is not an identity, and such that the implicit function theorem may be applied to (1.9) to determine $u(x, y)$ near $(0, 0)$, then only this solved-for u can be the solution passing through p . However, it still may not satisfy (1.6), in which case there is no solution through p .

If u does not appear in (1.9), then it gives a relation between x and y , and there is no solution defined on an open set around $(0, 0)$.

Remark 1.2.5. For more complicated systems of PDE, it is not as easy to determine if all mixed partials commute. The Cartan-Kähler Theorem (see Chapters 5 and 7) will provide an algorithm which tells us when to stop checking compatibilities.

Exercises 1.2.6:

1. Consider this special case of Example 1.2.3:

$$\begin{aligned}u_x &= A(x, y), \\ u_y &= B(x, y),\end{aligned}$$

where A and B satisfy $A(0, 0) = B(0, 0) = 0$. Verify that solving the initial value problems (1.7)–(1.8) gives

$$(1.10) \quad u(x, y) = u_0 + \int_{s=0}^x A(s, 0) ds + \int_{t=0}^y B(x, t) dt.$$

Under what condition does this function u satisfy (1.6)? Verify that the resulting condition is equivalent to (1.9) in this special case.

2. Rewrite (1.10) as a line integral involving the 1-form

$$\omega := A(x, y)dx + B(x, y)dy,$$

and determine the condition which ensures that the integral is independent of path.

3. Determine the space of solutions to (1.6) in the following special cases:

- (a) $A = -\frac{x}{u}, B = -\frac{y}{u}$.
- (b) $A = B = \frac{x}{u}$.
- (c) $A = -\frac{x}{u}, B = y$.

1.3. Introduction to differential equations without coordinates

Example 1.2.3 revisited. Instead of working on $\mathbb{R}^2 \times \mathbb{R}$ with coordinates $(x, y) \times (u)$, we will work on the larger space $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ with coordinates $(x, y) \times (u) \times (p, q)$, which we will denote $J^1(\mathbb{R}^2, \mathbb{R})$, or J^1 for short. This space, called the *space of 1-jets of mappings from \mathbb{R}^2 to \mathbb{R}* , is given additional structure and generalized in §1.9.

Let $u : U \rightarrow \mathbb{R}$ be a smooth function defined on an open set $U \subset \mathbb{R}^2$. We associate to u the surface in J^1 given by

$$(1.11) \quad u = u(x, y), \quad p = u_x(x, y), \quad q = u_y(x, y).$$

which we will refer to as the *lift* of u . The graph of u is the projection of the lift (1.11) in J^1 to $\mathbb{R}^2 \times \mathbb{R}$.

We will eventually work on J^1 without reference to coordinates. As a step in that direction, consider the differential forms

$$\theta := du - p dx - q dy, \quad \Omega := dx \wedge dy$$

defined on J^1 . Suppose $i : S \hookrightarrow J^1$ is a surface such that $i^*\Omega \neq 0$ at each point of S . Since dx, dy are linearly independent 1-forms on S , we may use x, y as coordinates on S , and the surface may be expressed as

$$u = u(x, y), \quad p = p(x, y), \quad q = q(x, y).$$

Suppose $i^*\theta = 0$. Then

$$i^*du = p dx + q dy.$$

On the other hand, since u restricted to S is a function of x and y , we have

$$du = u_x dx + u_y dy.$$

Because dx, dy are independent on S , these two equations imply that $p = u_x$ and $q = u_y$ on S . Thus, surfaces $i : S \hookrightarrow J^1$ such that $i^*\theta = 0$ and $i^*\Omega$ is nonvanishing correspond to lifts of maps $u : U \rightarrow \mathbb{R}$.

Now consider the 3-fold $j : \Sigma \hookrightarrow J^1$ defined by the equations

$$p = A(x, y, u), \quad q = B(x, y, u).$$

Let $i : S \hookrightarrow \Sigma$ be a surface such that $i^*\theta = 0$ and $i^*\Omega$ is nonvanishing. Then the projection of S to $\mathbb{R}^2 \times \mathbb{R}$ is the graph of a solution to (1.6). Moreover, all solutions to (1.6) are the projections of such surfaces, by taking S as the lift of the solution.

Thus we have a correspondence

$$\text{solutions to (1.6)} \Leftrightarrow \text{surfaces } i : S \hookrightarrow \Sigma \text{ such that } i^*\theta \equiv 0 \text{ and } i^*\Omega \neq 0.$$

On such surfaces, we also have $i^*d\theta \equiv 0$, but

$$\begin{aligned} d\theta &= -dp \wedge dx - dq \wedge dy, \\ j^*d\theta &= -(A_x dx + A_y dy + A_u du) \wedge dx - (B_x dx + B_y dy + B_u du) \wedge dy, \\ i^*d\theta &= (A_y - B_x + A_u B - B_u A) i^*(\Omega). \end{aligned}$$

(To obtain the second line we use the defining equations of Σ and to obtain the third line we use $i^*(du) = Adx + Bdy$.) Because $i^*\Omega \neq 0$, the equation

$$(1.12) \quad A_y - B_x + A_u B - B_u A = 0$$

must hold on S . This is precisely the same as the condition (1.9) obtained by checking that mixed partials commute.

If (1.12) does not hold identically on Σ , then it gives another equation which must hold for any solution. But since $\dim \Sigma = 3$, in that case (1.12) already describes a surface in Σ . If there is any solution surface S , it must be an open subset of the surface in Σ given by (1.12). This surface will only be a solution if θ pulls back to be zero on it. If (1.12) is an identity, then we may use the Frobenius Theorem (see below) to conclude that through any point of Σ there is a unique solution S (constructed, as in §1.2, by solving a succession of ODE's). In this sense, (1.12) implies that all higher partial derivatives commute.

We have now recovered our observations from §1.2.

The general game plan for treating a system of PDE as an *exterior differential system* (EDS) will be as follows:

One begins with a “universal space” (J^1 in the above example) where the various partial derivatives are represented by independent variables. Then one restricts to the subset Σ of the universal space defined by the system of PDE by considering it as a set of equations among independent variables.

Solutions to the PDE correspond to submanifolds of Σ on which the variables representing what we want to be partial derivatives actually are partial derivatives. These submanifolds are characterized by the vanishing of certain differential forms.

These remarks will be explained in detail in §1.9.

Picard's Theorem revisited. On \mathbb{R}^2 with coordinates (x, u) , consider $\theta = du - f(x, u)dx$. Then there is a one-to-one correspondence between solutions of the ODE (1.5) and curves $c : \mathbb{R} \rightarrow \mathbb{R}^2$ such that $c^*(\theta) = 0$ and $c^*(dx)$ is nonvanishing.

More generally, the flowbox coordinate theorem 1.2.2 implies:

Theorem 1.3.1. *Let M be a C^∞ manifold of dimension m , and let $\theta^1, \dots, \theta^{m-1} \in \Omega^1(M)$ be pointwise linearly independent in some open neighborhood $U \subset M$. Then through $p \in U$ there exists a curve $c : \mathbb{R} \rightarrow U$, unique up to reparametrization, such that $c^*(\theta^j) = 0$ for $1 \leq j \leq m - 1$.*

(For a proof, see [142].)

The Frobenius Theorem. In §1.9 we will prove the following result, which is a generalization, both of Theorem 1.3.1 and of the asserted existence of solutions to Example 1.2.3 when (1.9) holds, to an existence theorem for certain systems of PDE:

Theorem 1.3.2 (Frobenius Theorem, first version). *Let Σ be a C^∞ manifold of dimension m , and let $\theta^1, \dots, \theta^{m-n} \in \Omega^1(\Sigma)$ be pointwise linearly independent. If there exist 1-forms $\alpha_j^i \in \Omega^1(\Sigma)$ such that $d\theta^j = \alpha_j^i \wedge \theta^i$ for all j , then through each $p \in \Sigma$ there exists a unique n -dimensional manifold $i : N \hookrightarrow \Sigma$ such that $i^*(\theta^j) = 0$ for $1 \leq j \leq m - n$.*

In order to motivate our study of exterior differential systems, we reword the Frobenius Theorem more geometrically as follows: Let Σ be an m -dimensional manifold such that through each point $x \in \Sigma$ there is an n -dimensional subspace $E_x \subset T_x\Sigma$ which varies smoothly with x (such a structure is called a *distribution*). We consider the problem of finding submanifolds $X \subset \Sigma$ such that $T_xX = E_x$ for all $x \in X$.

Consider $E_x^\perp \subset T_x^*\Sigma$. Let θ_x^a , $1 \leq a \leq m - n$, be a basis of E_x^\perp . We may choose the θ_x^a to vary smoothly to obtain $m - n$ linearly independent forms $\theta^a \in \Omega^1(\Sigma)$. Let $\mathcal{I} = \{\theta^1, \dots, \theta^{m-n}\}_{\text{diff}}$ denote the *differential ideal* they generate in $\Omega^*(\Sigma)$ (see §B.4). The submanifolds X tangent to the distribution E are exactly the n -dimensional submanifolds $i : N \hookrightarrow \Sigma$ such that $i^*(\alpha) = 0$ for all $\alpha \in \mathcal{I}$. Call such a submanifold an *integral manifold* of \mathcal{I} .

To find integral manifolds, we already know that if there are any, their tangent space at any point $x \in \Sigma$ is already uniquely determined, namely it is E_x . The question is whether these n -planes can be “fitted together” to obtain an n -dimensional submanifold. This information is contained in the derivatives of the θ^a 's, which indicate how the n -planes “move” infinitesimally.

If we are to have $i^*(\theta^a) = 0$, we must also have $d(i^*\theta^a) = i^*(d\theta^a) = 0$. If there is to be an integral manifold through x , or even an n -plane $E_x \subset T_x\Sigma$ on which $\alpha|_{E_x} = 0$, $\forall \alpha \in \mathcal{I}$, the equations $i^*(d\theta^a) = 0$ cannot impose any additional conditions, i.e., we must have $d\theta^a|_{E_x} = 0$ because we already have a unique n -plane at each point $x \in \Sigma$. To recap, for all a we must have

$$(1.13) \quad d\theta^a = \alpha_1^a \wedge \theta^1 + \dots + \alpha_{m-n}^a \wedge \theta^{m-n}$$

for some $\alpha_b^a \in \Omega^1(\Sigma)$, because the forms θ_x^a span E_x^\perp .

Notation 1.3.3. Suppose \mathcal{I} is an ideal and ϕ and ψ are k -forms. Then we write $\phi \equiv \psi \pmod{\mathcal{I}}$ if $\phi = \psi + \beta$ for some $\beta \in \mathcal{I}$.

Let $\{\theta^1, \dots, \theta^{m-n}\}_{\text{alg}} \subset \Omega^*(\Sigma)$ denote the algebraic ideal generated by $\theta^1, \dots, \theta^{m-n}$ (see §B.4). Now (1.13) may be restated as

$$(1.14) \quad d\theta^a \equiv 0 \pmod{\{\theta^1, \dots, \theta^{m-n}\}_{\text{alg}}}.$$

The Frobenius Theorem states that this necessary condition is also sufficient:

Theorem 1.3.4 (Frobenius Theorem, second version). *Let \mathcal{I} be a differential ideal generated by the linearly independent 1-forms $\theta^1, \dots, \theta^{m-n}$ on an m -fold Σ , i.e., $\mathcal{I} = \{\theta^1, \dots, \theta^{m-n}\}_{\text{diff}}$. Suppose \mathcal{I} is also generated algebraically by $\theta^1, \dots, \theta^{m-n}$, i.e., $\mathcal{I} = \{\theta^1, \dots, \theta^{m-n}\}_{\text{alg}}$. Then through any $p \in \Sigma$ there exists an n -dimensional integral manifold of \mathcal{I} . In fact, in a sufficiently small neighborhood of p there exists a coordinate system y^1, \dots, y^m such that \mathcal{I} is generated by dy^1, \dots, dy^{m-n} .*

We postpone the proof until §1.9.

Definition 1.3.5. We will say a subbundle $I \subset T^*\Sigma$ is *Frobenius* if the ideal generated algebraically by sections of I is also a differential ideal. We will say a distribution $\Delta \subset \Gamma(T\Sigma)$ is *Frobenius* if $\Delta^\perp \subset T^*\Sigma$ is Frobenius. Equivalently (see Exercise 1.3.6.2 below), Δ is Frobenius if $\forall X, Y \in \Delta$, $[X, Y] \in \Delta$, where $[X, Y]$ is the Lie bracket.

If $\{\theta^a\}$ fails to be Frobenius, not all hope is lost for an n -dimensional integral manifold, but we must restrict ourselves to the subset $j : \Sigma' \hookrightarrow \Sigma$ on which (1.14) holds, and see if there are n -dimensional integral manifolds of the ideal generated by $j^*(\theta^a)$ on Σ' . (This was what we did in the special case of Example 1.2.3.)

Exercises 1.3.6:

1. Which of the following ideals are Frobenius?

$$\mathcal{I}_1 = \{dx^1, x^2 dx^3 + dx^4\}_{\text{diff}}$$

$$\mathcal{I}_2 = \{dx^1, x^1 dx^3 + dx^4\}_{\text{diff}}$$

2. Show that the differential forms and vector field conditions for being Frobenius are equivalent, i.e., $\Delta \subset \Gamma(T\Sigma)$ satisfies $[\Delta, \Delta] \subseteq \Delta$ if and only if $\Delta^\perp \subset T^*\Sigma$ satisfies $d\theta \equiv 0 \pmod{\Delta^\perp}$ for all $\theta \in \Gamma(\Delta^\perp)$.

3. On \mathbb{R}^3 let $\theta = A dx + B dy + C dz$, where $A = A(x, y, z)$, etc. Assume the differential ideal generated by θ is Frobenius, and explain how to find a function $f(x, y, z)$ such that the differential systems $\{\theta\}_{\text{diff}}$ and $\{df\}_{\text{diff}}$ are equivalent.

1.4. Introduction to geometry without coordinates: curves in \mathbb{E}^2

We will return to our study of surfaces in \mathbb{E}^3 in Chapter 2. To see how to use moving frames to obtain invariants, we begin with a simpler problem.

Let \mathbb{E}^2 denote the oriented Euclidean plane. Given two parametrized curves $c_1, c_2 : \mathbb{R} \rightarrow \mathbb{E}^2$, we ask two questions: When does there exist a Euclidean motion $A : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ (i.e., a rotation and translation) such that $A(c_1(\mathbb{R})) = c_2(\mathbb{R})$? And, when do there exist a Euclidean motion $A : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ and a constant c such that $A(c_1(t)) = c_2(t + c)$ for all t ?

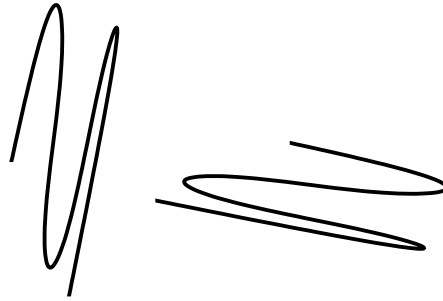


Figure 2. Are these two curves equivalent?

Instead of using coordinates at a point, we will use an *adapted frame*, i.e., for each t we take a basis of $T_{c(t)}\mathbb{E}^2$ that is “adapted” to Euclidean geometry. This geometry is induced by the group of Euclidean motions—the changes of coordinates of \mathbb{E}^2 preserving the inner product and orientation—which we will denote by $ASO(2)$.

In more detail, the group $ASO(2)$ consists of transformations of the form

$$(1.15) \quad \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto \begin{pmatrix} t^1 \\ t^2 \end{pmatrix} + R \begin{pmatrix} x^1 \\ x^2 \end{pmatrix},$$

where $R \in SO(2)$ is a rotation matrix. It can be represented as a matrix Lie group by writing

$$(1.16) \quad ASO(2) = \left\{ M \in GL(3, \mathbb{R}) \mid M = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & R \end{pmatrix}, \mathbf{t} \in \mathbb{R}^2, R \in SO(2) \right\}.$$

Then its action on \mathbb{E}^2 is given by $\mathbf{x} \mapsto M\mathbf{x}$, where we represent points in \mathbb{E}^2 by $\mathbf{x} = {}^t(1 \ x^1 \ x^2)$.

We may define a mapping from $ASO(2)$ to \mathbb{E}^2 by

$$(1.17) \quad \begin{pmatrix} 1 & 0 \\ x & R \end{pmatrix} \mapsto x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which takes each group element to the image of the origin under the transformation (1.15). The fiber of this map over every point is a left coset of $SO(2) \subset ASO(2)$, so \mathbb{E}^2 , as a manifold, is the quotient $ASO(2)/SO(2)$. Furthermore, $ASO(2)$ may be identified with the bundle of oriented orthonormal bases of \mathbb{E}^2 by identifying the columns of the rotation matrix $R = (e_1, e_2)$ with an oriented orthonormal basis of $T_x\mathbb{E}^2$, where x is the basepoint given by (1.17). (Here we use the fact that for a vector space V , we may identify V with T_xV for any $x \in V$.)

Returning to the curve $c(t)$, we choose an oriented orthonormal basis of $T_{c(t)}\mathbb{E}^2$ as follows: A natural element of $T_{c(t)}\mathbb{E}^2$ is $c'(t)$, but this may not be of unit length. So, we take $e_1(t) = c'(t)/|c'(t)|$, and this choice also determines $e_2(t)$. Of course, to do this we must assume that the curve is regular:

Definition 1.4.1. A curve $c(t)$ is said to be *regular* if $c'(t)$ never vanishes. More generally, a map $f : M \rightarrow N$ between differentiable manifolds is regular if df is everywhere defined and of rank equal to $\dim M$.

What have we done? We have constructed a map to the Lie group $ASO(2)$ as follows:

$$C : \mathbb{R} \rightarrow ASO(2),$$

$$t \mapsto \begin{pmatrix} 1 & 0 \\ c(t) & (e_1(t), e_2(t)) \end{pmatrix}.$$

We will obtain differential invariants of our curve by differentiating this mapping, and taking combinations of the derivatives that are invariant under Euclidean changes of coordinates.

Consider $v(t) = |c'(t)|$, called the *speed* of the curve. It is invariant under Euclidean motions and thus is a differential invariant. However, it is only an invariant of the mapping, not of the image curve (see Exercise 1.4.2.2). The speed measures how much (internal) distance is being distorted under the mapping c .

Consider $\frac{de_1}{dt}$. We must have $\frac{de_1}{dt} = \lambda(t)e_2(t)$ for some function $\lambda(t)$ because $|e_1(t)| \equiv 1$ (see Exercise 1.4.2.1 below). Thus $\lambda(t)$ is a differential invariant, but it again depends on the parametrization of the curve. To determine an invariant of the image alone, we let $\tilde{c}(t)$ be another parametrization of the same curve. We calculate that $\tilde{\lambda}(t) = \frac{\tilde{v}(t)}{v(t)}\lambda(t)$, so we set $\kappa(t) = \frac{\lambda(t)}{v(t)}$. This $\kappa(t)$, called the *curvature* of the curve, measures how much c is infinitesimally moving away from its tangent line at $c(t)$.

A necessary condition for two curves c, \tilde{c} to have equivalent images is that there exists a diffeomorphism $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\kappa(t) = \tilde{\kappa}(\psi(t))$. It will follow from Corollary 1.6.13 that the images of curves are locally classified up to congruence by their curvature functions, and that parametrized curves are locally classified by κ, v .

Exercises 1.4.2:

1. Let V be a vector space with a nondegenerate inner product $\langle \cdot, \cdot \rangle$. Let $v(t)$ be a curve in V such that $F(t) := \langle v(t), v(t) \rangle$ is constant. Show that $v'(t) \perp v(t)$ for all t . Show the converse is also true.
2. Suppose that c is regular. Let $s(t) = \int_0^t |c'(\tau)| d\tau$ and consider c parametrized by s instead of t . Since s gives the length of the image of $c : [0, s] \rightarrow \mathbb{E}^2$, s is called an *arclength parameter*. Show that in this preferred parametrization, $\kappa(s) = \left| \frac{de_1}{ds} \right|$.
3. Show that $\kappa(t)$ is constant iff the curve is an open subset of a line (if $\kappa = 0$) or circle of radius $\frac{1}{\kappa}$.
4. Let $c(t) = (x(t), y(t))$ be given in coordinates. Calculate $\kappa(t)$ in terms of $x(t), y(t)$ and their derivatives.
5. Calculate the function $\kappa(t)$ for an ellipse. Characterize the points on the ellipse where the maximum and minimum values of $\kappa(t)$ occur.
6. Can $\kappa(t)$ be unbounded if $c(t)$ is the graph of a polynomial?

Exercise 1.4.3 (Osculating circles):

- (a) Calculate the equation of a circle passing through three points in the plane.
- (b) Calculate the equation of a circle passing through two points in the plane and having a given tangent line at one of the points.

Parts (a) and (b) may be skipped; the exercise proper starts here:

- (c) Show that for any curve $c \subset \mathbb{E}^2$, at each point $x \in c$ one can define an *osculating circle* by taking the limit of the circle through the three points

$c(t), c(t_1), c(t_2)$ as $t_1, t_2 \rightarrow t$. (A line is defined to be a circle of infinite radius.)

(d) Show that one gets the same circle if one takes the limit as $t \rightarrow t_1$ of the circle through $c(t), c(t_1)$ that has tangent line at $c(t)$ parallel to $c'(t)$.

(e) Show that the radius of the osculating circle is $1/\kappa(t)$.

(f) Show that if $\kappa(t)$ is monotone, then the osculating circles are nested. \odot

1.5. Submanifolds of homogeneous spaces

Using the machinery we develop in this section and §1.6, we will answer the questions about curves in \mathbb{E}^2 posed at the beginning of §1.4. The quotient $\mathbb{E}^2 = ASO(2)/SO(2)$ is an example of a *homogeneous space*, and our answers will follow from a general study of classifying maps into homogeneous spaces.

Definition 1.5.1. Let G be a Lie group, H a closed Lie subgroup, and G/H the set of left cosets of H . Then G/H is naturally a differentiable manifold with the induced differentiable structure coming from the quotient map (see [77], Theorem II.3.2). The space G/H is called a *homogeneous space*.

Definition 1.5.2 (Left and right actions). Let G be a group that acts on a set X by $x \mapsto \sigma(g)(x)$. Then σ is called a *left action* if $\sigma(a) \circ \sigma(b) = \sigma(ab)$, or a *right action* if $\sigma(a) \circ \sigma(b) = \sigma(ba)$,

For example, the action of G on itself by left-multiplication is a left action, while left-multiplication by g^{-1} is a right action.

A homogeneous space G/H has a natural (left) G -action on it; the subgroup stabilizing $[e]$ is H , and the stabilizer of any point is conjugate to H . Conversely, a manifold X is a homogeneous space if it admits a smooth transitive action by a Lie group G . If H is the isotropy group of a point $x_0 \in X$, then $X \simeq G/H$, and x_0 corresponds to $[e] \in G/H$, the coset of the identity element. (See [77, 142] for additional facts about homogeneous spaces.)

In the spirit of Klein's *Erlanger Programm* (see [76, 92] for historical accounts), we will consider G as the group of *motions* of G/H . We will study the geometry of submanifolds $M \subset G/H$, where two submanifolds $M, M' \subset G/H$ will be considered *equivalent* if there exists a $g \in G$ such that $g(M) = M'$.

To determine necessary conditions for equivalence we will find differential invariants as we did in §1.1 and §1.4. (Note that we need to specify whether we are interested in invariants of a mapping or just of the image.) After finding invariants, we will then interpret them as we did in the exercises in §1.4.

Euclidean Geometry and Riemannian Geometry

In this chapter we return to the study of surfaces in Euclidean space $\mathbb{E}^3 = ASO(3)/SO(3)$. Our goal is not just to understand Euclidean geometry, but to develop techniques for solving equivalence problems for submanifolds of arbitrary homogeneous spaces. We begin with the problem of determining if two surfaces in \mathbb{E}^3 are locally equivalent up to a Euclidean motion. More precisely, given two immersions $f, \tilde{f} : U \rightarrow \mathbb{E}^3$, where U is a domain in \mathbb{R}^2 , when do there exist a local diffeomorphism $\phi : U \rightarrow U$ and a fixed $A \in ASO(3)$ such that $\tilde{f} \circ \phi = A \circ f$? Motivated by our results on curves in Chapter 1, we first try to find a complete set of *Euclidean differential invariants* for surfaces in \mathbb{E}^3 , i.e., functions I_1, \dots, I_r that are defined in terms of the derivatives of the parametrization of a surface, with the property that $f(U)$ differs from $\tilde{f}(U)$ by a Euclidean motion if and only if $(\tilde{f} \circ \phi)^* I_j = f^* I_j$ for $1 \leq j \leq r$.

In §2.1 we derive the Euclidean differential invariants Gauss curvature K and mean curvature H using moving frames. Unlike with curves in \mathbb{E}^3 , for surfaces in \mathbb{E}^3 there is not always a unique lift to $ASO(3)$, and we are led to define the *space of adapted frames*. (Our discussion of adapted frames for surfaces in \mathbb{E}^3 is later generalized to higher dimensions and codimensions in §2.5.) We calculate the functions H, K for two classical classes of surfaces in §2.2; developable surfaces and surfaces of revolution, and discuss basic properties of these surfaces.

Scalar-valued differential invariants turn out to be insufficient (or at least not convenient) for studying equivalence of surfaces and higher-dimensional submanifolds, and we are led to introduce *vector bundle valued invariants*. This study is motivated in §2.4 and carried out in §2.5, resulting in the definitions of the first and second fundamental forms, I and II . In §2.5 we also interpret II and Gauss curvature, define the Gauss map and derive the Gauss equation for surfaces.

Relations between intrinsic and extrinsic geometry of submanifolds of Euclidean space are taken up in §2.6, where we prove Gauss's theorem egregium, derive the Codazzi equation, discuss frames for C^∞ manifolds and Riemannian manifolds, and prove the fundamental lemma of Riemannian geometry. We include many exercises about connections, curvature, the Laplacian, isothermal coordinates and the like. We conclude the section with the fundamental theorem for hypersurfaces.

In §2.7 and §2.8 we discuss two topics we will need later on, space forms and curves on surfaces. In §2.9 we discuss and prove the Gauss-Bonnet and Poincaré-Hopf theorems. We conclude this chapter with a discussion of non-orthonormal frames in §2.10, which enables us to finally prove the formula (1.3) and show that surfaces with H identically zero are minimal surfaces.

The geometry of surfaces in \mathbb{E}^3 is studied further in §3.1 and throughout Chapters 5–7. Riemannian geometry is discussed further in Chapter 8.

2.1. Gauss and mean curvature via frames

Guided by Cartan's Theorem 1.6.11, we begin our search for differential invariants of immersed surfaces $f : U^2 \rightarrow \mathbb{E}^3$ by trying to find a lift $F : U \rightarrow SO(3)$ which is adapted to the geometry of $M = f(U)$. The most naïve lift would be to take

$$F(p) = \begin{pmatrix} 1 & 0 \\ f(p) & \text{Id} \end{pmatrix}.$$

Any other lift \tilde{F} is of the form

$$\tilde{F} = F \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$$

for some map $R : U \rightarrow SO(3)$.

Let $x = f(p)$; then $T_x\mathbb{E}^3$ has distinguished subspaces, namely $f_*(T_pU)$ and its orthogonal complement. We use our rotational freedom to adapt to this situation by requiring that e_3 always be normal to the surface, or equivalently that $\{e_1, e_2\}$ span T_xM . This is analogous to our choice of coordinates at our preferred point in Chapter 1, but is more powerful since it works on an open set of points in U .

Projective Geometry

This chapter may be considered as an update to the paper of Griffiths and Harris [69], which began a synthesis of modern algebraic geometry and moving frames techniques. Other than the first three sections, it may be skipped by readers anxious to arrive at the Cartan-Kähler Theorem. An earlier version of this chapter, containing more algebraic results than presented here, constituted the monograph [107].

We study the local geometry of submanifolds of projective space and applications to algebraic geometry. We begin in §3.1 with a discussion of Grassmannians, one of the most important classes of manifolds in all of geometry, and some uses of Grassmannians in Euclidean geometry. We define the Euclidean and projective Gauss maps. We then describe moving frames for submanifolds of projective space and define the projective second fundamental form in §3.2. In §3.3 we give some basic definitions from algebraic geometry. We give examples of homogeneous algebraic varieties and explain several constructions of auxiliary varieties from a given variety $X \subset \mathbb{P}V$: the secant variety $\sigma(X)$, the tangential variety $\tau(X)$ and the dual variety X^* . In §3.4 we describe the basic properties of varieties with degenerate Gauss maps and classify the surface case. We return in §§3.5–3.7 to discuss moving frames and differential invariants in more detail, with plenty of homogeneous examples in §3.6. We discuss osculating hypersurfaces and prove higher-order Bertini theorems in §3.7.

In §3.8 and §3.9, we apply our machinery respectively to study uniruled varieties and to characterize quadric hypersurfaces (Fubini's Theorem). Varieties with degenerate duals and associated varieties are discussed in §3.10 and §3.11 respectively. We prove the bounds of Zak and Landman on the dual defect from our differential-geometric perspective. In §3.12 we study

varieties with degenerate Gauss images in further detail. In §3.14 we state and prove *rank restriction theorems*: we show that the projective second fundamental form has certain genericity properties in small codimension if X is not too singular. We describe how to calculate $\dim \sigma(X)$ and $\dim \tau(X)$ infinitesimally, and state the Fulton-Hansen Theorem relating tangential and secant varieties. In §3.13 we state Zak's theorem classifying *Severi varieties*, the smooth varieties of minimal codimension having secant defects. Section §3.15 is dedicated to the proof of Zak's theorem. In §3.16 we generalize Fubini's Theorem to higher codimension, and finally in §3.17 we discuss applications to the study of complete intersections.

In this chapter, when we work over the complex numbers, all tangent, cotangent, etc., spaces are the holomorphic tangent, cotangent, etc., spaces (see Appendix C). We will generally use X to denote an algebraic variety and M to denote a complex manifold.

Throughout this chapter we often commit the following abuse of notation: We omit the \circ in symmetric products and the \otimes when the product is clear from the context. For example, we will often write $\omega_0^\alpha e_\alpha$ for $\omega_0^\alpha \otimes e_\alpha$.

3.1. Grassmannians

In projective geometry, Grassmannians play a central role, so we begin with a study of Grassmannians and the Plücker embedding. We also give applications to Euclidean geometry.

We fix index ranges $1 \leq i, j \leq k$, and $k+1 \leq s, t, u \leq n$ for this section.

Let V be a vector space over \mathbb{R} or \mathbb{C} and let $G(k, V)$ denote the *Grassmannian* of k -planes that pass through the origin in V . To specify a k -plane E , it is sufficient to specify a basis v_1, \dots, v_k of E . We continue our notational convention that $\{v_1, \dots, v_k\}$ denotes the span of the vectors v_1, \dots, v_k . After fixing a reference basis, we identify $GL(V)$ with the set of bases for V , and define a map

$$\begin{aligned} \pi : GL(V) &\rightarrow G(k, V), \\ (e_1, \dots, e_n) &\mapsto \{e_1, \dots, e_k\}, \end{aligned}$$

If we let $\tilde{e}_1, \dots, \tilde{e}_n$ denote the standard basis of V , i.e., \tilde{e}_A is a column vector with a 1 in the A -th slot and zeros elsewhere, the fiber of this mapping over $\pi(\text{Id}) = \{\tilde{e}_1, \dots, \tilde{e}_k\}$, is the subgroup

$$P_k = \left\{ g = \begin{pmatrix} g_j^i & g_s^i \\ 0 & g_t^s \end{pmatrix} \mid \det(g) \neq 0 \right\} \subset GL(V).$$

More generally, for $g \in GL(V)$, $\pi^{-1}(\pi(g)) = gP_k g^{-1}$.

Of particular importance is *projective space* $\mathbb{P}V = G(1, V)$, the space of all lines through the origin in V . We define a *line* in $\mathbb{P}V$ to be the

Cartan-Kähler I: Linear Algebra and Constant-Coefficient Homogeneous Systems

We have seen that differentiating the forms that generate an exterior differential system often reveals additional conditions that integral manifolds must satisfy (e.g., the Gauss and Codazzi equations for a surface in Euclidean space). The conditions are consequences of the fact that mixed partials must commute. What we did not see was a way of telling when one has differentiated enough to find all hidden conditions. We do know the answer in two cases: If a system is in Cauchy-Kowalevski form there are no extra conditions. In the case of the Frobenius Theorem, if the system passes a first-order test, then there are no extra conditions.

What will emerge over the next few chapters is a test, called *Cartan's Test*, that will tell us when we have differentiated enough.

The general version of Cartan's Test is described in Chapter 7. For a given integral element $E \in \mathcal{V}_n(\mathcal{I})_x$ of an exterior differential system \mathcal{I} on a manifold Σ , it guarantees existence of an integral manifold to the system with tangent plane E if E passes the test.

In Chapter 5, we present a version of Cartan's Test valid for a class of exterior differential systems with independence condition called *linear Pfaffian systems*. These are systems that are generated by 1-forms and have the additional property that the variety of integral elements through a

point $x \in \Sigma$ is an affine space. The class of linear Pfaffian systems includes all systems of PDE expressed as exterior differential systems on jet spaces. One way in which a linear Pfaffian system is simpler than a general EDS is that an integral element $E \in \mathcal{V}_n(\mathcal{I}, \Omega)_x$ passes Cartan's Test iff all integral elements at x do.

In this chapter we study first-order, constant-coefficient, homogeneous systems of PDE for analytic maps $f : V \rightarrow W$ expressed in terms of *tableaux*. We derive *Cartan's Test* for this class of systems, which determines if the initial data one might naïvely hope to specify (based on counting equations) actually determines a solution.

We dedicate an entire chapter to such a restrictive class of EDS because at each point of a manifold Σ with a linear Pfaffian system there is a naturally defined tableau, and the system passes Cartan's Test for linear Pfaffian systems at a point $x \in \Sigma$ if and only if its associated tableau does and the *torsion* of the system (defined in Chapter 5) vanishes at x .

In analogy with the inverse function theorem, Cartan's Test for linear Pfaffian systems (and even in its most general form) implies that if the linear algebra at the infinitesimal level works out right, the rest follows. What we do in this chapter is determine what it takes to get the linear algebra to work out right.

Throughout this chapter, V is an n -dimensional vector space, and W is an s -dimensional vector space. We use the index ranges $1 \leq i, j, k \leq n$, $1 \leq a, b, c \leq s$. V has the basis v_1, \dots, v_n and V^* the corresponding dual basis v^1, \dots, v^n ; W has basis w_1, \dots, w_s and W^* the dual basis w^1, \dots, w^s .

4.1. Tableaux

Let $x = x^i v_i, u = u^a w_a$ denote elements of V and W respectively. We will consider (x^1, \dots, x^n) , respectively (u^1, \dots, u^n) , as coordinate functions on V and W respectively. Any first-order, constant-coefficient, homogeneous system of PDE for maps $f : V \rightarrow W$ is given in coordinates by equations

$$(4.1) \quad B_a^{ri} \frac{\partial u^a}{\partial x^i} = 0, \quad 1 \leq r \leq R,$$

where the B_a^{ri} are constants. For example, the Cauchy-Riemann system $u_{x_1}^1 - u_{x_2}^2 = 0, u_{x_2}^1 + u_{x_1}^2 = 0$ has $B_1^{11} = 1, B_2^{12} = -1, B_1^{12} = 0, B_2^{11} = 0, B_2^{21} = 1, B_1^{22} = 1, B_1^{21} = 0$ and $B_2^{22} = 0$.

Cartan-Kähler II: The Cartan Algorithm for Linear Pfaffian Systems

We now generalize the test from Chapter 4 to a test valid for a large class of exterior differential systems called *linear Pfaffian systems*, which are defined in §5.1. In §§5.2–5.4 we present three examples of linear Pfaffian systems that lead us to Cartan’s algorithm and the definitions of torsion and prolongation, all of which are given in §5.5. For easy reference, we give a summary and flowchart of the algorithm in §5.6. Additional aspects of the theory, including characteristic hyperplanes, Spencer cohomology and the Goldschmidt version of the Cartan-Kähler Theorem, are given in §5.7. In the remainder of the chapter we give numerous examples, beginning with elementary problems coming mostly from surface theory in §5.8, then an example motivated by variation of Hodge structure in §5.9, then the Cartan-Janet Isometric Immersion Theorem in §5.10, followed by a discussion of isometric embeddings of space forms in §5.11 and concluding with a discussion of calibrations and calibrated submanifolds in §5.12.

5.1. Linear Pfaffian systems

Recall that a Pfaffian system on a manifold Σ is an exterior differential system generated by 1-forms, i.e., $\mathcal{I} = \{\theta^a\}_{\text{diff}}$, $\theta^a \in \Omega^1(\Sigma)$, $1 \leq a \leq s$. If $\Omega = \omega^1 \wedge \dots \wedge \omega^n$ represents an independence condition, let $J := \{\theta^a, \omega^i\}$

and $I := \{\theta^a\}$. We will often use J to indicate the independence condition in this chapter, and refer to the system as (I, J) .

Definition 5.1.1. (I, J) is a *linear Pfaffian system* if $d\theta^a \equiv 0 \pmod{J}$ for all $1 \leq a \leq s$.

Exercise 5.1.2: Let (I, J) be a linear Pfaffian system as above. Let π^ϵ , $1 \leq \epsilon \leq \dim \Sigma - n - s$, be a collection of 1-forms such that $T^*\Sigma$ is locally spanned by $\theta^a, \omega^i, \pi^\epsilon$. Show that there exist functions A_{ei}^a, T_{ij}^a defined on Σ such that

$$(5.1) \quad d\theta^a \equiv A_{ei}^a \pi^\epsilon \wedge \omega^i + T_{ij}^a \omega^i \wedge \omega^j \pmod{I}.$$

Example 5.1.3. The canonical contact system on $J^2(\mathbb{R}^2, \mathbb{R}^2)$ is a linear Pfaffian system because

$$\begin{aligned} d(du - p_1^1 dx - p_2^1 dy) &= -dp_1^1 \wedge dx - dp_2^1 \wedge dy \\ &\equiv 0 \pmod{\{dx, dy, du - p_1^1 dx - p_2^1 dy, dv - p_1^2 dx - p_2^2 dy\}}, \\ d(dv - p_1^2 dx - p_2^2 dy) &= -dp_1^2 \wedge dx - dp_2^2 \wedge dy \\ &\equiv 0 \pmod{\{dx, dy, du - p_1^1 dx - p_2^1 dy, dv - p_1^2 dx - p_2^2 dy\}}, \end{aligned}$$

and the same calculation shows that the pullback of this system to any submanifold $\Sigma \subset J^2(\mathbb{R}^2, \mathbb{R}^2)$ is linear Pfaffian. More generally, we have

Example 5.1.4. Any system of PDE expressed as the pullback of the contact system on $J^k(M, N)$ to a subset Σ is a linear Pfaffian system. If M has local coordinates (x^1, \dots, x^n) and N has local coordinates (u^1, \dots, u^s) , then $J^k = J^k(M, N)$ has local coordinates (x^i, u^a, p_i^a, p_L^a) , where $L = (l_1, \dots, l_p)$ is a symmetric multi-index of length $p \leq k-1$. In these coordinates, the contact system I is $\{\theta^a = du^a - p_i^a dx^i, \theta_L^a = dp_L^a - p_{Lj}^a dx^j\}$, and $J = \{\theta^a, \theta_L^a, dx^i\}$. On $J^k(M, N)$,

$$\left. \begin{aligned} d\theta^a &= -dp_j^a \wedge dx^j \\ d\theta_L^a &= -dp_{Lj}^a \wedge dx^j \end{aligned} \right\} \equiv 0 \pmod{J},$$

and these equations continue to hold when we restrict to any subset $\Sigma \subset J^k$.

Example 5.1.5. On \mathbb{R}^6 , let $\theta = y^1 dy^2 + y^3 dy^4 + y^5 dx$, let $I = \{\theta\}$ and $J = \{\theta, dx\}$. Then

$$\begin{aligned} d\theta &= dy^1 \wedge dy^2 + dy^3 \wedge dy^4 + dy^5 \wedge dx \\ &\equiv (dy^3 - \frac{y^3}{y^1} dy^1) \wedge dy^4 \pmod{\{\theta, dx\}}. \end{aligned}$$

In this case, (I, J) is not a linear Pfaffian system.

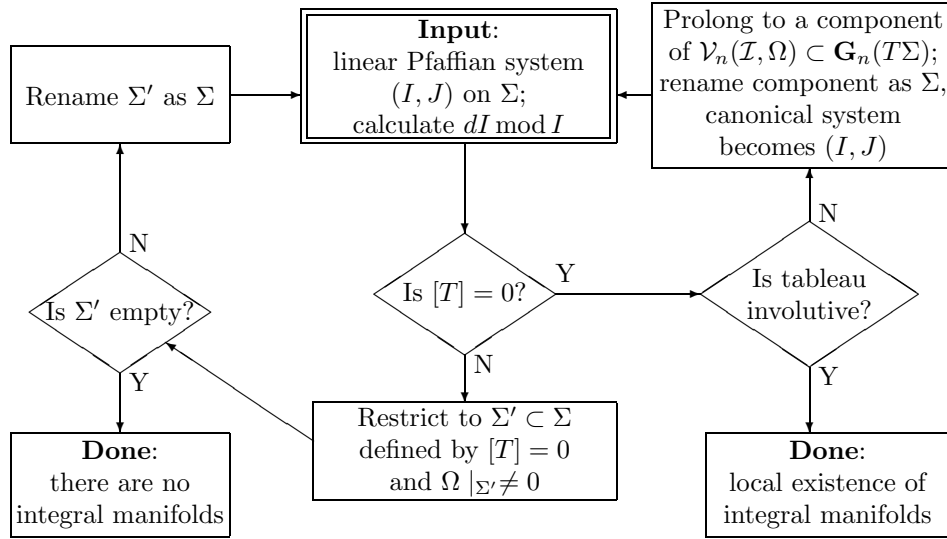


Figure 1. Our final flowchart.

Warning: Where the algorithm ends up (i.e., in which “Done” box in the flowchart) may depend on the component one is working on.

Summary. Let (I, J) be a linear Pfaffian system on Σ . We summarize Cartan’s algorithm:

- (1) Take a local coframing of Σ adapted to the filtration $I \subset J \subset T^*\Sigma$. Let $x \in \Sigma$ be a general point. Let $V^* = (J/I)_x$, $W^* = I_x$, and let $v^i = \omega_x^i, w^a = \theta_x^a$ and v_i, w_a be the corresponding dual basis vectors.
- (2) Calculate $d\theta^a$; since the system is linear, these are of the form

$$d\theta^a \equiv A_{\epsilon i}^a \pi^\epsilon \wedge \omega^i + T_{ij}^a \omega^i \wedge \omega^j \text{ mod } I.$$

Define the *tableau* at x by

$$A = A_x := \{A_{\epsilon i}^a v^i \otimes w_a \in V^* \otimes W \mid 1 \leq \epsilon \leq r\} \subseteq W \otimes V^*.$$

Let δ denote the natural skew-symmetrization map $\delta : W \otimes V^* \otimes V^* \rightarrow W \otimes \Lambda^2 V^*$ and let

$$H^{0,2}(A) := W \otimes \Lambda^2 V^* / \delta(A \otimes V^*).$$

The *torsion* of (I, J) at x is

$$[T]_x := [T_{ij}^a w_a \otimes v^i \wedge v^j] \in H^{0,2}(A).$$

- (3) If $[T]_x \neq 0$, then start again on $\Sigma' \subset \Sigma$ defined by the equations $[T] = 0$, with the additional requirement that J/I has rank n over

Σ' . Since the additional requirement is a transversality condition, it will be generically satisfied as long as $\dim \Sigma' \geq n$. In practice one works infinitesimally, using the equations $d[T] = 0$, and checks what relations $d[T] \equiv 0 \pmod I$ imposes on the forms π^ϵ used before.

- (4) Assume $[T]_x = 0$. Let $A_k := A \cap (\text{span}\{v^{k+1}, \dots, v^n\} \otimes W)$. Let $A^{(1)} := (A \otimes V^*) \cap (W \otimes S^2 V^*)$, the *prolongation* of the tableau A . Then

$$\dim A^{(1)} \leq \dim A + \dim A_1 + \dots + \dim A_{n-1}$$

and A is *involutive* if equality holds.

Warning: One can fail to obtain equality even when the system is involutive if the bases were not chosen sufficiently generically. In practice, one does the calculation with a natural, but perhaps nongeneric basis and takes linear combinations of the columns of A to obtain genericity. If the bases are generic, then equality holds iff A is involutive.

When doing calculations, it is convenient to define the *characters* s_k by $s_1 + \dots + s_k = \dim A - \dim A_k$, in which case the inequality becomes $\dim A^{(1)} \leq s_1 + 2s_2 + \dots + ns_n$. If $s_p \neq 0$ and $s_{p+1} = 0$, then s_p is called the *character* of the tableau and p the *Cartan integer*. If A is involutive, then the Cartan-Kähler Theorem applies, and one has local integral manifolds depending on s_p functions of p variables.

- (5) If A is not involutive, prolong, i.e., start over on the pullback of the canonical system on the Grassmann bundle to the space of integral elements. In calculations this amounts to adding the elements of $A^{(1)}$ as independent variables, and adding differential forms $\theta_i^a := A_{ci}^a \pi^\epsilon - p_{ij}^a \omega^j$ to the ideal, where $p_{ij}^a v^i v^j \otimes w_a \in A^{(1)}$.

5.7. Additional remarks on the theory

Another interpretation of $A^{(1)}$. We saw in Chapter 4 that for a constant-coefficient, first-order, homogeneous system defined by a tableau A , the prolongation $A^{(1)}$ is the space of admissible second-order terms $p_{ij}^a x^i x^j$ in a power series solution of the system. This was because the constants p_{ij}^a had to satisfy $p_{ij}^a w_a \otimes v^i \otimes v^j \in A^{(1)}$.

The following proposition, which is useful for computing $A^{(1)}$, is the generalization of this observation to linear Pfaffian systems:

Proposition 5.7.1. *After fixing $x \in \Sigma$ and a particular choice of 1-forms $\pi_i^a \pmod I$ satisfying $d\theta^a \equiv \pi_i^a \wedge \omega^i \pmod I$, $A^{(1)}$ may be identified with the space of 1-forms $\tilde{\pi}_i^a \pmod I$ satisfying $d\theta^a \equiv \tilde{\pi}_i^a \wedge \omega^i \pmod I$, as follows: any such*

Applications to PDE

Introduction. Consider the well-known closed-form solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ due to d'Alembert:

$$(6.1) \quad u(x, t) = f(x + ct) + g(x - ct),$$

where f and g are arbitrary C^2 functions. It is rare that all solutions of a given PDE are obtained by a single formula (especially, one which does not involve integration). The key to obtaining the d'Alembert solution is to rewrite the equation in “characteristic coordinates” $\eta = x + ct, \xi = x - ct$, yielding $u_{\eta\xi} = 0$. By integrating in η or in ξ , we get

$$u_{\eta} = F(\eta), \quad u_{\xi} = G(\xi),$$

where F and G are independent arbitrary functions; then (6.1) follows by another integration. For which other PDE do such solution formulas exist? And, how can we find them in a systematic way?

In this chapter we study invariants of exterior differential systems that aid in constructing integral manifolds, and we apply these to the study of first and second-order partial differential equations, and classical surface theory. (For second-order PDE, we will restrict attention to equations for one function of two variables.) All functions and forms are assumed to be smooth.

In §6.1 we define *symmetry vector fields* and in particular *Cauchy characteristic vector fields* for EDS. We discuss the general properties of Cauchy characteristics, and use them to recover the classical result that any first-order PDE can be solved using ODE methods. In §6.2 we define the *Monge characteristic systems* associated to second-order PDE, and discuss *hyperbolic exterior differential systems*. In §6.3 we discuss a systematic

method, called *Darboux's method*, which helps uncover solution formulas like d'Alembert's (when they exist), and more generally determines when a given PDE is solvable by ODE methods. We also define the *derived systems* associated to a Pfaffian system.

In §6.4 we treat Monge-Ampère systems, focusing on several geometric examples. We show how solutions of the sine-Gordon equation enable us to explicitly parametrize surfaces in \mathbb{E}^3 with constant negative Gauss curvature. We show how consideration of complex characteristics, for equations for minimal surfaces and for surfaces of constant mean curvature (CMC), leads to solutions for these equations in terms of holomorphic data. In particular, we derive the Weierstrass representation for minimal surfaces and show that any CMC surface has a one-parameter family of non-congruent deformations.

In §6.5 we discuss integrable extensions and Bäcklund transformations. Examples discussed include the Cole-Hopf and Miura transformations, the KdV equation, and Bäcklund's original transformation for pseudospherical surfaces. We also prove Theorem 6.5.14, relating Bäcklund transformations to Darboux-integrability.

6.1. Symmetries and Cauchy characteristics

Infinitesimal symmetries. One of Lie's contributions to the theory of ordinary differential equations was to put the various solution methods for special kinds of equations in a uniform context, based on *infinitesimal symmetries* of the equation—i.e., vector fields whose flows take solutions to solutions. This generalizes to EDS when we let the vector fields act on differential forms via the Lie derivative operator \mathcal{L} (see Appendix B):

Definition 6.1.1. Let \mathcal{I} be an EDS on Σ . A vector field $\mathbf{v} \in \Gamma(T\Sigma)$ is an (infinitesimal) symmetry of \mathcal{I} if $\mathcal{L}_{\mathbf{v}}\psi \in \mathcal{I}$ for all $\psi \in \mathcal{I}$.

Exercises 6.1.2:

1. The Lie bracket of any two symmetries is a symmetry; thus, symmetries of a given EDS form a Lie algebra. \odot
2. To show that \mathbf{v} is a symmetry, it suffices to check the condition in Definition 6.1.1 on a set of forms which generate \mathcal{I} differentially. \odot
3. Let \mathcal{I} be the Pfaffian system generated by the contact form $\theta = dy - zdx$ on \mathbb{R}^3 . Verify that

$$(6.2) \quad \mathbf{v} = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + (g_x + zg_y - zf_x - z^2 f_y) \frac{\partial}{\partial z},$$

for any functions $f(x, y), g(x, y)$, is a symmetry. Then, find the most general symmetry vector field for \mathcal{I} . \odot

Cartan-Kähler III: The General Case

In this chapter we will discuss the Cartan-Kähler Theorem, which guarantees the existence of integral manifolds for arbitrary exterior differential systems in involution. This theorem is a generalization of the Cauchy-Kowalevski Theorem (see Appendix D), which gives conditions under which an analytic system of partial differential equations has an analytic solution (defined in the neighborhood of a given point) satisfying a Cauchy problem, i.e., initial data for the solution specified along a hypersurface in the domain. Similarly, the “initial data” for the Cartan-Kähler Theorem is an integral manifold of dimension n , which we want to extend to an integral manifold of dimension $n + 1$. In Cauchy-Kowalevski, the equations are assumed to be of a special form, which has the feature that no conflicts arise when one differentiates them and equates mixed partials. The condition of involutivity is a generalization of this, guaranteeing that no new integrability conditions arise when one looks at the equations that higher jets of solutions must satisfy.

The reader may wonder why one bothers to consider any exterior differential systems other than linear Pfaffian systems. For, as remarked in Chapter 5, the prolongation of any exterior differential system is a linear Pfaffian system, so theoretically it is sufficient to work with such systems. However, in practice it is generally better to work on the smallest space possible. In fact, certain spectacular successes of the EDS machinery were obtained by cleverly rephrasing systems that were naïvely expressed as linear Pfaffian systems as systems involving generators of higher degree on a smaller manifold. One elementary example of this is the study of linear

Weingarten surfaces (see §6.4); a more complex example is Bryant's proof of the existence of manifolds with holonomy G_2 [17].

We begin this chapter with a more detailed study of the space of integral elements of an EDS. Before proving the full Cartan-Kähler Theorem, in §7.2 we give an example that shows how one can use the Cauchy-Kowalevski Theorem to construct triply orthogonal systems of surfaces. (This also serves as a model for the proof in §7.3.) In §7.4 we discuss Cartan's Test, a procedure by which one can test for involution, and which was already described in Chapter 5 for the special case of linear Pfaffian systems. Then in §7.5 we give a few more examples that illustrate how one applies Cartan's Test in the non-Pfaffian case.

7.1. Integral elements and polar spaces

Suppose \mathcal{I} is an exterior differential system on Σ ; we will assume that \mathcal{I} contains no 0-forms (otherwise, we could restrict to subsets of Σ on which the 0-forms vanish). Recall from §1.9 that $\mathcal{V}_n(\mathcal{I})_p \subset G(n, T_p\Sigma)$ denotes the space of n -dimensional integral elements in $T_p\Sigma$ and $\mathcal{V}_n = \mathcal{V}_n(\mathcal{I}) \subset G_n(T\Sigma)$ the space of all n -dimensional integral elements. In this chapter we will obtain a criterion that guarantees that a given $E \in \mathcal{V}_n(\mathcal{I})_p$ is tangent to an integral manifold. We think of this as "extending" (p, E) to an integral manifold.

Coordinates on $G_n(T\Sigma)$. To study the equations that define \mathcal{V}_n , we use local coordinates on the Grassmann bundle $G_n(T\Sigma)$. Given $E \in G_n(T_p\Sigma)$, there are coordinates x^1, \dots, x^n and y^1, \dots, y^s on Σ near p such that E is spanned by the vectors $\partial/\partial x^i$. By continuity, there is a neighborhood of E in $G_n(T\Sigma)$ consisting of n -planes \tilde{E} such that $dx^1 \wedge \dots \wedge dx^n|_{\tilde{E}} \neq 0$. For each \tilde{E} , there are numbers p_i^a such that $dy^a|_{\tilde{E}} = p_i^a dx^i|_{\tilde{E}}$; these p_i^a , along with the x 's and y 's, form a local coordinate system on $G_n(T\Sigma)$.

Recall from Exercise 1.9.4 that $E \in \mathcal{V}_n(\mathcal{I})$ if and only if every $\psi \in \mathcal{I}^n$ vanishes on E . Each such ψ has some expression

$$\psi = \sum_{I, J} f_{IJ} dy^I \wedge dx^J,$$

where I and J are multi-indices with components in increasing order, such that $|I| + |J| = n$, and the f_{IJ} are smooth functions on Σ . Then $\psi|_{\tilde{E}} = F_\psi dx^1 \wedge \dots \wedge dx^n|_{\tilde{E}}$, where the F_ψ are polynomials in the p_i^a , given by

$$F = \sum_{I, J, L} f_{I, J, L}(x, y) p_{l_1}^{i_1} \dots p_{l_k}^{i_k} dx^L \wedge dx^J,$$

Geometric Structures and Connections

We study the *equivalence problem* for geometric structures. That is, given two geometric structures (e.g., pairs of Riemannian manifolds, pairs of manifolds equipped with foliations, etc.), we wish to find differential invariants that determine existence of a local diffeomorphism preserving the geometric structures. We begin in §8.1 with two examples, 3-webs in the plane and Riemannian geometry, before concluding the section by defining G -structures. In order to find differential invariants, we will need to take derivatives in some geometrically meaningful way, and we spend some time (§§8.2–8.4) figuring out just how to do this. In §8.3 we define connections on coframe bundles and briefly discuss a general algorithm for finding differential invariants of G -structures. In §8.5 we define and discuss the holonomy of a Riemannian manifold. In §8.6 we present an extended example of the equivalence problem, finding the differential invariants of a path geometry. Finally, in §8.7 we discuss generalizations of G -structures and recent work involving these generalizations.

8.1. G -structures

In this section we present two examples of G -structures and then give a formal definition.

First example: 3-webs in \mathbb{R}^2 .

First formulation of the question. Let $\mathcal{L} = \{L_1, L_2, L_3\}$ be a collection of three pairwise transverse foliations of an open subset $U \subseteq \mathbb{R}^2$. Such a structure is called a *3-web*; see Figure 1.

Let $\tilde{\mathcal{L}} = \{\tilde{L}_1, \tilde{L}_2, \tilde{L}_3\}$ be another 3-web on an open subset $\tilde{U} \subset \mathbb{R}^2$.

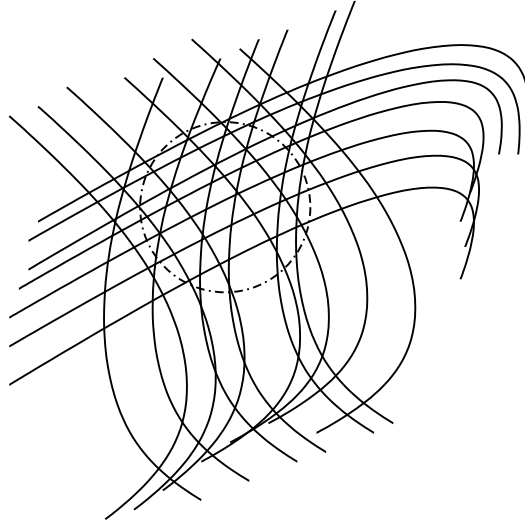


Figure 1. U is the region inside the circle

Problem 8.1.1. When does there exist a diffeomorphism $\phi : U \rightarrow \tilde{U}$ such that $\phi^*(\tilde{L}_j) = L_j$?

If there exists such a ϕ , we will say the webs $\mathcal{L}, \tilde{\mathcal{L}}$ are *equivalent*. We would like to find differential invariants that determine when two webs are equivalent, as we did for local equivalence of submanifolds of homogeneous spaces.

In particular, let \mathcal{L}^0 be the 3-web

$$L_1^0 = \{y = \text{const}\}, \quad L_2^0 = \{x = \text{const}\}, \quad L_3^0 = \{y - x = \text{const}\};$$

call this the *flat case*. We ask: when is a 3-web locally equivalent to the flat case?

Second formulation of the question. Let $y' = F(x, y)$ be an ordinary differential equation in the plane. Let $y' = \tilde{F}(x, y)$ be another.

Problem 8.1.2. When does there exist a change of coordinates $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the form $\psi(x, y) = (\alpha(x), \beta(y))$ such that $\psi^*\tilde{F} = (\beta'/\alpha')F$? (I.e., so that solutions to one ODE are carried to solutions of the other.)

In particular, given F , is it equivalent to $y' = 1$ via a change of coordinates of the form of ψ ?

Exercise 8.1.3: Determine local equivalence of first-order ordinary differential equations in the plane $y' = F(x, y)$ under arbitrary changes of coordinates. ©

To see that Problems 8.1.1, 8.1.2 are the same, note that any two transverse foliations can be used to give local coordinates x, y , and the space of integral curves of an ODE in coordinates provides the third foliation. The diffeomorphisms of \mathbb{R}^2 that preserve the two coordinate foliations are exactly those of the form of ψ .

In order to study the local equivalence of webs, we would like to associate a coframe to a 3-web. For example, we could take a coframe $\{\underline{\omega}^1, \underline{\omega}^2\}$ such that

- (a) $\underline{\omega}^1$ annihilates L_1 ,
- (b) $\underline{\omega}^2$ annihilates L_2 , and
- (c) $\underline{\omega}^1 - \underline{\omega}^2$ annihilates L_3 .

In the case of an ODE in coordinates, we could similarly take $\underline{\omega}^1 = F(x, y)dx, \underline{\omega}^2 = dy$.

Remark 8.1.4. Note that we are imitating the flat model (\mathcal{L}^0) on the infinitesimal level. This is what we did in Chapter 2 for Riemannian geometry when we took a basis of the cotangent space corresponding to the standard flat structure on the infinitesimal level. Just as any Riemannian metric looks flat to first order, so does any 3-web in the plane.

Just as in the case of choosing a frame for a submanifold of a homogeneous space, we need to determine how unique our choice of adapted frame is, and we will then work on the space of adapted frames.

Any other frame satisfying conditions (a) and (b) must satisfy

$$\begin{aligned}\tilde{\omega}^1 &= \lambda^{-1}\underline{\omega}^1, \\ \tilde{\omega}^2 &= \mu^{-1}\underline{\omega}^2\end{aligned}$$

for some nonvanishing functions λ, μ . Any frame satisfying (c) must be of the form

$$\tilde{\omega}^2 - \tilde{\omega}^1 = \nu^{-1}(\underline{\omega}^2 - \underline{\omega}^1).$$

Combining these three conditions, we see $\lambda = \mu = \nu$. Let $\mathcal{F}_{\mathcal{L}} \subset \mathcal{F}_{\text{GL}}(U)$ be the space of coframes satisfying (a),(b),(c), a fiber bundle with fiber $\simeq \mathbb{R}^*$.

Dually, if we write a point of $\mathcal{F}_{\mathcal{L}}$ as a frame $f = (p, e_1, e_2)$, then e_1 is tangent to $L_{2,p}$, e_2 is tangent to $L_{1,p}$, and $e_1 + e_2$ is tangent to $L_{3,p}$.

Fixing a section $(\underline{\omega}^1, \underline{\omega}^2)$, we may use local coordinates (x, y, λ) on $\mathcal{F}_{\mathcal{L}}$. On $\mathcal{F}_{\mathcal{L}}$ we have tautological forms

$$\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} \underline{\omega}^1 \\ \underline{\omega}^2 \end{pmatrix}.$$

Exercise 8.1.5: Show that these tautological forms are the pullbacks of the tautological forms of $\mathcal{F}_{\text{GL}}(U)$. Thus, they are independent of our initial choice of $\underline{\omega}^1, \underline{\omega}^2$.

For submanifolds of homogeneous spaces, there was a canonical coframing of the space of adapted frames. Here, we have two 1-forms, but $\dim \mathcal{F}_{\mathcal{L}} = 3$ so we seek a third 1-form. When we faced the problem of completing a set of geometrically determined vectors to a basis before (as with submanifolds of homogeneous spaces), we differentiated. We do the same here:

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \frac{d\lambda}{\lambda^2} & 0 \\ 0 & \frac{d\lambda}{\lambda^2} \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \lambda \begin{pmatrix} d\underline{\omega}^1 \\ d\underline{\omega}^2 \end{pmatrix}.$$

Since $\lambda d\underline{\omega}^j$ is semi-basic for the projection to \mathbb{R}^2 , we may write $\lambda d\underline{\omega}^j = T^j \omega^1 \wedge \omega^2$ for some functions $T^1, T^2 : \mathcal{F}_{\mathcal{L}} \rightarrow \mathbb{R}$. Write $\theta = \frac{d\lambda}{\lambda}$. Our equations now have the form

$$(8.1) \quad d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} T^1 \omega^1 \wedge \omega^2 \\ T^2 \omega^1 \wedge \omega^2 \end{pmatrix}.$$

In analogy with the situation in §5.5, we will refer to the terms T^1, T^2 as “apparent torsion”. More precisely, as we will see in §3, this is the *torsion of θ* . The forms $\omega^1, \omega^2, \theta$ give a coframing of $\mathcal{F}_{\mathcal{L}}$, but θ is not uniquely determined. In fact the choice of a θ satisfying (8.1) is unique up to modification by ω^1, ω^2 : any other choice must be of the form $\tilde{\theta} = \theta + a\omega^1 + b\omega^2$. In particular, if we choose $\tilde{\theta} = \theta - T^2\omega^1 + T^1\omega^2$, our new choice has the effect that the apparent torsion is zero; there is a unique such form θ . So, renaming $\tilde{\theta}$ as θ , we have

Proposition 8.1.6. *There exists a unique form $\theta \in \Omega^1(\mathcal{F}_{\mathcal{L}})$ such that the equations*

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

are satisfied.

Any choice of θ such that the derivative of the tautological forms is of the form (8.1) will be called a *connection* (or connection form), and a choice of θ such that the torsion of θ is zero will be called a *torsion-free connection*.

The canonical coframing $(\omega^1, \omega^2, \theta)$ that we have constructed on $\mathcal{F}_{\mathcal{L}}$ enables us to begin to solve Problem 8.1.1:

Corollary 8.1.7. *Let $\phi : U \rightarrow U'$ be a diffeomorphism such that $\phi^*(\tilde{L}_j) = L_j$, and let $\Phi : \mathcal{F}_{\text{GL}}(U) \rightarrow \mathcal{F}_{\text{GL}}(\tilde{U})$ be the induced diffeomorphism on the coframe bundles (i.e., Φ takes a coframe to its pullback under ϕ^{-1}). Then $\Phi(\mathcal{F}_{\mathcal{L}}) = \Phi(\tilde{\mathcal{F}}_{\mathcal{L}})$ and $\Phi^*(\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\theta}) = (\omega^1, \omega^2, \theta)$.*

Exercise 8.1.8: Prove this corollary.

Bibliography

- [1] L. Ahlfors, *Complex Analysis*, McGraw-Hill, 1966.
- [2] D.N. Akhiezer, *Lie group actions in complex analysis*, Vieweg, 1995.
- [3] M. Akivis, Webs and almost Grassmann structures, *Soviet Math. Dokl.* 21 (1980), 707–709.
- [4] M. Akivis, V. Goldberg, On the structure of submanifolds with degenerate Gauss maps, *Geom. Dedicata* 86 (2001), 205–226.
- [5] I. Anderson, N. Kamran, The variational bicomplex for hyperbolic second-order scalar partial differential equations in the plane, *Duke Math. J.* 87 (1997), 265–319.
- [6] I. Anderson, N. Kamran, P. Olver, Internal, External, and Generalized Symmetries, *Adv. Math* 100 (1993), 53–100.
- [7] M. Audin, J. Lafontaine (eds.), *Holomorphic curves in symplectic geometry*, Birkhäuser, 1994.
- [8] W. Barth, M. Larsen, On the homotopy groups of complex projective algebraic manifolds *Math. Scand.* 30 (1972), 88–94.
- [9] R. Baston, M. Eastwood, *The Penrose transform. Its interaction with representation theory*, Oxford University Press, 1989.
- [10] E. Berger, R. Bryant, P. Griffiths, The Gauss equations and rigidity of isometric embeddings, *Duke Math. J.* 50 (1983) 803–892.
- [11] A.L. Besse, *Einstein Manifolds*, Springer, 1987.
- [12] R. Bishop, There is more than one way to frame a curve, *Am. Math. Monthly* 82 (1975), 246–251.
- [13] A. Bobenko, Exploring surfaces through methods from the theory of integrable systems: Lectures on the Bonnet Problem, preprint (1999), available at <http://arXiv.org/math.DG/9909003>
- [14] R. Bott, L. Tu, *Differential forms in algebraic topology*, Springer, 1982.
- [15] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 4–6, Hermann, 1968.
- [16] R. Bryant, Conformal and Minimal Immersions of Compact Surfaces into the 4-sphere, *J. Diff. Geom.* 17 (1982), 455–473.
- [17] —, Metrics with exceptional holonomy, *Ann. of Math.* 126 (1987), 525–576.

- [18] —, Rigidity and quasi-rigidity of extremal cycles in Hermitian symmetric spaces, preprint available at [arXiv: math.DG/0006186](https://arxiv.org/abs/math/0006186).
- [19] R. Bryant, S.-S. Chern, P. Griffiths, Exterior Differential Systems, pp. 219–338 in *Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations*, Science Press, Beijing, 1982.
- [20] R. Bryant, S.-S. Chern, R.B. Gardner, H. Goldschmidt, P. Griffiths, *Exterior Differential Systems*, MSRI Publications, Springer, 1990.
- [21] R. Bryant, P. Griffiths, Characteristic Cohomology of Differential Systems (II): Conservation Laws for a Class of Parabolic Equations, *Duke Math. J.* 78 (1995), 531–676.
- [22] R. Bryant, P. Griffiths, L. Hsu, Hyperbolic exterior differential systems and their conservation laws (I), *Selecta Mathematica (N.S.)* 1 (1995), 21–112.
- [23] —, Toward a geometry of differential equations, pp. 1–76 in *Geometry, topology, and physics*, International Press, 1995.
- [24] R. Bryant, S. Salamon, On the construction of some complete metrics with exceptional holonomy, *Duke Math. J.* 58 (1989), 829–850.
- [25] J. Carlson, D. Toledo, Generic Integral Manifolds for weight two period domains, *Trans. AMS*, to appear.
- [26] E. Cartan, Sur la structure des groupes infinis de transformations, *Ann. Sci. Ecole Norm. Sup.* 26 (1909), 93–161.
- [27] —, Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, *Ann. Ecole Norm. Sup.* 27 (1910), 109–192.
- [28] —, Sur les variétés de courbure constante d'un espace euclidien ou non euclidien, *Bull. Soc. Math France* 47 (1919) 125–160 and 48 (1920), 132–208; see also pp. 321–432 in *Oeuvres Complètes* Part 3, Gauthier-Villars, 1955.
- [29] —, Sur les variétés à connexion projective, *Bull. Soc. Math. Fr.* 52 (1924), 205–241.
- [30] —, Sur la théorie des systèmes en involution et ses applications à la relativité, *Bull. Soc. Math. Fr.* 59 (1931), 88–118; see also pp. 1199–1230 in *Oeuvres Complètes*, Part 2.
- [31] —, *Les Systèmes Extérieurs et leurs Applications Géométriques*, Hermann, 1945.
- [32] P. Chaput, Severi varieties, *Math. Z.* 240 (2002), 451–459.
- [33] S.-S. Chern, *Selected Papers* Vols. 1–4, Springer, 1985/1989.
- [34] —, A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds, *Ann. Math* 45 (1944), 747–752.
- [35] —, Pseudo-groupes continus infinis, pp. 119–136 in *Colloques Internationaux du Centre National de la Recherche Scientifique: Géométrie différentielle*, C.N.R.S, 1953.
- [36] —, *Complex Manifolds without Potential Theory*, Springer, 1979.
- [37] —, Deformations of surfaces preserving principal curvatures, pp. 155–163 in *Differential Geometry and Complex Analysis*, Springer, 1984. (See also *Selected Papers*, vol. IV.)
- [38] S.-S. Chern, R. Osserman, Remarks on the Riemannian metric of a minimal submanifold pp. 49–90 in *Geometry Symposium, Utrecht 1980*, Lecture Notes in Math. 894, Springer, 1981.
- [39] S.-S. Chern, C.-L. Terng, An analogue of Bäcklund's theorem in affine geometry, *Rocky Mountain Math J.* 10 (1980), 105–124.
- [40] J. Clelland, T. Ivey, Parametric Bäcklund Transformations I: Phenomenology, preprint available at <http://arXiv.org/abs/math/0208035>
- [41] D. Cox, S. Katz, *Mirror symmetry and algebraic geometry*, AMS, 1999.
- [42] G. Darboux, *Leçons sur la Théorie Générale des Surfaces* (3rd ed.), Chelsea, 1972.

- [43] M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.
- [44] L. Ein, Varieties with small dual varieties, I, *Inventiones Math.* 86 (1986), 63–74.
- [45] —, Varieties with small dual varieties, II, *Duke Math. J.* 52 (1985), 895–907.
- [46] Ph. Ellia, D. Franco, On codimension two subvarieties of \mathbf{P}^5 and \mathbf{P}^6 , *J. Algebraic Geom.* 11 (2002), 513–533.
- [47] Y. Eliashberg, L. Traynor (eds.) *Symplectic Geometry and Topology*, AMS, 1999.
- [48] L. Evans, *Partial Differential Equations*, AMS, 1998.
- [49] A.R. Forsyth, *Theory of Differential Equations (Part IV): Partial Differential Equations*, Cambridge University Press, 1906; also, Dover Publications, 1959.
- [50] G. Fubini, E. Cech, *Géométrie projective différentielle des surfaces*, Gauthier-Villars, 1931.
- [51] W. Fulton, J. Hansen, A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings, *Ann. of Math. (2)* 110 (1979), 159–166.
- [52] W. Fulton, J. Harris, *Representation theory. A first course*, Springer, 1991.
- [53] R.B. Gardner, Invariants of Pfaffian Systems, *Trans. A.M.S.* 126 (1967), 514–533.
- [54] —, *The Method of Equivalence and Its Applications*, SIAM, 1989.
- [55] —, Differential geometric methods interfacing control theory, pp. 117–180 in *Differential Geometric Control Theory*, Birkhäuser, 1983.
- [56] R.B. Gardner, W.F. Shadwick, The GS algorithm for exact linearization, *IEEE Trans. Autom. Control* 37 (1992), 224–230.
- [57] J. Gasqui, Sur la résolubilité locale des équations d'Einstein, *Compositio Math.* 47 (1982), 43–69.
- [58] —, Formal integrability of systems of partial differential equations, pp. 21–36 in *Nonlinear equations in classical and quantum field theory*, Lecture Notes in Phys. 226, Springer, 1985.
- [59] I. Gel'fand, S. Fomin, *Calculus of Variations*, Prentice-Hall, 1963.
- [60] I. Gel'fand, M. Kapranov, A. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, 1994.
- [61] H. Goldschmidt, Existence theorems for analytic linear partial differential equations, *Ann. of Math.* 86 (1967), 246–270.
- [62] A.B. Goncharov, Generalized Conformal Structures on Manifolds, *Selecta Math. Sov.* 6 (1987), 307–340.
- [63] E. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, Gauthier-Villars, 1890.
- [64] —, Recherches sur quelques équations aux dérivées partielles du second ordre, *Annales de la Faculté de Toulouse, deuxième série* 1 (1899), 31–78.
- [65] Mark L. Green, Generic initial ideals, pp. 119–186 in *Six lectures on commutative algebra* (J. Elias et al, eds.), *Progr. Math.* 166, Birkhäuser, 1998.
- [66] P. Griffiths, Some aspects of exterior differential systems, pp. 151–173 in *Complex geometry and Lie theory*, Proc. Sympos. Pure Math. 53 (1991), AMS.
- [67] —, *Exterior Differential Systems and the Calculus of Variations*, Birkhäuser, 1983.
- [68] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley, 1978.
- [69] —, Algebraic geometry and local differential geometry, *Ann. Sci. Ecole Norm. Sup.* 12 (1979), 355–432.

- [70] P. Griffiths, G. Jensen, *Differential systems and isometric embeddings*, Princeton University Press, 1987.
- [71] V. Guillemin, The integral geometry of line complexes and a theorem of Gel'fand-Graev, pp. 135–149 in *The mathematical heritage of Elie Cartan, Astérisque* Numero Hors Serie (1985).
- [72] J. Harris, *Algebraic geometry, a first course*, Springer, 1995.
- [73] Robin Hartshorne, Varieties of small codimension in projective space, *Bull. Amer. Math. Soc.* 80 (1974), 1017–1032.
- [74] F.R. Harvey, *Spinors and Calibrations*, Academic Press, 1990.
- [75] F.R. Harvey, H.B. Lawson, Calibrated geometries, *Acta Math.* 148 (1982), 47–157.
- [76] T. Hawkins, *Emergence of the theory of Lie groups. An essay in the history of mathematics 1869–1926*, Springer, 2000.
- [77] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, 1978.
- [78] N. Hitchin, Complex manifolds and Einstein's equations, pp. 73–99 in *Twistor geometry and nonlinear systems*, Lecture Notes in Math. 970, Springer, 1982.
- [79] W.V.D. Hodge, D. Pedoe, *Methods of Algebraic Geometry* Vol. 2, Cambridge University Press, 1954.
- [80] J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, 1972.
- [81] J.M. Hwang and N. Mok, Uniruled projective manifolds with irreducible reductive G -structures, *J. reine angew. Math.* 490 (1997), 55–64.
- [82] J.M. Hwang and N. Mok, Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kahler deformation, *Invent. Math.* 131 (1998), 393–418.
- [83] B. Ilic and J.M. Landsberg, On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties, *Math. Ann.* 314 (1999), 159–174.
- [84] T. Ivey, Surfaces with orthogonal families of circles, *Proc. AMS* **123** (1995), 865–872.
- [85] F. John, *Partial Differential Equations* (4th ed.), Springer, 1982.
- [86] D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2 , I, II, *J. Diff. Geom.* 43 (1996), 291–375.
- [87] D. Joyce, Compact 8-manifolds with holonomy $\text{Spin}(7)$, *Invent. Math.* 123 (1996), 507–552.
- [88] M. Juráš, I. Anderson, Generalized Laplace invariants and the method of Darboux, *Duke Math. J.* 89 (1997), 351–375.
- [89] E. Kähler, *Einführung in die Theorie der Systeme von Differentialgleichungen*, Teubner, 1934.
- [90] N. Kamran, An elementary introduction to exterior differential systems, pp. 151–173 in *Geometric approaches to differential equations*, Cambridge University Press, 2000.
- [91] S. Kleiman, Tangency and duality, pp. 163–225 in *Proceedings of the 1984 Vancouver conference in algebraic geometry*, CMS Conf. Proc., vol. 6, Amer. Math. Soc., 1986.
- [92] M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972.
- [93] A. Knapp, *Lie groups beyond an introduction*, Birkhäuser, 1996.
- [94] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vols. 1,2, Wiley, 1963/1969.
- [95] S. Kobayashi, T. Ochiai, Holomorphic structures modeled after compact Hermitian symmetric spaces, pp. 207–222 in *Manifolds and Lie groups*, Birkhäuser, 1981.

- [96] J.Krasil'shchick, A.M. Vinogradov (eds.), *Symmetries and conservation laws for differential equations of mathematical physics*, American Math. Society, 1999.
- [97] M. Kuranishi, *Lectures on exterior differential systems*, Tata Institute, Bombay, 1962.
- [98] M. Kuranishi, CR geometry and Cartan geometry, *Forum Math.* 7 (1995), 147–205.
- [99] J.M. Landsberg, Minimal submanifolds defined by first-order systems of PDE, *J. Diff. Geom.* 36 (1992), 369–415.
- [100] —, On second fundamental forms of projective varieties, *Inventiones Math.* 117 (1994), 303–315.
- [101] —, On degenerate secant and tangential varieties and local differential geometry, *Duke Mathematical Journal* 85 (1996), 605–634.
- [102] —, Differential-geometric characterizations of complete intersections, *J. Diff. Geom.* 44 (1996), 32–73.
- [103] —, Introduction to G -structures via three examples, pp. 133–149 in *Web theory and related topics*, World Scientific, 2001,
- [104] —, Exterior differential systems: a geometric approach to pde, pp. 77–101 in *Topology and Geometry*, Proc. Workshop Pure Math. vol. 17, Korean Academic Council, 1998.
- [105] —, On the infinitesimal rigidity of homogeneous varieties, *Compositio Math.* 118 (1999), 189–201.
- [106] —, Is a linear space contained in a submanifold? - On the number of derivatives needed to tell, *J. reine angew. Math.* 508 (1999), 53–60.
- [107] —, *Algebraic geometry and projective differential geometry*, Lecture Notes Series, vol. 45., Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1999.
- [108] —, Lines on algebraic varieties, to appear in *J. reine angew. Math.*.
- [109] —, Griffiths-Harris rigidity of compact Hermitian symmetric spaces, preprint.
- [110] J.M. Landsberg, L. Manivel, On the projective geometry of homogeneous varieties, *Commentari Math. Helv.* 78 (2003), 65–100.
- [111] —, Construction and classification of complex simple Lie algebras via projective geometry, *Selecta Math. (N.S.)* 8 (2002), 137–159.
- [112] —, The projective geometry of Freudenthal's magic square, *J. Algebra* 239 (2001), 477–512.
- [113] H.B. Lawson, M. Michelsohn, *Spin geometry*, Princeton University Press, 1989.
- [114] R. Lazarsfeld, *Positivity in algebraic geometry*, preprint.
- [115] M. van Leeuwen, LiE, a computer algebra package,
available at <http://young.sp2mi.univ-poitiers.fr/marc/LiE/>.
- [116] S. L'vovsky, On Landsberg's criterion for complete intersections, *Manuscripta Math.* 88 (1995), 185–189.
- [117] B. Malgrange, L'involutivité générique des systèmes différentiels analytiques, *C. R. Acad. Sci. Paris, Ser. I* 326 (1998), 863–866.
- [118] R. Mayer, Coupled contact systems and rigidity of maximal dimensional variations of Hodge structure, *Trans. AMS* 352 (2000), 2121–2144.
- [119] R. McLachlan, A Gallery of Constant-Negative-Curvature Surfaces, *Math. Intelligencer* 16 (1994), 31–37.
- [120] M. Melko, I. Sterling, Integrable systems, harmonic maps and the classical theory of surfaces, pp. 129–144 in *Harmonic Maps and Integrable Systems*, Vieweg, 1994.

- [121] J. Milnor, *Topology from the differentiable viewpoint*, U. Virginia Press, 1965.
- [122] D. Mumford, *Algebraic Geometry I: Complex projective varieties*, Springer, 1976.
- [123] R. Muñoz, Varieties with degenerate dual variety, *Forum Math.* 13 (2001), 757–779.
- [124] I. Nakai, Web geometry and the equivalence problem of the first order partial differential equations, pp. 150–204 in *Web theory and related topics*, World Scientific, 2001.
- [125] T. Ochiai, Geometry associated with semisimple flat homogeneous spaces. *Trans. AMS* 152 (1970), 159–193.
- [126] C. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective spaces*, Progress in Mathematics 3, Birkhauser, 1980.
- [127] P. Olver, *Applications of Lie Groups to Differential Equations* (2nd ed.), Springer, 1993.
- [128] —, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, 1995.
- [129] G. Pappas, J. Lygeros, D. Tilbury, S. Sastry, Exterior differential systems in control and robotics, pp. 271–372 in *Essays on mathematical robotics*, Springer, 1998.
- [130] J. Piontkowski, Developable varieties of Gauss rank 2, *Internat. J. Math.* 13 (2002), 93–110.
- [131] Z. Ran, On projective varieties of codimension 2 *Invent. Math.* 73 (1983), 333–336.
- [132] C. Rogers, Bäcklund transformations in soliton theory, pp. 97–130 in *Soliton theory: a survey of results* (A. P. Fordy, ed.), St. Martin's Press, 1990.
- [133] C. Rogers, W. Schief, *Bäcklund and Darboux Transformations*, Cambridge University Press, 2002.
- [134] C. Rogers, W. Shadwick, *Bäcklund Transformations and Their Applications*, Academic Press, 1982.
- [135] E. Sato, Uniform vector bundles on a projective space, *J. Math. Soc. Japan* 28 (1976), 123–132.
- [136] B. Segre, Bertini forms and Hessian matrices, *J. London Math. Soc.* 26 (1951), 164–176.
- [137] C. Segre, Preliminari di una teoria delle varietà luoghi di spazi, *Rend. Circ. Mat. Palermo* XXX (1910), 87–121.
- [138] W. Shadwick, The KdV Prolongation Algebra, *J. Math. Phys.* 21 (1980), 454–461.
- [139] R. Sharpe, *Differential geometry: Cartan's generalization of Klein's Erlangen program*, Springer, 1997.
- [140] G.F. Simmons, *Differential Equations, with Applications and Historical Notes*, McGraw-Hill, 1972.
- [141] M. Spivak, *Calculus on Manifolds*, Benjamin, 1965.
- [142] —, *A Comprehensive Introduction to Differential Geometry* (3rd ed.), Publish or Perish, 1999.
- [143] O. Stormark, *Lie's Structural Approach to PDE Systems*, Encyclopedia of Mathematics and its Applications, v. 80, Cambridge University Press, 2000.
- [144] V. Strassen, Relative bilinear complexity and matrix multiplication, *J. Reine Angew. Math.* 413 (1991), 127–180.
- [145] S. Sternberg, *Lectures on differential geometry*, Chelsea, 1983.
- [146] D.J. Struik, *Lectures on Classical Differential Geometry* (2nd ed.), Addison-Wesley, 1961; Dover, 1988.

-
- [147] A. Terracini, Alcune questioni sugli spazi tangenti e osculatori ad una varieta, I, II, III, *Atti della Societa dei Naturalisti e Matematici Torino* 49 (1914), 214–247.
- [148] E. Tevelev, Projectively Dual Varieties, preprint available at <http://arXiv.org/math.AG/0112028>
- [149] P.J. Vassiliou, Darboux Integrability and Symmetry, *Trans. AMS* 353 (2001), 1705–1739.
- [150] M. Wadati, H. Sanuki, and K. Konno, Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws *Progr. Theoret. Phys.* 53 (1975), 419–436.
- [151] H. Wahlquist, F. Estabrook, Prolongation structures of nonlinear evolution equations. *J. Math. Phys.* 16 (1975), 1–7.
- [152] —, Prolongation structures, connection theory and Bäcklund transformations, pp. 64–83 in *Nonlinear evolution equations solvable by the spectral transform*, Res. Notes in Math., vol. 26, Pitman, 1978.
- [153] F. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer, 1983.
- [154] K. Yamaguchi, Differential Systems Associated with Simple Graded Lie Algebras, pp. 413–494 in *Progress in Differential Geometry*, Adv. Stud. Pure Math., vol 22., Math. Soc. Japan, 1993.
- [155] D. Yang, Involutive hyperbolic differential systems, *A.M.S. Memoirs # 370* (1987).
- [156] K. Yang, *Exterior differential systems and equivalence problems*, Kluwer, 1992.
- [157] F. Zak, *Tangents and Secants of Algebraic Varieties*, Translations of mathematical monographs vol. 127, AMS, 1993.
- [158] M.Y. Zvyagin, Second order equations reducible to $z_{xy} = 0$ by a Bäcklund transformation, *Soviet Math. Dokl.* 43 (1991) 30–34.

Index

- II , Euclidean second fundamental form, 46
- II , projective second fundamental form, 77
- $|II_{M,x}|$, 80
- III , projective third fundamental form, 96
- III^v , 129
- $\Gamma(E)$, smooth sections of E , 335
- $\Gamma_{\alpha,i}^\beta$, 277
- $\Delta r_{\alpha\beta\gamma}^\mu$, 95
- $\Lambda^2 V$, 313
- $\Lambda^k V$, 314
- Ξ_A , characteristic variety of a tableau, 157
- $\Omega^k(M), \Omega^*(M)$, 336
- $\Omega^k(M, V)$, 338
- $\Omega^{(p,q)}(M)$, 345
- $\delta_\sigma(X)$, secant defect, 129
- $\delta_\tau(X)$, tangential defect, 129
- δ_* , dual defect, 120
- ϕ^* , pullback by ϕ , 337
- ϕ_* , pushforward by ϕ , 337
- κ_g , 59
- κ_n , 60
- $\tau(X)$, tangential variety, 86
- $\tau(Y, X)$, 131
- τ_g , 60
- $A^{(1)}$, 147
- $A^{(l)}$, 147
- $\text{Ann}(v)$, 129
- $ASO(2)$, 12
- $ASO(3)$, 23
 - as space of frames, 24
- Baseloc $|II_{M,x}|$, 80
- $C^\infty(M)$, 335
- c_k , codimension of polar space, 256
- $Cl(V, Q)$, 331
- d , exterior derivative, 337
- \underline{d}^k , 97
- \det , 102
- \det , 315
- \mathbb{E}^3 , Euclidean three-space, 2
- E_6 , exceptional Lie group, 102
- $\text{End}(V)$, 312
- $\mathcal{F}(M)$, 49
- \mathcal{F}^1
 - Euclidean, 37
 - projective, 78
- F_4 , exceptional Lie group, 102
- F_4 , differential invariant, 107
- F_k , 108
- \mathbb{FF}^k , 97
- $|\mathbb{FF}^k|$, 97
- \mathfrak{g} , Lie algebra of Lie group G , 17
- G_2 , exceptional Lie group, 323
- $G(k, V)$, Grassmannian, 72
- $\mathbf{G}(n, m)$, 198
- $\mathbf{G}(n, T\Sigma)$, 177
- $GL(V)$, 316
- $Gr(k, V)$, orthogonal Grassmannian, 75
- $H^{0,2}(A)$, 175
- $H^{i,j}(A)$, 180
- $\mathcal{H}^{i,j}(\mathfrak{g})$, 283
- Hol_u^θ , 287
- $\text{Hom}(V, W)$, 312
- \mathcal{I} , differential ideal, 340
- \mathcal{I}^k , k -th homogeneous component of \mathcal{I} , 340
- $I^{(1)}$, derived system, 216
- (I, J) , linear Pfaffian system, 164
- $J(Y, Z)$, join of varieties, 86
- $\mathcal{K}(V)$, 330
- \mathcal{L}_X , Lie derivative, 339
- m_x , functions vanishing at x , 335
- $O(V, Q)$, orthogonal group, 317

- (p, q) -forms, 345
- $[R_\theta]$, 282
- S^2V , 313
- S^kV , 314
- s_k
 - characters of a tableau, 154
 - characters of an EDS, 258
- \mathbb{S}_m , spinor variety, 106
- Singloc $| II_{M,x} |$, 80
- $SL(V)$, SL_n , special linear group, 317
- $SO(V, Q)$, special orthogonal group, 317
- $Sp(V, \omega)$, symplectic group, 317
- $SU(n)$, special unitary group, 319
- $\mathbb{T}(V)$, 273
- TM , tangent bundle, 335
- T^*M , cotangent bundle, 335
- T_xM , tangent space, 335
- T_x^*M , cotangent space, 335
- $U(n)$, unitary group, 319
- $V_{\mathbb{C}}$, complexification of V , 343
- X_{smooth} , 82
- $[X, Y]$, 336
- \lrcorner , interior product, 315
- ∇ , 277
- \otimes , tensor product, 312
- \sharp , 53
- $\{ \}$, linear span, 340
- $\{ \}_{\text{alg}}$, 340
- $\{ \}_{\text{diff}}$, 340
- abuse of notation, 29, 72, 170
- adjoint representation, 321
- affine connection, 285
- affine tangent space, 76
- algebraic variety, 82
 - degree of, 82
 - dimension of, 82
 - general point of, 83
 - ideal of, 82
- almost complex manifold, 274, 282, 344
- almost complex structure, 344
- almost symplectic manifold, 274
- Ambrose-Singer Theorem, 290
- apparent torsion, 165
- arclength parameter, 14
- associated hypersurface, 124
- associated varieties, 123
- associative submanifolds, 201, 265
- associator, 325
- asymptotic directions, 80
- asymptotic line, 60, 226, 238
- Bäcklund transformations, 235–241
- Bäcklund's Theorem, 237
- basic differential form, 339
- Bertini Theorem, 112
 - higher-order, 112
- Bertrand curve, 26
- Bezout's Theorem, 82
- Bianchi identities, 53–54
- Bonnet surface, 44, 231
- Burger's equation, 208, 232
- calibrated submanifold, 198
- calibration, 197
 - associative, 201
 - Cayley form, 202
 - coassociative, 201
 - special Lagrangian, 200
- canonical system
 - on Grassmann bundle, 177
 - on space of jets, 28
- Cartan geometry, 296
- Cartan integer, 156, 179
- Cartan Lemma, 314
- Cartan system, 209
- Cartan's algorithm for linear Pfaffian systems, 178
- Cartan's five variables paper, 217
- Cartan's Test, 256
- Cartan-Dieudonné Theorem, 331
- Cartan-Janet Theorem, 192
- Cartan-Kähler Theorem, 254–256
 - for linear Pfaffian systems, 176
 - for tableaux, 156
 - Goldschmidt version, 181
- catenoid, 43
- Cauchy problem, 349
- Cauchy-Kowalevski form, 350
- Cauchy-Kowalevski Theorem, 243, 351
- Cauchy-Riemann equations, 347
 - tableau, 144, 156
- Cayley submanifold, 202
- character of a tableau, 156
- characteristic hyperplane, 181
- characteristic systems (Monge), 213
- characteristic variety, 157
 - dimension and degree of, 159
- characteristics
 - Cauchy, 205, 259
 - quotient by, 210
 - confounded, 213
 - first-order, 214
 - method of, 207–208
 - Monge, 213
- characters, 258
 - of linear Pfaffian system, 179
 - of tableau, 154
- Chebyshev net, 227
- Christoffel symbols, 277
- Clifford algebras, 331
 - fundamental lemma of, 332
- Clifford torus, 58
- co-roots, 329

- coassociative submanifold, 201
- Codazzi equation
 - for Darboux frames, 43
 - matrix form, 49
- codimension, 245
- coisotropic hypersurface, 124
- complete intersection, 140
- complex characteristic variety, 158
- complex contact structure, 348
- complex manifold, 343, 344
- complex structure, 318, 344
- complexification
 - of a real vector space, 318
- cone, 44
 - characterization of, 125
 - over a variety, 86
- connection
 - affine, 285
 - on coframe bundle, 278–283
 - on induced vector bundles, 284
 - on vector bundle, 277
 - symmetric, 285
- connection form, 279
- conormal space, of submanifold in \mathbb{P}^N , 77
- contact manifold, 33
- contact system
 - on space of jets, 28
- contact, order of, 83
- cotangent
 - bundle, 335
 - space, 335
- covariant differential operator, 54, 277
- cubic form, 94
- curvature
 - Gauss, 38
 - geometric interpretation of, 47
 - in coordinates, 4
 - mean, 38
 - geometric interpretation of, 68
 - in coordinates, 4
 - of curve in \mathbb{E}^2 , 14
 - of curve in \mathbb{E}^3 , 25
 - of G -structure, 280
 - Ricci, 53, 262
 - scalar, 53, 262, 266, 330
 - sectional, 53
 - traceless Ricci, 330
 - Weyl, 330
- curvature-line coordinates, 188
- curve
 - arclength parameter, 14
 - Bertrand, 26
 - regular, 13
 - speed of, 14
- curve in \mathbb{E}^2
 - curvature, 14
 - osculating circle, 14
- curve in \mathbb{E}^3
 - curvature, 25
 - differential invariants, 25–26
 - torsion, 25
- cylinder, 44
- Darboux
 - integrable, 218, 239
 - method of, 217–222
 - semi-integrable, 222
- Darboux frame, 42
- Darboux's Theorem, 32
- de Rham Splitting Theorem, 289
- decomposable tensor, 312
- derived flag, 216
- derived system, 216
- determinant
 - of linear endomorphism, 315
- developable surface, 40
- differential form, 336
 - basic, semi-basic, 339
 - closed, 338
 - homogeneous, 340
 - left-invariant, 17
 - vector-valued, 338
- differential ideal, 340
- differential invariant
 - Euclidean, 3
- dual basis, 311
- dual variety, 87, 118
 - defect of, 120
 - reflexivity, 119
- dual vector space, 311
- Dupin
 - cyclides of, 361
 - theorem of, 253
- e -structure, 304
- embedded tangent space, 76
- Engel structure, 217
- equivalent
 - G -structures, 275
 - webs, 268
- Euclidean group, 23
- Euler characteristic, 62
- exterior derivative, 337–338
- exterior differential system, 29
 - hyperbolic, 214–215
 - linear Pfaffian, 164
 - Pfaffian, 341
 - symmetries, 204–205
 - with independence condition, 27
- face of calibration, 199
- first fundamental form (Riemannian), 46
- first-order adapted frames (Euclidean), 45

- flag
 - A -generic, 154
 - complete, 85
 - derived, 216
 - partial, 85
- flag variety, 85, 316
- flat
 - G -structure, 275
 - 3-web, 268
 - path geometry, 296
 - Riemannian manifold, 52
 - isometric immersions of, 194
 - surface, 41
- flow of a vector field, 6
- flowbox coordinates, 6
- flowchart for Cartan's algorithm, 178
- focal hypersurface, 89
- focal surface, 237, 266
- frame
 - Darboux, 42
- frame bundle
 - general, 49
 - orthonormal, 50
- Frenet equations, 25
- Frobenius ideal, 11
- Frobenius structure, 308
- Frobenius system
 - tableau of, 146
- Frobenius Theorem, 10–12, 30
 - proof, 30
- Fubini cubic form, 94
- Fubini forms, 94, 107
- Fulton-Hansen Theorem, 130
- fundamental form
 - effective calculation of, 97
 - k -th, 97
 - prolongation property of, 97
 - via spectral sequences, 98
- G -structure, 267–275
 - 1-flat, 280
 - 2-flat, 281
 - curvature, 280, 282
 - definition, 274
 - flat, 275
 - prolongation, 281
- G/H -structure of order two, 296
- Gauss curvature
 - geometric interpretation of, 47
 - in coordinates, 4
 - via frames, 36–38
- Gauss equation, 47
- Gauss image, 77
 - characterization of, 93
- Gauss map
 - algebraic, 55
 - Euclidean, 46
 - projective, 77
 - varieties with degenerate, 89
- Gauss' theorema egregium, 48
- Gauss-Bonnet formula, 64
- Gauss-Bonnet theorem, 62
 - for compact hypersurfaces, 64
 - local, 60
- Gauss-Bonnet-Chern Theorem, 65
- general point, 83
- generalized conformal structure, 309
- generalized Monge system, 139
- generic point, 83
- geodesic, 59
 - of affine connection, 285
- geodesic curvature, 59
- geodesic torsion, 60
- Grassmann bundle, 177
 - canonical system on, 177
- Grassmannian, 72, 316
 - isotropic, 84
 - tangent space of, 73
- half-spin representation, 107
- Hartshorne's conjecture, 140
- heat equation, 350
- helicoid, 39
- Hermitian form, 319
- Hermitian inner product, 319
- hexagonality, 271
- higher associated hypersurface, 124
- holomorphic map, 345
- holonomy, 286–295
- holonomy bundle, 287
- holonomy group, 287
- homogeneous space, 15
- Hopf differential, 230
- horizontal curve, 287
- horizontal lift, 287
- hyperbolic space, 58
 - isometric immersions of, 197
- hyperplane section of a variety, 88
- hypersurfaces in \mathbb{E}^N
 - fundamental theorem for, 55
- ideal
 - algebraic, 340
 - differential, 340
 - Frobenius, 11
- incidence correspondence, 88
- independence condition, 27
- index of a vector field, 61
- index of relative nullity, 80
- induced vector bundle, 283
- initial data, 349
- initial value problem, 349
- integrable extension, 232
 - via conservation law, 233

- integral
 - intermediate/general, 219
- integral curve, 5
- integral element, 27
 - Kähler-ordinary, 245
 - Kähler-regular, 249
 - ordinary, 256
- integral manifold, 27, 29
- interior product, 315
- involutive
 - integral element, 256
 - linear Pfaffian system, 176
 - tableau, 155
- isometric embedding, 169–173
- isothermal coordinates, 57
 - existence of, 185
- isotropic Grassmannian, 84
- isotropy representation, 16

- Jacobi identity, 320
- jets, 27
- join of varieties, 86

- Kähler manifold, 199
- KdV equation, 234, 236
 - prolongation algebra, 235
- Killing form, 323

- Laplace system
 - tableau for, 157
- Laplace's equation, 223
- Laplacian, 56
- left action, 15
- left-invariant
 - differential form, 17
 - vector field, 17, 320
- level, 155
- Lie algebra, 320
 - of a Lie group, 17
 - semi-simple, 327
 - simple, 327
- Lie bracket, 336
- Lie derivative, 339
- Lie group, 316
 - linear representation of, 316
 - matrix, 16, 316–318
 - Maurer-Cartan form of, 17
- lift, 16
 - first-order adapted, 37
- line congruence, 237
- line of curvature, 60, 253
 - isothermal coordinates along, 188
- linear map, 311
 - transpose/adjoint of, 312
- linear normality
 - Zak's theorem on, 128
- linear Pfaffian systems, 164
 - Cartan's algorithm for, 178
 - involutivity, 176
 - linear projection of variety, 88
 - linear syzygy, 111
 - Liouville's equation, 218, 237
 - locally ruled variety, 89
 - locally symmetric, 290

- majorants, 150
- manifold
 - contact, 33
 - restraining, 255
 - symplectic, 31
- matrix Lie groups, 316–318
- Maurer-Cartan equation, 18
- Maurer-Cartan form
 - of a matrix Lie group, 17
 - of an arbitrary Lie group, 17
- maximal torus, 327
- mean curvature
 - geometric interpretation of, 68
 - in coordinates, 4
 - via frames, 36–38
- mean curvature vector, 69
- minimal hypersurfaces, 266
- minimal submanifold, 197
- minimal surface, 68, 228–229
 - Riemannian metric of, 186
- minimizing submanifold, 197
- minuscule variety, 104
- modified KdV equation, 234
- Monge's method, 224
- Monge-Ampère
 - equation, 222
 - system, 223
- moving frame, 4
 - adapted, 12
- multilinear, 312
- multiplicity of intersection, 83
- musical isomorphism, 53

- Newlander-Nirenberg Theorem, 345
- Nijenhuis tensor, 346
- non-characteristic initial data, 157
- nondegenerate quadratic form, 322
- normal bundle, 46, 66
- normal curvature, 60
- normal space, of submanifold in \mathbb{P}^N , 77

- octonions, 324–326
- orthogonal Grassmannian, 75
- orthogonal group, 317
- orthogonal involutive Lie algebra, 291
- osculating circle, 14
- osculating hypersurface, 109, 111
- osculating quadric hypersurface, 109

- parabolic subgroup, 84, 104
- parallel surfaces, 225
- parallel transport, 287
- path geometry, 295–308
 - definition of, 295, 298
 - dual, 297
 - flat, 296
- Pfaff's Theorem, 33
- Pfaffian, 322
- Pfaffian system, 341
 - linear, 164
- Picard's Theorem, 5, 10
- Poincaré-Hopf Theorem, 62
- point transformation, 295
- polar spaces, 246–248
- principal curvatures, 39
- principal framing, 42
- principal symbol, 145
- projective differential invariants
 - in coordinates, 108
- projective second fundamental form, 77
 - coordinate description of, 81
 - frame definition of, 79
- projective structure, 286
- prolongation, 147, 177, 214, 220
 - of a G -structure, 281
- prolongation property, 97
 - strict, 105
- prolongation structures, 233
- pseudospherical surfaces, 226–227
 - Bäcklund transformation for, 237
 - of revolution, 227
- pullback, 337
- pushforward, 337
- rank
 - of a Lie algebra, 327
 - of a Pfaffian system, 341
 - of a tensor, 313
- rational homogeneous variety, 83
- reductive
 - Lie group/Lie algebra, 327
- refined third fundamental form, 129
- regular curve, 13
- regular second-order PDE, 174
- relative tangent star, 131
- representation
 - isotropy, 16
 - of Lie algebra, 320
 - of Lie group, 316
- restraining manifold, 255
- retracting space, 209
- Ricci curvature, 53, 262
- Riemann curvature tensor, 52–55, 273
- Riemann invariant, 217
- Riemann surface, 346
- Riemannian geometry, 271–273
 - fundamental lemma, 50–51, 273
- Riemannian manifold, 47
 - flat, 52
- Riemannian metric, 46, 47
- right action, 15
- root, 328
- root system, 328
- ruled surface, 41
- ruled variety, 113
- S -structure, 309
- scalar curvature, 53, 262, 266, 330
- Schur's Lemma, 317
- Schwarzian derivative, 22
- secant defect, 129
- secant variety, 86
- second fundamental form
 - base locus of, 80
 - Euclidean, 46
 - projective, 77
 - singular locus of, 80
- second-order PDE
 - characteristic variety, 182
 - classical notation, 174
 - tableau, 175
- section
 - of vector bundle, 335
- sectional curvature, 53
- Segre product of varieties, 84
 - fundamental forms of, 101
- Segre variety, 84, 159
 - fundamental forms of, 100
- semi-basic form, 339
- semi-Riemannian manifold, 274
- semi-simple Lie algebra, 327
- Severi variety, 102
 - fundamental form of, 103
 - Zak's theorem on, 128
- signature
 - of quadratic form, 322
- simple Lie algebra, 327
- sine-Gordon equation, 223, 226, 235
- singular solutions, 191
- space form, 57
 - isometric immersions of, 194
- special Lagrangian submanifolds, 200, 265
- special linear group, 317
- special orthogonal group, 317
- special unitary group, 319
- Spencer cohomology, 180
- spin representation, 106, 107
- spinor variety, 85, 106
- stabilizer type, 282
- submanifold
 - associative, 265
 - Lagrangian, 185, 264
 - special Lagrangian, 200, 265

- surface
 - Bonnet, 44
 - catenoid, 43
 - cone, 44
 - constant mean curvature, 229–231
 - cylinder, 44
 - developable, 40
 - flat, 41
 - focal, 237, 266
 - helicoid, 39
 - isothermal coordinates on, 57
 - linear Weingarten, 183, 224, 261
 - minimal, 68, 228
 - of revolution, 41, 227
 - parallel, 225
 - pseudospherical, 226
 - ruled, 41
 - warp of, 4
 - with degenerate Gauss image, 91
- symbol mapping, 157
- symbol relations, 145, 174
- symmetric connection, 285
- symmetric Lie algebra, 291
- symmetric space, 290
- symmetries, 241
- symplectic form, 32, 185, 199, 212, 264, 317
- symplectic group, 317
- symplectic manifold, 31

- tableau, 145
 - determined, 158
 - of linear Pfaffian system, 174
 - of order p , 147
- tangent
 - bundle, 335
 - space, 335
- tangent star, 86
- tangential defect, 129
 - critical, 135
- tangential surface, 40
- tangential variety, 86
 - dimension of, 128
- tautological EDS
 - for torsion-free G -structures, 293
- tautological form
 - for coframe bundle, 49
- tensor product, 312
- Terracini's Lemma, 87
- third fundamental form
 - projective, 96
- torsion
 - of connection, 279
 - of curve in \mathbb{E}^3 , 25
 - of G -structure, 280
 - of linear Pfaffian system, 165, 175
- transformation
 - Bäcklund, 232, 236
 - Cole-Hopf, 232, 238
 - fractional linear, 20
 - Lie, 231
 - Miura, 234
 - triangulation, 61
 - triply orthogonal systems, 251–254
- umbilic point, 39
- uniruled complex manifold, 310
- uniruled variety, 113
- unitary group, 319

- variation of Hodge structure, 189
- variety
 - algebraic, 82
 - dual, 87, 118
 - flag, 85
 - miniscule, 104
 - rational homogeneous, 83
 - ruled, 113
 - secant, 86
 - Segre, 84
 - spinor, 85, 106
 - tangential, 86
 - uniruled, 113
 - Veronese, 85
- vector bundle
 - induced, 283
- vector field, 335
 - flow of a, 6
 - left-invariant, 17
- Veronese embedding, 85
- Veronese re-embedding, 85, 109
- Veronese variety, 85
 - fundamental forms of, 99
- vertical vector, 339
- volume form, 46

- Waring problems, 313
- warp of a surface, 4
- wave equation, 203, 349
- web, 267
 - hexagonality of, 271
- wedge product, 314
 - matrix, 18
- Weierstrass representation, 228–229
- weight, 327
 - highest, 329
 - multiplicity of, 327
- weight diagram for invariants, 305
- weight lattice, 329
- weight zero invariant, 300
- Weingarten equation, 224
- Weingarten surface, linear, 183, 224, 261
- Weyl curvature, 330
- Wirtinger inequality, 199

Zak's theorem
 on linear normality, 128
 on Severi varieties, 128
 on tangencies, 131