THE LOCAL EQUIVALENCE PROBLEM FOR 7-DIMENSIONAL, 2-NONDEGENERATE CR

MANIFOLDS WHOSE CUBIC FORM IS OF CONFORMAL UNITARY TYPE

A Dissertation

by

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ABSTRACT

We apply E. Cartan's method of equivalence to classify 7-dimensional, 2-nondegenerate CR manifolds M up to local CR equivalence in the case that the cubic form of M satisfies a certain symmetry property with respect to the Levi form of M. The solution to the equivalence problem is given by a parallelism on a principal bundle over M which takes values in $\mathfrak{su}(2,2)$ or $\mathfrak{su}(3,1)$, depending on the signature of the nondegenerate part of the Levi form. Differentiating this parallelism provides a complete set of local invariants of M. We exhibit an explicit example of a real hypersurface in $\mathbb{C}^4$ whose invariants are nontrivial.

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1. INTRODUCTION

A CR manifold M of CR-dimension n and CR-codimension c is intrinsically defined to abstract the structure of a smooth, real, codimension-c submanifold of a complex manifold of complex dimension $n + c$. The most trivial example of such a submanifold is $\mathbb{C}^n \times \mathbb{R}^c \subset \mathbb{C}^{n+c}$, and the obstruction to the existence of a local CR equivalence $M \to \mathbb{C}^n \times \mathbb{R}^c$ is the Levi form $\mathcal{L}$ of M, a $\mathbb{C}^c$-valued Hermitian form on the CR bundle of M whose signature in the $c = 1$ case is a basic invariant of M's CR structure. Accordingly, attempts to classify CR manifolds of hypersurface-type ($c = 1$) fundamentally depend on the degree of degeneracy of $\mathcal{L}$.

The primary instrument for achieving such classification is the method of equivalence, a broadly applicable procedure for constructing invariants of smooth manifolds under a specified notion of local isomorphism. When a geometric structure on a manifold M is amenable to the procedure, the method of equivalence constructs a principal bundle $B \to M$ and a parallelism $\omega \in \Omega^1(B, \mathfrak{g})$ taking values in a Lie algebra $\mathfrak{g}$. The *curvature tensor* $d\omega + \omega \wedge \omega \in \Omega^2(B, \mathfrak{g})$ along with its higher derivatives then provides a complete set of local invariants of the geometric structure under consideration.

When the curvature tensor vanishes identically, M is locally equivalent to the *flat model* of the geometry – a homogeneous manifold G/Q where G is a Lie group with Lie algebra $\mathfrak{g}$ and $Q \subset G$ is a closed Lie subgroup isomorphic to the structure group of B. The "flat model" terminology may be understood by analogy with the case of Riemannian geometry, wherein ω is the affine extension of the Levi-Civita connection of a Riemannian manifold M, and the curvature tensor measures the obstruction to M being locally isometric to a "flat" Euclidean vector space.

The method of equivalence has been successfully implemented to classify CR manifolds whose Levi form is nondegenerate, and this classification naturally extends to *straightenable* CR manifolds, which are locally CR equivalent to the Cartesian product of a Levi-nondegenerate CR manifold and a complex vector space. For those CR manifolds that are Levi-degenerate but carry no such local product structure, classification is so far limited to dimension five. This dissertation treats a generalization of the 5-dimensional case to dimension seven. We now proceed to a description of the contents of our report.

§2 offers an overview of CR geometry by tracing the history of the subject and its interactions

with complex analysis, partial differential equations, and the theory of Lie groups. After motivating the basic definitions in §2.1, we review necessary background material on CR manifolds in §2.2 and use CR structures to illustrate the efficacy of the method of equivalence for studying differential geometry in general. In §2.3 we discuss the Levi-degenerate case, including the known results in dimension five. We also introduce the cubic form $\mathcal{C}$ – a higher-order analogue of $\mathcal{L}$ that detects obstruction to CR straightening – and define what it means for $\mathcal{C}$ to be of *conformal unitary type*, thus arriving at a formal statement of the 7-dimensional problem to be solved in the present work. All of the structures described theretofore are reformulated in §2.4 in terms of local, adapted coframings on a 7-dimensional CR manifold M.

The technical core of the dissertation is §3, in which the 7-dimensional equivalence problem is solved by the construction of a principal bundle $B_4^{(1)} \to M$ and a parallelism $\omega \in \Omega^1(B_4^{(1)}, \mathfrak{su}_\star)$ taking values in the Lie algebra of $SU_\star = SU(2,2)$ or $SU_\star = SU(3,1)$, depending on the signature of the nondegenerate part of $\mathcal{L}$. A standard reference for the algorithmic procedure of the classical method of equivalence is [Gar89]. The author also greatly benefited from the exposition of [BGG03], wherein the general theory is illuminated by the extended examples of Monge-Ampère equations and conformal geometry.

The parallelism ω and the invariants encoded in its curvature tensor are the subject of §4. When the curvature tensor vanishes, M is locally CR equivalent to the flat model $M_\star := SU_\star/P_\star$ described in §4.1. Moreover, we demonstrate in §4.2 that the lowest order invariants appearing in the curvature tensor suffice to detect local flatness. In §4.3, we show that $P_\star$ is isomorphic to the structure group of $B_4^{(1)}$, and that ω fails to satisfy a certain equivariance property with respect to the principal $P_\star$-action on $B_4^{(1)}$, as evidenced by the presence of two-forms in the curvature tensor that are not semibasic for the bundle projection $B_4^{(1)} \to M$. Finally, in §4.4 we exhibit a real hypersurface $M \subset \mathbb{C}^4$ that is not locally isomorphic to $M_\star$, demonstrating the existence of so-called "non-flat" CR manifolds which satisfy our hypotheses. Our work in sections 3 and 4 constitutes a proof of the main result of this dissertation, which may be summarized as follows.

Theorem 1.1 *Let M be a hypersurface-type CR manifold of CR dimension 3 such that $\ker \mathcal{L}$ has constant rank 1 and $\mathcal{C}$ is of conformal unitary type. There exists a principal $P_\star$-bundle $B_4^{(1)} \to M$ and an absolute parallelism $\omega \in \Omega^1(B, \mathfrak{su}_\star)$. Differentiating ω provides a complete set of local invariants of M which measure the obstruction to M being locally CR equivalent to $SU_\star/P_\star$.*

2. HISTORY AND PERSPECTIVES

Cauchy-Riemann (CR) geometry studies boundaries of domains in complex vector spaces and their generalizations. In one complex dimension, the Riemann mapping theorem shows that any simply connected domain which is not the entire complex line $\mathbb{C}$ is biholomorphically equivalent to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Already in $\mathbb{C}^2$, however, there are elementary examples of diffeomorphic yet holomorphically inequivalent domains. Figure 2 depicts the boundaries of the bidisk $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ and open ball $\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ by graphing the moduli of the standard coordinates z_1, z_2 of $\mathbb{C}^2$.
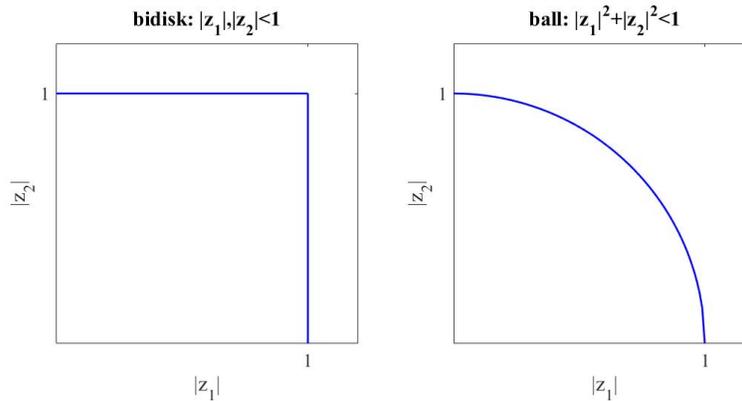
Figure 2.1: The bidisk and ball in $\mathbb{C}^2$

The boundary $\partial \mathbb{D}^2$ of the bidisk contains a copy of $\mathbb{D}$, while the boundary $\partial \mathbb{B}$ of the ball does not. If there existed a biholomorphic map $F : \mathbb{D}^2 \to \mathbb{B}$, one could choose a sequence in $\mathbb{D}^2$ of open disks that converge to the boundary disk, and the restriction of F to the disks in this sequence would form a normal family of holomorphic functions on $\mathbb{D}$. This normal family must have a subsequence which converges to a holomorphic function $f : \mathbb{D} \subset \partial \mathbb{D}^2 \to \partial \mathbb{B}$. Now $f(z) = (f_1(z), f_2(z))$ takes values in the sphere, so taking the Laplacian of the equation $|f_1(z)|^2 + |f_2(z)|^2 = 1$ reveals that f is constant, which in turn implies that F is constant on $\mathbb{D}^2$, a contradiction. (The details of this proof – based on ideas from R. Remmert and K. Stein's [RS60] as presented in [Nar71] – may be found in [Ran86, Thm I.2.7]). Hence, no such F can exist.

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Though the forgoing argument was rather specific to the two domains in question, this example illustrates a crucial principle: namely, that useful information about a complex domain may be educed from the differential geometry of its boundary. The central role played by domains in the field of several complex variables therefore provides ample motivation to study boundaries of domains – or more generally, hypersurfaces – in complex manifolds. In order to understand the geometry such a hypersurface inherits from its ambient space, we examine some features of a complex structure.

2.1   Complex Structure of $\mathbb{C}^m$

We denote $\mathbf{i} := \sqrt{-1}$, so that linear coordinates $z_1, \ldots, z_m$ on $\mathbb{C}^m$ ($m \in \mathbb{N}$) can be expressed in terms of their real and imaginary parts $z_j = x_j + \mathbf{i} y_j$ ($1 \leq j \leq m$). As a smooth manifold, $\mathbb{C}^m$ is diffeomorphic to its underlying real vector space $\mathbb{R}^{2m}$, and the $\mathbb{R}$-valued coordinates x_j, y_j determine a smooth, global coordinate chart on $\mathbb{C}^m \simeq \mathbb{R}^{2m}$. In particular, the tangent bundle is parallelized by coordinate vector fields

$$T_z \mathbb{R}^{2m} = \mathrm{span}_{\mathbb{R}} \left\{ \left. \frac{\partial}{\partial x_j} \right|_z, \left. \frac{\partial}{\partial y_j} \right|_z \right\}_{j=1}^m; \qquad\qquad z \in \mathbb{C}^m,$$

and the complex-algebraic notion of multiplication by $\mathbf{i}$ is recovered infinitesimally in this real-geometric category by a bundle endomorphism

$$J : T\mathbb{R}^{2m} \to T\mathbb{R}^{2m}$$
$$\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial y_j},$$
$$\frac{\partial}{\partial y_j} \mapsto -\frac{\partial}{\partial x_j}.$$

Evidently, $J^2 = -\mathbb{1}$ where $\mathbb{1}$ is the identity map on $T\mathbb{R}^{2m}$, whence the induced action of J on the complexified tangent bundle

$$\mathbb{C}T\mathbb{R}^{2m} := T\mathbb{R}^{2m} \otimes \mathbb{C}$$

splits its fibers into $\pm\mathbf{i}$-eigenspaces defining holomorphic and anti-holomorphic bundles $H, \overline{H} \subset$

$\mathbb{C}T\mathbb{R}^{2m}$:

$$H := \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - \mathbf{i} \frac{\partial}{\partial y_j} \right) \right\}_{j=1}^{m}, \quad \overline{H} := \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + \mathbf{i} \frac{\partial}{\partial y_j} \right) \right\}_{j=1}^{m}.$$

These bundles have some remarkable properties. We have already remarked that they split the complexified tangent bundle,

$$\mathbb{C}T\mathbb{R}^{2m} = H \oplus \overline{H}.$$

Moreover, the Lie bracket of two local sections of H is once again a section of H – a fact which we abbreviate

$$[H, H] \subset H,$$

and by taking complex conjugates we can similarly say $[\overline{H}, \overline{H}] \subset \overline{H}$.

The tangent bundle of a hypersurface $M \subset \mathbb{C}^m$ will also admit a restricted action of J, whereby the complexified tangent bundle $\mathbb{C}TM$ will have intersection with the distinguished subbundles $H, \overline{H} \subset \mathbb{C}T\mathbb{R}^{2m}$. It is exactly this tangential structure which motivates the definition of a CR manifold. However, a submanifold $M \subset \mathbb{C}^m$ is an extrinsically defined object, so in order to work intrinsically we must formulate the definition without reference to an ambient space.

2.2 CR Manifolds, the Levi Form, and the Method of Equivalence

Let M be a smooth (C^∞) manifold of real dimension $2n + c$ for $n, c \in \mathbb{N}$. For any vector bundle $p : E \to M$, $E_x := p^{-1}(x)$ denotes the fiber of E over $x \in M$, $\Gamma(E)$ denotes the sheaf of smooth (local) sections of E, and $\mathbb{C}E$ denotes the complexified vector bundle whose fiber over x is $\mathbb{C}E_x := E_x \otimes_{\mathbb{R}} \mathbb{C}$. References for background material in CR geometry include [Jac90] and [Bog91].

A CR structure of CR dimension n and codimension c is determined by a rank-$2n$ subbundle D of the tangent bundle TM, and an *almost complex* structure J on D; i.e., a smooth bundle endomorphism $J : D \to D$ which satisfies $J^2 = -\mathbb{1}_D$, where $\mathbb{1}_D$ denotes the identity map of D. The induced action of J on $\mathbb{C}D$ splits each fiber $\mathbb{C}D_x = H_x \oplus \overline{H}_x$, where $H \subset \mathbb{C}D$ denotes the smooth, $\mathbb{C}$-rank-n subbundle of $\mathbf{i}$-eigenspaces of J, while $\overline{H}$ is that of $-\mathbf{i}$-eigenspaces. We refer to H and $\overline{H}$ as the CR and anti-CR bundles of M, respectively.

If M_1, M_2 are two CR manifolds with respective CR structures $(D_1, J_1), (D_2, J_2)$ determining CR bundles H_1, H_2, then a *CR map* is a smooth map $F : M_1 \to M_2$ whose pushforward $F_* : TM_1 \to TM_2$ satisfies $F_*(D_1) \subset D_2$ and $F_* \circ J_1 = J_2 \circ F_*$. Equivalently, a smooth map F is a CR map if the induced action of F_* on $\mathbb{C}TM_1$ satisfies $F_*(H_1) \subset H_2$. A *CR equivalence* is a local diffeomorphism which is a CR map. We write $M_1 \cong_{\mathrm{CR}} M_2$ when M_1 is CR equivalent to M_2, bearing in mind that this is a strictly local condition in our lexicon; i.e., $M_1 \cong_{\mathrm{CR}} M_2$ when every $x \in M_1$ is contained in an open neighborhood that is CR equivalent to an open subset of M_2.

Local sections $\Gamma(H)$ of the CR bundle are called *CR vector fields*. A CR structure is *integrable* if the Lie bracket of CR vector fields is again a CR vector field, often abbreviated $[H, H] \subset H$ (or by conjugating, $[\overline{H}, \overline{H}] \subset \overline{H}$). We restrict our attention to integrable CR structures. Note that CR integrability does not imply that D is an integrable subbundle of TM, which would additionally require $[H, \overline{H}] \subset H \oplus \overline{H}$. When the latter holds, the Newlander-Nirenberg theorem implies that the almost-complex structure on D locally integrates to a complex structure, so that

$$M \cong_{\mathrm{CR}} \mathbb{C}^n \times \mathbb{R}^c. \tag{2.1}$$

In this most trivial instance (2.1), we say that M is *Levi-flat*, as the obstruction to this triviality is the familiar *Levi form*, the sesquilinear bundle map

$$\mathcal{L} : H \times H \to \mathbb{C}TM/\mathbb{C}D,$$

defined as follows. For $X_x, Y_x \in H_x$ and $X, Y \in \Gamma(H)$ such that $X|_x = X_x$ and $Y|_x = Y_x$,

$$\mathcal{L}(X_x, Y_x) := \mathbf{i}[X, \overline{Y}]|_x \mod \mathbb{C}D.$$

Though this is defined by the Lie bracket of local extensions of X_x, Y_x, the quotient projection $\mathbb{C}TM \to \mathbb{C}TM/\mathbb{C}D$ ensures $\mathcal{L}$ is tensorial. In particular, when $c = 1$ so that D has corank-1 in TM, $\mathcal{L}$ takes values in a complex line bundle and may locally be considered a Hermitian form on H whose signature remains invariant under CR equivalence.

When $c = 1$, we say that M is of *hypersurface-type*, and though all of the structures to be defined in the sequel can be formulated for higher CR codimension, the results we will discuss apply exclusively to the hypersurface-type case, so we will assume henceforth that $c = 1$. Because

CR codimension is defined to generalize the notion of codimension of a real submanifold $M \subset \mathbb{C}^m$, the hypersurface-type case includes the following class of examples.

Example 2.1 *Suppose a CR hypersurface $M \subset \mathbb{C}^{n+1}$ is given locally by a level set of a smooth, real-valued function with nonvanishing gradient. Submitting $\mathbb{C}^{n+1}$ to a biholomorphic change of coordinates if necessary, it is no loss of generality to assume that M is the level set*

$$f(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, z_{n+1} + \overline{z}_{n+1}) = -\tfrac{\mathbf{i}}{2}(z_{n+1} - \overline{z}_{n+1}) \qquad \textit{for some} \qquad f : \mathbb{C}^n \to \mathbb{R}.$$

The Levi form $\mathcal{L}$ of M is represented as the $n \times n$ Hermitian matrix of second-order partial derivatives

$$\mathcal{L} = \left[\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j} \right]_{i,j \leq n}.$$

A biholomorphic transformation of $\mathbb{C}^{n+1}$ restricts to give a CR equivalence on M. Though the matrix representation of $\mathcal{L}$ may change under such a transformation, its signature remains invariant.

In 1910, E.E. Levi showed ([Lev10]) that for a domain in $\mathbb{C}^2$ (later generalized to $\mathbb{C}^{n+1}$ by J. Krzoska's [Krz33]) bounded by M as in Example 2.1, the pseudoconvexity property characterizing a domain of holomorphy is equivalent to the condition that the matrix $\mathcal{L}$ is positive-semidefinite, with the positive-definite case defining strongly pseudoconvex domains. Levi's result exemplifies the rich interaction between the fields of CR geometry, several complex variables, and partial differential equations.

Perhaps even more fundamental than its interplay with analysis, CR geometry has decidedly algebraic facets as well. Three years before Levi's proof was published, the seminal work [Poi07] of H. Poincaré demonstrated that two real hypersurfaces in $\mathbb{C}^2$ can have distinct automorphism groups, thus precluding biholomorphic equivalence and indicating the existence of invariants which distinguish inequivalent hypersurfaces. Poincaré devoted particular attention to the 3-dimensional hypersphere and its symmetry group. In arbitrary dimension, the hypersphere is one of an especially important class of hypersurfaces known as the *real hyperquadrics*.

Example 2.2 *To specialize Example 2.1, let $f : \mathbb{C}^n \to \mathbb{R}$ be given by*

$$f(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n) = h^{ij} z_i \overline{z}_j; \qquad\qquad h^{ij} = h^{ji} \in \mathbb{R},$$

in accordance with the summation convention. By what we have seen in the general case,

$$\mathcal{L} = \left[h^{ij} \right]_{i,j \leq n},$$

and we say that $\mathcal{L}$ has signature (p, q) if the real, symmetric matrix $[h]$ has p strictly positive and q strictly negative eigenvalues. When $\mathcal{L}$ is nondegenerate so that $p + q = n$, M is called a real hyperquadric, and may be exhibited as the homogeneous quotient

$$M = SU(p + 1, q + 1)/P$$

of the special unitary group in signature $(p + 1, q + 1)$ by a parabolic subgroup P.

Late 19th-century mathematics witnessed the generalization of Euclidean geometry in two separate directions ([Sha97, Preface]). In one direction, F. Klein's Erlangen program abstracted and contextualized the familiar Euclidean geometry and novel non-Euclidean geometries by modeling them globally as homogeneous spaces of appropriate Lie groups of automorphisms. On the other hand, the advent of smooth manifolds and vector bundles along with the innovation of covariant differentiation allowed for "curved" Riemannian manifolds to generalize "flat" Euclidean vector spaces, and gave rise to Ricci's computationally convenient tensor calculus in local coordinates.

These two perspectives were unified and clarified under E. Cartan's notion of *espaces généralisés*, now called Cartan geometries. After Poincaré indicated that real hypersurfaces in $\mathbb{C}^2$ should have local invariants under biholomorphic transformations, Cartan produced ([Car33]) a complete set of such invariants by constructing a Cartan geometry over any $M \subset \mathbb{C}^2$ for which $\mathcal{L} \neq 0$ ($\mathcal{L}$ is a scalar in dimension three, so this only excludes the Levi-flat case).

Specifically, Cartan constructed a principal bundle $B \to M$ with structure group $P \subset SU(2, 1)$ as in Example 2.2, along with an absolute parallelism of B. The absolute parallelism is given by a globally defined one-form $\omega \in \Omega^1(B, \mathfrak{su}(2, 1))$ taking values in the Lie algebra $\mathfrak{su}(2, 1)$ of $SU(2, 1)$. Invariants of M are obtained in the form of an $\mathfrak{su}(2, 1)$-valued CR curvature tensor

by differentiating ω, just as the curvature tensor of an n-dimensional Riemannian manifold is obtained by differentiating the $\mathfrak{so}(n)$-valued Levi-Civita connection. And just as a Riemannian manifold whose curvature tensor vanishes is locally isometric to a flat Euclidean vector space, vanishing of M's CR curvature tensor implies that M is locally CR-equivalent to the hypersphere $SU(2,1)/P$. In this sense, the hypersphere is the "flat" or *homogeneous model* of 3-dimensional CR geometry, and the CR curvature tensor measures the obstruction to the existence of a CR equivalence $M \cong_{\mathrm{CR}} SU(2,1)/P$.

The procedure Cartan used to construct B and ω is his *method of equivalence*, an algorithmic application of his exterior differential calculus that has been used to classify a wide range of geometric structures up to local equivalence, including conformal, projective, and Finsler manifolds as well as generic distributions arising from geometric PDE. Beginning in the early 1960's, N. Tanaka developed a sophisticated modification of Cartan's method that facilitated the uniform construction of Cartan geometries and description of their invariants in broad generality ([Tan62, Tan65, Tan67, Tan70, Tan76, Tan79]).

In particular, Tanaka extended Cartan's result in 1962 by constructing Cartan geometries corresponding to Levi-nondegenerate, hypersurface-type CR manifolds of any CR dimension ([Tan62]). However, the technical details underlying Tanaka's work are forbidding, and his result languished in relative obscurity until S.S. Chern replicated it in 1974 using Cartan's classical method in joint work with J. Moser ([CM74]). We summarize the Tanaka-Chern-Moser (TCM) solution to the CR equivalence problem in the language of Example 2.2 with the following

Theorem 2.3 (TCM classification) *Let M be a hypersurface-type CR manifold of CR dimension n whose Levi form has signature (p,q) with $p + q = n$. There exists a principal P-bundle $B \to M$ and an absolute parallelism $\omega \in \Omega^1(B, \mathfrak{su}(p+1, q+1))$. Differentiating ω provides a complete set of local invariants of M which measure the obstruction to $M \cong_{CR} SU(p+1, q+1)/P$.*

As stated, Theorem 2.3 is actually weaker than what Tanaka and Chern proved, since it merely presents the "solution to the equivalence problem" rather than the assignment of a Cartan geometry to the given CR structure. The distinction between these two statements depends on the parallelism ω. If ω satisfies a certain equivariance condition with respect to the principal P-action on B, then it determines a *Cartan connection* which is the defining ingredient of a Cartan geometry.

In general, equivariance is unnecessary for the purposes of identifying the homogeneous model

of a geometric structure or producing invariants that distinguish inequivalent manifolds. However, Cartan connections have many desirable properties. For example, Cartan connections induce linear connections on vector bundles associated to the principal bundle constructed by the method of equivalence, giving rise to an invariant "tractor calculus" ([CG14, ČG02, ČG03, ČG08]) analogous to Ricci's tensor calculus for appropriate geometries. We will elaborate on the equivariance condition in §4.3.

2.3   CR Straightening vs 2-Nondegeneracy and the Cubic Form

The TCM classification settles the equivalence problem for CR manifolds whose Levi form is nondegenerate, but this does not even exhaust all smooth boundaries of pseudoconvex domains, for example, so there are many more cases to consider. To approach the middle ground between Levi-nondegeneracy and Levi-flatness, define the Levi kernel,

$$K_x := \{X_x \in H_x \mid \mathcal{L}(X_x, Y_x) = 0 \ \forall Y_x \in H_x; \ x \in M\},$$

which we assume has constant rank $0 < k < n$ so that K is a smooth subbundle of H (and by conjugating, $\overline{K} \subset \overline{H}$). By definition of K, the Levi form of M descends to a well-defined, nondegenerate Hermitian form on the quotient bundle

$$\underline{\mathcal{L}} : H/K \times H/K \to \mathbb{C}TM/\mathbb{C}D.$$

An application of the Newlander-Nirenberg theorem ([Fre74, Thm 1.1]) reveals that $K \oplus \overline{K} \subset \mathbb{C}TM$ is the complexification of a J-invariant, integrable subbundle $D^\circ \subset D$, so that M is foliated by complex manifolds of complex dimension k. Thus, a coordinate chart adapted to this *Levi foliation* provides a local diffeomorphism $F : M \to \underline{M} \times \mathbb{C}^k$, where $\underline{M}$ is a CR manifold of CR dimension $n - k$. It is not true in general that F must be a CR equivalence onto its image, however, and M. Freeman studied this phenomenon ([Fre77a]) in the years immediately following the publication of the Chern-Moser paper, leading to the notion of *CR straightening* (see also, [Chi91]).

Definition 2.4 *A CR manifold M with $rank_\mathbb{C} K = k$ is* straightenable *if there exists a CR manifold $\underline{M}$ such that $M \cong_{CR} \underline{M} \times \mathbb{C}^k$.*

When M is straightenable, the Levi form $\mathcal{L}$ of M descends to the Levi form $\underline{\mathcal{L}}$ of $\underline{M}$. Since $\underline{\mathcal{L}}$

is nondegenerate, the TCM classification of $\underline{M}$ extends to a classification of M, as the factor $\mathbb{C}^k$ is Levi-flat and so contributes trivially to the CR structure of M. It therefore remains to classify non-straightenable M, for which purpose we must first ascertain when straightening can fail. To this end, we consider the transverse structure to the Levi foliation. CR integrability of M dictates

$$[\overline{H}, \overline{H}] \subset \overline{H},$$

hence the CR structure of $\underline{M}$ is integrable when

$$[\overline{H} \mod K \oplus \overline{K}, \; \overline{H} \mod K \oplus \overline{K}] \subset \overline{H} \mod K \oplus \overline{K} \qquad \Longleftrightarrow \qquad [K, \overline{H}] \subset K \oplus \overline{H}.$$

Thus, it is necessary that $[K, \overline{H}] \subset K \oplus \overline{H}$ in order for M to be straightenable.

Let us rephrase this condition in terms of a family of antilinear operators on $\mathbb{C}TM$. For $X \in \Gamma(K)$ and $Y \in \Gamma(H)$, $\underline{Y}$ denotes the image of Y under the quotient projection $H \to H/K$, and we define

$$\mathrm{ad}_X : H/K \to H/K$$

$$\underline{Y} \mapsto [X, \overline{Y}] \mod K \oplus \overline{H}.$$

As with the Levi form, the quotient projection ensures ad_X is well-defined and tensorial in both X and Y, and we write ad_K for the collection of all ad_X. Now we can say that M fails to admit any straightening when the operators $\mathrm{ad}_K : H/K \to H/K$ are nontrivial. Clearly, H must have rank at least two in order for $H/K \neq 0$, so the lowest dimension in which straightening can fail is $\dim_{\mathbb{R}} M = 5$. The prototypical example of a nonstraightenable CR hypersurface in dimension five is the *tube over the future light cone* ([Fre77b],[IZ13],[MS14]).

Example 2.5 *Let $M \subset \mathbb{C}^3$ be given by*

$$(z_1 + \overline{z}_1)^2 + (z_2 + \overline{z}_2)^2 = (z_3 + \overline{z}_3)^2; \qquad\qquad (z_3 + \overline{z}_3) > 0.$$

The CR bundle of $H \subset \mathbb{C}TM$ can be explicitly parametrized as

$$c_1 \frac{\partial}{\partial z_1} + c_2 \frac{\partial}{\partial z_2} + \frac{c_1(z_1 + \overline{z}_1) + c_2(z_2 + \overline{z}_2)}{(z_3 + \overline{z}_3)} \frac{\partial}{\partial z_3},$$

where c_1, c_2 are $\mathbb{C}$-valued fiber coordinates. The subbundle $K \subset H$ is defined by the constraint

$$c_1(z_2 + \overline{z}_2) = c_2(z_1 + \overline{z}_1).$$

Let $X \in \Gamma(K)$ be given by $c_1 = (z_1 + \overline{z}_1)$, $c_2 = (z_2 + \overline{z}_2)$ and $Y \in \Gamma(H)$ by $c_1 = 1$ and $c_2 \in \{0, 1\}$ so that $\underline{Y} \in \Gamma(H/K)$ as above. Then

$$[X, \overline{Y}] = -Y \implies ad_X(\underline{Y}) = -\underline{Y},$$

which is nontrivial everywhere on M, as $Y \notin K$ for at least one of $c_2 = 0, 1$.

Thus we see that M does not admit any CR straightening. Furthermore, M may be exhibited (c.f. [IZ13],[MS14]) as the homogeneous quotient

$$M = SO^{\circ}(3, 2)/Q$$

of the identity component of the special orthogonal group in signature $(3, 2)$ by a subgroup $Q \subset SO^{\circ}(3, 2)$.

The method of equivalence was first employed to classify 5-dimensional, non-straightenable CR manifolds by P. Ebenfelt in 2001 ([Ebe01]), though his proof was valid only for a restricted class of CR maps ([Ebe06]). In 2013, S. Pocchiola constructed parallelisms over embedded 5-dimensional hypersurfaces ([Poc13]), and the general 5-dimensional case was treated by A. Isaev and D. Zaitsev ([IZ13]) using techniques adapted from the Chern-Moser paper. A year later, C. Medori and A. Spiro presented an alternative proof ([MS14]) based on a variation of Tanaka's construction. It is notable that the Isaev-Zaitsev solution to the equivalence problem does not satisfy the equivariance condition to determine a Cartan geometry, while the Medori-Spiro solution does. In any case, the solution may be summarized with the notation of Example 2.5 as follows.

Theorem 2.6 *Let M be a hypersurface-type CR manifold of CR dimension 2 whose Levi form has constant rank 1. There exists a principal Q-bundle $B \to M$ and an absolute parallelism $\omega \in \Omega^1(B, \mathfrak{so}(3, 2))$. Differentiating ω provides a complete set of local invariants of M which measure the obstruction to $M \cong_{CR} SO^{\circ}(3, 2)/Q$.*

In dimension five, both of K and H/K have rank 1, so the action $ad_K : H/K \to H/K$ is by

scalar multiplication which is either zero or nonzero. In higher dimensions, the question of straightenability is more nuanced. Fortunately, Freeman's [Fre77a] provides a definitive characterization of straightenability in terms of higher-order generalizations of the Levi form. When the Levi kernel K of M has constant, nonzero rank, we can define what is sometimes called the *cubic form* ([Web95]) or *third order tensor* ([Ebe98]):

$$\mathcal{C} : K \times H \times H \to \mathbb{C}TM/\mathbb{C}D.$$

For $X_x \in K_x$ and $Y_x, Z_x \in H_x$ with CR vector fields $X \in \Gamma(K)$ and $Y, Z \in \Gamma(H)$ which locally extend them, we define

$$\mathcal{C}(X_x, Y_x, Z_x) := \mathbf{i}[[X, \overline{Y}], \overline{Z}]|_x \quad \mod \mathbb{C}D.$$

Now $\ker \mathcal{C} \subset K$ is defined to be the kernel in the first factor of $\mathcal{C}$'s domain, and Freeman showed that M is straightenable exactly when this kernel is all of K. At the other extreme, we have

Definition 2.7 M *is called* 2-nondegenerate *when* $\ker \mathcal{C} = 0$.

In the intermediate case $0 \neq \ker \mathcal{C} \subsetneq K$, Freeman's argument may be iterated to define higher-order analogues of $\mathcal{L}, \mathcal{C}$, leading to higher-nondegeneracy conditions (see also, [BER99, Ch.XI]) and more refined notions of straightening. However, when $k = \mathrm{rank}_{\mathbb{C}}K = 1$, 2-nondegeneracy is synonymous with non-straightenability. In order to make contact with our *ad hoc* condition of non-straightenability via the ad_K maps, we first note that integrability of the Levi kernel shows that $\mathcal{C}$ descends to be well-defined ([Fre77a, Thm 4.4]) on the quotient

$$\underline{\mathcal{C}} : K \times H/K \times H/K \to \mathbb{C}TM/\mathbb{C}D$$
$$(X, \underline{Y}, \underline{Z}) \mapsto \mathcal{C}(X, Y, Z).$$

We therefore adduce CR integrability and the definition of the Levi kernel to write

$$\underline{\mathcal{C}}(X, \underline{Y}, \underline{Z}) = \mathbf{i}[[X, \overline{Y}], \overline{Z}]|_x \quad \mod \mathbb{C}D$$
$$= \underline{\mathcal{L}}(\mathrm{ad}_X(\underline{Y}), \underline{Z}).$$

As such, $\mathcal{C}$ may be interpreted as the collection of the $\mathrm{ad}_K : H/K \to H/K$ operators into a

single tensor by way of the nondegenerate Hermitian form $\mathcal{L}$ on H/K. To further explore this perspective, we make use of the Jacobi identity to calculate

$$\underline{\mathcal{L}}(\mathrm{ad}_X(\underline{Y}), \underline{Z}) = \mathbf{i}(-\underbrace{[[\overline{Y}, \overline{Z}], X]|_x}_{\in H \oplus \overline{H}} - [[\overline{Z}, X], \overline{Y}]|_x) \mod \mathbb{C}D$$

$$= -\underline{\mathcal{L}}(\underline{Y}, \mathrm{ad}_X(\underline{Z})),$$

whence we see that ad_K determines a family of normal (albeit antilinear) operators on H/K. Distinguished among the set of normal operators on a Hermitian inner product space is the group of unitary operators that act bijectively and preserve the inner product. More generally, we could consider those invertible operators which preserve the inner product up to some nonzero conformal factor, and it is in this vein that we offer the following definition.

Definition 2.8 *The cubic form $\mathcal{C}$ of a 2-nondegenerate CR manifold M is said to be of* conformal unitary type *if*

$$\underline{\mathcal{L}}(ad_X(\underline{Y}), ad_X(\underline{Z})) = \lambda\underline{\mathcal{L}}(\underline{Y}, \underline{Z}), \qquad\qquad \forall X \in K;\ Y, Z \in H,$$

where λ is a non-vanishing, $\mathbb{C}$-valued function on M.

Note that the cubic form of a 5-dimensional, 2-nondegenerate CR manifold is automatically of conformal unitary type. Thus, the most direct generalization to higher CR dimension of the hypotheses in Theorem 2.6 may be summarized as follows.

Statement of the Problem 1 *Let M be a 2-nondegenerate, hypersurface-type CR manifold with*

$$\dim_{\mathbb{R}} M = 7, \qquad\qquad rank_{\mathbb{C}} K = 1,$$

such that $\mathcal{C}$ is of conformal unitary type. Determine a complete set of local invariants of M under any CR equivalence.

2.4   7-Dimensional Case: Local Coframing Formulation

In this section as in the sequel, we adhere to the summation convention. We have already observed that $\mathcal{L}$ takes values in a complex line bundle when M has CR codimension 1, whence

$\underline{\mathcal{L}}$ is locally represented as a nondegenerate Hermitian form on H/K. To achieve such a local representation in a neighborhood of $x \in M$, we choose a nonvanishing one-form $\theta^0 \in \Omega^1(M) \subset \Omega^1(M, \mathbb{C})$ that annihilates $D \subset TM$, which we denote $\theta^0 \in \Gamma(D^\perp)$. Incorporating this choice into our notation, we express the resulting Hermitian form

$$\begin{aligned}
\underline{\mathcal{L}}_0 : H \times H &\to \mathbb{C} \\
(X, Y) &\mapsto \mathbf{i}\theta^0([X, Y]) = -\mathbf{i}d\theta^0(X, Y).
\end{aligned} \tag{2.2}$$

To understand the local formulation of the hypotheses presented in the Statement of the Problem as articulated in the preceding section, we extend θ^0 to a full local coframing around x.

Definition 2.9 *A 0-adapted coframing θ in a neighborhood of $x \in M$ consists of local one-forms $\theta^0, \theta^1, \theta^2, \theta^3 \in \Gamma(\overline{H}^\perp) \subset \Omega^1(M, \mathbb{C})$ – and their complex conjugates – so that θ satisfies*

$$\theta^0 \in \Gamma(D^\perp) \subset \Omega^1(M), \qquad\qquad \theta^1, \theta^2 \in \Gamma(K^\perp) \subset \Omega^1(M, \mathbb{C}),$$

$$\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^{\overline{1}} \wedge \theta^{\overline{2}} \wedge \theta^{\overline{3}} \neq 0.$$

Here, $\theta^{\overline{j}}$ denotes the complex conjugate $\overline{\theta^j}$ of a $\mathbb{C}$-valued form. CR integrability $[\overline{H}, \overline{H}] \subset \overline{H}$ is equivalent to

$$d\theta^i \equiv 0 \mod \{\theta^0, \theta^1, \theta^2, \theta^3\}; \ 0 \leq i \leq 3, \tag{2.3}$$

while the integrability of D° (recall that $\mathbb{C}D^\circ = K \oplus \overline{K}$) additionally gives

$$d\theta^l \equiv 0 \mod \{\theta^0, \theta^1, \theta^2, \theta^{\overline{1}}, \theta^{\overline{2}}\}; \ 0 \leq l \leq 2. \tag{2.4}$$

Furthermore, since θ^0 is $\mathbb{R}$-valued,

$$d\theta^0 \equiv \mathbf{i}\ell_{j\overline{k}}\theta^j \wedge \theta^{\overline{k}} \mod \{\theta^0\}; \ (1 \leq j, k \leq 2), \tag{2.5}$$

for some $\ell_{j\overline{k}} \in C^\infty(M)$, where $\ell := \begin{bmatrix} \ell_{1\overline{1}} & \ell_{1\overline{2}} \\ \ell_{2\overline{1}} & \ell_{2\overline{2}} \end{bmatrix}$ is real, symmetric, nondegenerate, and provides a local matrix representation of $\underline{\mathcal{L}}_0$ (as a Hermitian form) as in (2.2). In order to consider the most general case, we let $\epsilon = \pm 1$ and note that by changing the sign of θ^0 if necessary, the matrix ℓ may

be diagonalized with diagonal entries $1, \epsilon$. If we also let $\delta_1 = 0$ and $\delta_{-1} = 1$ so that $\epsilon = (-1)^{\delta_\epsilon}$, then we can say in general that the signature of $\underline{\mathcal{L}}_0$ is $(2 - \delta_\epsilon, \delta_\epsilon)$.

We invoke (2.3) and (2.4) to write

$$\mathrm{d}\theta^j \equiv u^j_{\overline{k}} \theta^3 \wedge \theta^{\overline{k}} \mod \{\theta^0, \theta^1, \theta^2\}; \ (1 \le j, k \le 2),$$

for some $u^j_{\overline{k}} \in C^\infty(M, \mathbb{C})$, so that $u := \begin{bmatrix} u^1_{\overline{1}} & u^1_{\overline{2}} \\ u^2_{\overline{1}} & u^2_{\overline{2}} \end{bmatrix}$ is a local matrix representation of ad_{X_3}, where $X_3 \in \Gamma(K)$ is dual to θ^3 in our coframing θ – i.e., $\theta^3(X_3) = 1$ while $\theta^l(X_3) = \theta^{\overline{i}}(X_3) = 0$ for $0 \le l \le 2$ and $1 \le i \le 3$. The hypothesis of 2-nondegeneracy merely says that the matrix u is not zero, but the hypothesis that the cubic form of M is of conformal unitary type implies that u is conformally unitary with respect to the 2×2 matrix ℓ – specifically, u is invertible and

$$\overline{u}^t \ell u = \lambda \ell, \tag{2.6}$$

for some $\lambda \in C^\infty(M, \mathbb{C})$.

Expressing θ as the column vector $[\theta^0, \theta^1, \theta^2, \theta^3]^t$ and fixing index ranges $1 \le j, k \le 2$, we can summarize our analysis in this section thusly:

$$\mathrm{d}\theta = \begin{bmatrix} \mathrm{d}\theta^0 \\ \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \\ \mathrm{d}\theta^3 \end{bmatrix} \equiv \begin{bmatrix} \mathbf{i}\ell_{j\overline{k}}\theta^j \wedge \theta^{\overline{k}} \\ u^1_{\overline{k}}\theta^3 \wedge \theta^{\overline{k}} \\ u^2_{\overline{k}}\theta^3 \wedge \theta^{\overline{k}} \\ 0 \end{bmatrix} \mod \begin{Bmatrix} \theta^0 \\ \theta^0, \theta^1, \theta^2 \\ \theta^0, \theta^1, \theta^2 \\ \theta^0, \theta^1, \theta^2, \theta^3 \end{Bmatrix}. \tag{2.7}$$

We conclude this section with a remark about notation. As we have above, we will continue to denote the conjugate of every $\mathbb{C}$-valued one-form by putting overlines on its indices. By contrast, we indicate the conjugate of a $\mathbb{C}$-valued function with an overline on the name of the function itself, without changing the indices. For example, the conjugate of the second identity in (2.7) would be written

$$\mathrm{d}\theta^{\overline{1}} \equiv \overline{u}^1_{\overline{1}}\theta^{\overline{3}} \wedge \theta^1 + \overline{u}^1_{\overline{2}}\theta^{\overline{3}} \wedge \theta^2 \mod \{\theta^0, \theta^{\overline{1}}, \theta^{\overline{2}}\}.$$

3. THE EQUIVALENCE PROBLEM

The construction carried out in this section will serve as a proof of Theorem 1.1. Because of the technical nature of the calculation, we offer a brief outline of the steps involved.

In §3.1, the filtration on $\mathbb{C}TM$ determined by the CR bundle and Levi kernel is encoded in a principal bundle B_0 of complex coframes on M adapted to this filtration – an "order zero" adaptation. The structure group G_0 of B_0 is 21-dimensional, and the globally defined tautological forms on B_0 are extended to a full coframing of B_0 over any local trivialization $B_0 \cong G_0 \times M$ by the Maurer-Cartan forms of G_0. These Lie-algebra-valued "pseudoconnection" forms are only locally determined up to combinations of the tautological forms which take values in the same Lie algebra.

We gradually eliminate this ambiguity in the pseudoconnection forms when we restrict to subbundles of B_0 defined by coframes that are adapted to higher order, as this reduces the dimension of the structure group and its Lie algebra. Therefore, in §3.2, we perform the first such reductions. Restricting to the subbundle $B_1 \subset B_0$ of coframes which are "orthonormal" for the nondegenerate part of $\mathcal{L}$ reduces the structure group to a 17-dimensional subgroup $G_1 \subset G_0$. Similarly, our hypothesis on the cubic form implies there is a subbundle $B_2 \subset B_1$ of coframes which are analogously adapted to $\mathcal{C}$, and the structure group $G_2 \subset G_1$ has dimension 13.

In §3.3, we exploit the ambiguity in the pseudoconnection forms on B_2 in order to simplify the expressions of the exterior derivatives of the tautological forms. This process is known as absorbing torsion, and simplifying the equations facilitates the final two reductions in §3.4. The subbundles $B_4 \subset B_3 \subset B_2$ constructed therein have structure groups $G_4 \subset G_3 \subset G_2$ reduced from dimension 13 to $\dim G_3 = 9$, and ultimately to $\dim G_4 = 7$. At this point, no further reduction is possible without destroying the tautological forms, but the pseudoconnection forms on B_4 are still not uniquely defined.

To finish the calculation, in §3.5 we prolong to the bundle $B_4^{(1)}$ over B_4 that parameterizes the remaining ambiguity of the pseudoconnection forms on B_4 in the same way that B_4 parameterizes the ambiguity in our adapted coframes of M. In this sense we begin the method of equivalence anew, but the structure group of $B_4^{(1)}$ as a bundle over B_4 is only 1-dimensional. After finding expressions for the derivatives of the tautological forms on $B_4^{(1)}$, the ambiguity in the pseudoconnection form on $B_4^{(1)}$ is completely eliminated by absorbing torsion in these expressions.

3.1   Initial G-Structure

Let $V = \mathbb{R} \oplus \mathbb{C}^3$, presented as column vectors

$$V = \left\{ \begin{bmatrix} r \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} : r \in \mathbb{R}; z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

For $x \in M$, a coframe $v_x : T_x M \xrightarrow{\simeq} V$ is a linear isomorphism that will be called *0-adapted* if

- $v_x(D_x) = \left\{ \begin{bmatrix} 0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}$,

- $v_x|_{D_x} \circ J = \mathbf{i} v_x|_{D_x}$,

- $v_x(D_x^\circ) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ z_3 \end{bmatrix} : z_3 \in \mathbb{C} \right\}.$

Let $\underline{\pi} : B_0 \to M$ denote the bundle of all 0-adapted coframes, where $\underline{\pi}(v_x) = x$. A local section $s : M \to B_0$ in a neighborhood of x with $s(x) = v_x$ is a 0-adapted coframing θ, written as a column vector like in §2.4, so that $\theta|_x = v_x$. The *tautological one-form* $\eta \in \Omega^1(B_0, V)$ is intrinsically (therefore globally) defined by

$$\eta|_{v_x}(X) := v_x(\underline{\pi}_*(X|_{v_x})), \qquad \forall X \in \Gamma(TB_0). \tag{3.1}$$

It follows directly from the definition of η that if θ is a 0-adapted coframing given by a local section s of B_0, then the tautological form satisfies the so-called *reproducing property*: $\theta = s^*\eta$. Naturally, the reproducing property extends to

$$\mathrm{d}\theta = s^*\mathrm{d}\eta. \tag{3.2}$$

We will find a local expression for η by locally trivializing B_0 in a neighborhood of any $x \in M$. To this end, first note that if $v_x, \tilde{v}_x \in B_0$ are two coframes in the fiber over x, then by the definition of 0-adaptation, it must be that

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$$\tilde{v}_x = \begin{bmatrix} t & 0 & 0 & 0 \\ c^1 & a_1^1 & a_2^1 & 0 \\ c^2 & a_1^2 & a_2^2 & 0 \\ c^3 & b_1 & b_2 & b_3 \end{bmatrix} v_x; \qquad \text{where} \qquad \begin{cases} t \in \mathbb{R} \setminus \{0\}, \\ \\ c^j, b_k \in \mathbb{C} \ (b_3 \neq 0); \quad 1 \leq j, k \leq 3, \\ \\ \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \in GL_2\mathbb{C}. \end{cases} \tag{3.3}$$

Call the subgroup of $GL(V)$ given by all such matrices G_0, and its Lie algebra $\mathfrak{g}_0$. G_0 acts transitively on the fibers of B_0, so fixing a 0-adapted coframing $\theta_\mathbb{1}$ in a neighborhood of x determines a local trivialization $B_0 \cong G_0 \times M$, as every other θ may be written

$$\begin{bmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \begin{bmatrix} t & 0 & 0 & 0 \\ c^1 & a_1^1 & a_2^1 & 0 \\ c^2 & a_1^2 & a_2^2 & 0 \\ c^3 & b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \theta_\mathbb{1}^0 \\ \theta_\mathbb{1}^1 \\ \theta_\mathbb{1}^2 \\ \theta_\mathbb{1}^3 \end{bmatrix} \tag{3.4}$$

for some G_0-valued matrix of smooth functions defined on our neighborhood of x. In this trivialization, the fixed coframing $\theta_\mathbb{1}$ corresponds to the identity matrix $\mathbb{1} \in G_0$, and by restricting to $\theta|_x$, $\theta_\mathbb{1}|_x$ on each side of (3.4), we see that the G_0-valued matrix entries parametrize all $v_x \in B_0$ in the fiber over x, hence furnish local fiber coordinates for B_0.

By the reproducing property, the tautological V-valued one-form η on B_0 may now be expressed locally as

$$\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = \begin{bmatrix} t & 0 & 0 & 0 \\ c^1 & a_1^1 & a_2^1 & 0 \\ c^2 & a_1^2 & a_2^2 & 0 \\ c^3 & b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \underline{\pi}^*\theta_\mathbb{1}^0 \\ \underline{\pi}^*\theta_\mathbb{1}^1 \\ \underline{\pi}^*\theta_\mathbb{1}^2 \\ \underline{\pi}^*\theta_\mathbb{1}^3 \end{bmatrix}, \tag{3.5}$$

or more succinctly,

$$\eta = g^{-1}\underline{\pi}^*\theta_\mathbb{1}. \tag{3.6}$$

The matrix in (3.5) is considered to be the inverse $g^{-1} \in C^\infty(B_0, G_0)$ in (3.6) so that left-multiplication on coframes defines a right-principal G_0 action on B_0. Differentiating (3.6) yields

the structure equation

$$\mathrm{d}\eta = -g^{-1}\mathrm{d}g \wedge \eta + g^{-1}\underline{\pi}^*\mathrm{d}\theta_{\mathbb{1}}. \tag{3.7}$$

The *pseudoconnection form* $g^{-1}\mathrm{d}g$ takes values in the Lie algebra $\mathfrak{g}_0$. We see from the parametrization (3.3) of G_0 that $\mathfrak{g}_0$ may be presented as matrices of the form

$$\begin{bmatrix} \tau & 0 & 0 & 0 \\ \gamma^1 & \alpha_1^1 & \alpha_2^1 & 0 \\ \gamma^2 & \alpha_1^2 & \alpha_2^2 & 0 \\ \gamma^3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix},$$

where all of the entries are independent, $\tau \in \mathbb{R}$, and the rest of the entries take arbitrary complex values. For later convenience, we prefer instead to use the following, less obvious choice of parametrization for $\mathfrak{g}_0$:

$$\begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \alpha_1^1 & \alpha_2^1 & 0 \\ \gamma^2 & \alpha_1^2 & \alpha_2^2 & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 - \beta_1 & \mathbf{i}\gamma^1 - \beta_2 & \beta_3 \end{bmatrix}.$$

By taking the entries of this matrix to be forms in $\Omega^1(B_0, \mathbb{C})$ which complete η to a local coframing of B_0, the structure equation (3.7) can be written

$$\mathrm{d}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = -\begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \alpha_1^1 & \alpha_2^1 & 0 \\ \gamma^2 & \alpha_1^2 & \alpha_2^2 & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 - \beta_1 & \mathbf{i}\gamma^1 - \beta_2 & \beta_3 \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} + \begin{bmatrix} \Xi^0 \\ \Xi^1 \\ \Xi^2 \\ \Xi^3 \end{bmatrix}, \tag{3.8}$$

where the semibasic two-form $\Xi := g^{-1}\underline{\pi}^*\mathrm{d}\theta_{\mathbb{1}} \in \Omega^2(B_0, V)$ is apparent torsion. Note that the left-hand side of (3.7) is a globally defined two-form, while the terms on the right-hand side each depend on our local trivialization of B_0. In particular, the pseudoconnection forms in the matrix $g^{-1}\mathrm{d}g$ are determined only up to $\mathfrak{g}_0$-compatible combinations of the semibasic one-forms $\{\eta^j, \eta^{\bar{j}}\}_{j=0}^3$, which will in turn affect the presentation of the apparent torsion forms. We will use this ambiguity to

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simplify our local expression for Ξ, but first we must find what it is.

Fix index ranges $1 \leq j, k \leq 2$. The differential reproducing property (3.2) and the identities (2.7) imply

$$\Xi^0 = \mathbf{i}L_{j\overline{k}}\eta^j \wedge \eta^{\overline{k}} + \xi_0^0 \wedge \eta^0,$$

$$\Xi^j = U_{\overline{k}}^j \eta^3 \wedge \eta^{\overline{k}} + \xi_0^j \wedge \eta^0 + \xi_1^j \wedge \eta^1 + \xi_2^j \wedge \eta^2,$$

$$\Xi^3 = \xi_0^3 \wedge \eta^0 + \xi_1^3 \wedge \eta^1 + \xi_2^3 \wedge \eta^2 + \xi_3^3 \wedge \eta^3,$$

for some unknown, semibasic one-forms $\xi \in \Omega^1(B_0, \mathbb{C})$ (with ξ_0^0 $\mathbb{R}$-valued) and functions $L_{j\overline{k}} \in C^\infty(B_0)$, $U_{\overline{k}}^j \in C^\infty(B_0, \mathbb{C})$ whose value along the coframing θ described in §2.4 would be

$$L_{j\overline{k}}(\theta|_x) = \ell_{j\overline{k}}(x) \qquad \text{and} \qquad U_{\overline{k}}^j(\theta|_x) = u_{\overline{k}}^j(x). \qquad (3.9)$$

We will "absorb" as much of Ξ into our pseudoconnection forms as possible. It is a standard notational abuse to recycle the name of a pseudoconnection form after altering it to absorb apparent torsion. We will try to minimize confusion by denoting modified forms with hats, and then dropping the hats from the notation as each phase of the absorption process terminates. For example, the top line of (3.8) reads

$$\mathrm{d}\eta^0 = -2\tau \wedge \eta^0 + \mathbf{i}L_{j\overline{k}}\eta^j \wedge \eta^{\overline{k}} + \xi_0^0 \wedge \eta^0$$

$$= -(2\tau - \xi_0^0) \wedge \eta^0 + \mathbf{i}L_{j\overline{k}}\eta^j \wedge \eta^{\overline{k}},$$

so if we let $2\hat{\tau} = 2\tau - \xi_0^0$, we have simplified the expression to

$$\mathrm{d}\eta^0 = -2\hat{\tau} \wedge \eta^0 + \mathbf{i}L_{j\overline{k}}\eta^j \wedge \eta^{\overline{k}}.$$

Observe that $2\hat{\tau}$ must remain $\mathbb{R}$-valued for this absorption to be $\mathfrak{g}_0$-compatible, which is exactly the case as ξ_0^0 is $\mathbb{R}$-valued. To absorb the rest of the ξ's, set

$$\hat{\alpha}_k^j = \alpha_k^j - \xi_k^j, \qquad \hat{\gamma}^j = \gamma^j - \xi_0^j, \qquad \hat{\gamma}^3 = \gamma^3 - \xi_0^3,$$

$$\hat{\beta}_1 = \beta_1 - \mathbf{i}\xi_0^2 + \xi_1^3, \qquad \hat{\beta}_2 = \beta_2 - \mathbf{i}\xi_0^1 + \xi_2^3, \qquad \hat{\beta}_3 = \beta_3 - \xi_3^3.$$

Now the structure equations (3.8) may be written

$$\mathrm{d}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = -\begin{bmatrix} 2\hat{\tau} & 0 & 0 & 0 \\ \hat{\gamma}^1 & \hat{\alpha}_1^1 & \hat{\alpha}_2^1 & 0 \\ \hat{\gamma}^2 & \hat{\alpha}_1^2 & \hat{\alpha}_2^2 & 0 \\ \hat{\gamma}^3 & \mathbf{i}\hat{\gamma}^2 - \hat{\beta}_1 & \mathbf{i}\hat{\gamma}^1 - \hat{\beta}_2 & \hat{\beta}_3 \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} + \begin{bmatrix} \mathbf{i}L_{j\bar{k}}\eta^j \wedge \eta^{\bar{k}} \\ U_{\bar{k}}^1\eta^3 \wedge \eta^{\bar{k}} \\ U_{\bar{k}}^2\eta^3 \wedge \eta^{\bar{k}} \\ 0 \end{bmatrix}. \qquad (3.10)$$

3.2   First Two Reductions

We are done absorbing torsion for the moment, so we will drop the hats off of the pseudoconnection forms in (3.10). The remaining torsion terms are not absorbable, but we can normalize them by first ascertaining how the functions L, U in (3.10) vary along the fiber over fixed points of M, then choosing agreeable values from among those that L, U achieve in each fiber, and finally restricting to a subbundle of B_0 determined by the subgroup of G_0 which stabilizes the chosen torsion tensor over each fiber. To proceed, first differentiate the equation for $\mathrm{d}\eta^0$ and reduce modulo $\eta^0, \eta^3, \eta^{\bar{3}}$.

$$\begin{aligned}
0 &= \mathrm{d}(\mathrm{d}\eta^0) \\
&\equiv \mathbf{i}(\mathrm{d}L_{1\bar{1}} + L_{1\bar{1}}(2\tau - \alpha_1^1 - \alpha_{\bar{1}}^{\bar{1}}) - L_{1\bar{2}}\alpha_{\bar{1}}^{\bar{2}} - L_{2\bar{1}}\alpha_1^2) \wedge \eta^1 \wedge \eta^{\bar{1}} \\
&\quad + \mathbf{i}(\mathrm{d}L_{1\bar{2}} + L_{1\bar{2}}(2\tau - \alpha_1^1 - \alpha_{\bar{2}}^{\bar{2}}) - L_{1\bar{1}}\alpha_{\bar{2}}^{\bar{1}} - L_{2\bar{2}}\alpha_1^2) \wedge \eta^1 \wedge \eta^{\bar{2}} \\
&\quad + \mathbf{i}(\mathrm{d}L_{2\bar{1}} + L_{2\bar{1}}(2\tau - \alpha_2^2 - \alpha_{\bar{1}}^{\bar{1}}) - L_{1\bar{1}}\alpha_2^1 - L_{2\bar{2}}\alpha_{\bar{1}}^{\bar{2}}) \wedge \eta^2 \wedge \eta^{\bar{1}} \\
&\quad + \mathbf{i}(\mathrm{d}L_{2\bar{2}} + L_{2\bar{2}}(2\tau - \alpha_2^2 - \alpha_{\bar{2}}^{\bar{2}}) - L_{1\bar{2}}\alpha_2^1 - L_{2\bar{1}}\alpha_{\bar{2}}^{\bar{1}}) \wedge \eta^2 \wedge \eta^{\bar{2}} \qquad \mathrm{mod}\ \{\eta^0, \eta^3, \eta^{\bar{3}}\}.
\end{aligned}$$

If we momentarily agree that $j \neq k$, we can summarize these conditions

$$\left.\begin{aligned}
\mathrm{d}L_{j\bar{j}} &\equiv -L_{j\bar{j}}(2\tau - \alpha_j^j - \alpha_{\bar{j}}^{\bar{j}}) + L_{j\bar{k}}\alpha_{\bar{j}}^{\bar{k}} + L_{k\bar{j}}\alpha_j^k \\
\mathrm{d}L_{j\bar{k}} &\equiv -L_{j\bar{k}}(2\tau - \alpha_j^j - \alpha_{\bar{k}}^{\bar{k}}) + L_{j\bar{j}}\alpha_{\bar{k}}^{\bar{j}} + L_{k\bar{k}}\alpha_j^k
\end{aligned}\right\} \quad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\bar{1}}, \eta^{\bar{2}}, \eta^{\bar{3}}\}. \quad (3.11)$$

We will restrict to the subbundle $B_1 \subset B_0$ given by the level sets $L_{1\bar{1}} = 1$, $L_{2\bar{2}} = \epsilon$ (as in §2.4) and $L_{1\bar{2}} = L_{2\bar{1}} = 0$, which is simply the bundle of 0-adapted coframes in which θ^1, θ^2 are dual to CR vector fields that are orthonormal for the Levi form. Such coframings must exist, as one can apply Gram-Schmidt orthonormalization to nonvanishing CR vector fields which are not in the Levi kernel. In the notation of §2.4, B_1 is determined by local 0-adapted coframings θ which additionally

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satisfy

$$
\mathrm{d} \begin{bmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \begin{bmatrix} \mathbf{i}\theta^1 \wedge \theta^{\overline{1}} + \epsilon\mathbf{i}\theta^2 \wedge \theta^{\overline{2}} \\ u^1_{\overline{1}}\theta^3 \wedge \theta^{\overline{1}} + u^1_{\overline{2}}\theta^3 \wedge \theta^{\overline{2}} \\ u^2_{\overline{1}}\theta^3 \wedge \theta^{\overline{1}} + u^2_{\overline{2}}\theta^3 \wedge \theta^{\overline{2}} \\ 0 \end{bmatrix} \mod \begin{Bmatrix} \theta^0 \\ \theta^0, \theta^1, \theta^2 \\ \theta^0, \theta^1, \theta^2 \\ \theta^0, \theta^1, \theta^2, \theta^3 \end{Bmatrix}. \tag{3.12}
$$

We call such coframings *1-adapted*, and fix a new $\theta_{\mathbb{1}}$ among them to locally trivialize B_1. Computing directly with the coordinates of G_0 as in (3.4), one finds that any such θ with its Levi form so normalized differs from $\theta_{\mathbb{1}}$ by an element in G_0 with

$$
t = |a^1_1|^2 + \epsilon|a^2_1|^2 = \epsilon|a^1_2|^2 + |a^2_2|^2 \qquad \text{and} \qquad a^1_1\overline{a}^1_2 + \epsilon a^2_1\overline{a}^2_2 = 0, \tag{3.13}
$$

(which together imply $|a^1_1|^2 = |a^2_2|^2$). This subgroup $G_1 \subset G_0$ is therefore the stabilizer of our choice of torsion normalization, and the structure group of the subbundle $B_1 \subset B_0$. When restricted to B_1, we see by (3.11) that the pseudoconnection forms satisfy

$$
2\tau \equiv \alpha^1_1 + \alpha^{\overline{1}}_{\overline{1}} \equiv \alpha^2_2 + \alpha^{\overline{2}}_{\overline{2}}, \qquad \alpha^1_2 + \epsilon\alpha^{\overline{2}}_{\overline{1}} \equiv 0 \qquad \mod \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\}. \tag{3.14}
$$

Let $\iota_1 : B_1 \hookrightarrow B_0$ be the inclusion map. When we pull back our coframing of B_0 along ι_1 to get a coframing of B_1, we introduce new names for some one-forms, but we also recycle many of the current names. For those being recycled, we view the following definition as recursive. Those being recycled are

$$
\begin{bmatrix} \eta \\ \tau \\ \gamma^j \\ \beta_k \end{bmatrix} := \iota^*_1 \begin{bmatrix} \eta \\ \tau \\ \gamma^j \\ \beta_k \end{bmatrix}; \qquad (1 \leq j, k \leq 3),
$$

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while we also introduce

$$
\begin{bmatrix} \varrho \\ \varsigma \\ \alpha^1 \\ \xi_1^1 \\ \zeta_1^2 \\ \xi_2^2 \end{bmatrix} := \iota_1^* \begin{bmatrix} -\frac{\mathbf{i}}{2}(\alpha_1^1 - \alpha_{\bar{1}}^{\bar{1}}) \\ -\frac{\mathbf{i}}{2}(\alpha_2^2 - \alpha_{\bar{2}}^{\bar{2}}) \\ \alpha_2^1 \\ \tau - \frac{1}{2}(\alpha_1^1 + \alpha_{\bar{1}}^{\bar{1}}) \\ -(\alpha_1^2 + \epsilon\alpha_{\bar{2}}^{\bar{1}}) \\ \tau - \frac{1}{2}(\alpha_2^2 + \alpha_{\bar{2}}^{\bar{2}}) \end{bmatrix} .
\tag{3.15}
$$

Note that ξ_1^1 and ξ_2^2 are $\mathbb{R}$-valued, and by (3.14), we know

$$
\xi_1^1, \zeta_1^2, \xi_2^2 \equiv 0 \mod \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\bar{1}}, \eta^{\bar{2}}, \eta^{\bar{3}}\}.
\tag{3.16}
$$

If we keep the names $U_{\bar{k}}^j := \iota_1^* U_{\bar{k}}^j$, then pulling back (3.10) to B_1 yields new structure equations

$$
\mathrm{d} \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = - \begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\varrho & \alpha^1 & 0 \\ \gamma^2 & -\epsilon\alpha^{\bar{1}} & \tau + \mathbf{i}\varsigma & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 - \beta_1 & \mathbf{i}\gamma^1 - \beta_2 & \beta_3 \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} + \begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\bar{1}} + \epsilon\mathbf{i}\eta^2 \wedge \eta^{\bar{2}} \\ U_{\bar{k}}^1 \eta^3 \wedge \eta^{\bar{k}} + \xi_1^1 \wedge \eta^1 \\ U_{\bar{k}}^2 \eta^3 \wedge \eta^{\bar{k}} + \zeta_1^2 \wedge \eta^1 + \xi_2^2 \wedge \eta^2 \\ 0 \end{bmatrix} .
\tag{3.17}
$$

We turn our attention to normalizing the $U_{\bar{k}}^j$. Differentiating $\mathrm{d}\eta^0$ and reducing modulo η^0, η^1, η^2 will reveal that these functions are not independent on B_1.

$$
0 = \mathrm{d}(\mathrm{d}\eta^0) \equiv \mathbf{i}(U_{\bar{2}}^1 - \epsilon U_{\bar{1}}^2)\eta^3 \wedge \eta^{\bar{2}} \wedge \eta^{\bar{1}} \mod \{\eta^0, \eta^1, \eta^2\},
$$

so $U_{\bar{2}}^1 = \epsilon U_{\bar{1}}^2$, and we can declutter some notation by naming

$$
U := U_{\bar{1}}^2 = \epsilon U_{\bar{2}}^1, \qquad\qquad U^1 := U_{\bar{1}}^1, \qquad\qquad U^2 := U_{\bar{2}}^2.
$$

To see how these functions vary in a fiber over a fixed point of M, we differentiate $\mathrm{d}\eta^1$ and $\mathrm{d}\eta^2$ and reduce modulo η^0, η^1, η^2.

$$
0 = \mathrm{d}(\mathrm{d}\eta^1)
$$

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$$\equiv (\mathrm{d}U^1 - U^1(\beta_3 - 2\mathbf{i}\varrho) + 2U\alpha^1 + \epsilon U\zeta_{\overline{1}}^{\overline{2}}) \wedge \eta^3 \wedge \eta^{\overline{1}}$$

$$+ (\epsilon \mathrm{d}U - \epsilon U(\beta_3 - \mathbf{i}\varrho - \mathbf{i}\varsigma) - U^1\alpha^{\overline{1}} + U^2\alpha^1 + \epsilon U(\xi_2^2 - \xi_1^1)) \wedge \eta^3 \wedge \eta^{\overline{2}} \qquad \mathrm{mod} \ \{\eta^0, \eta^1, \eta^2\},$$

and similarly

$$0 = \mathrm{d}(\mathrm{d}\eta^2)$$

$$\equiv (\mathrm{d}U - U(\beta_3 - \mathbf{i}\varrho - \mathbf{i}\varsigma) - \epsilon U^1\alpha^{\overline{1}} + \epsilon U^2\alpha^1 + U(\xi_1^1 - \xi_2^2) - U^1\zeta_1^2 + U^2\zeta_{\overline{1}}^{\overline{2}}) \wedge \eta^3 \wedge \eta^{\overline{1}}$$

$$+ (\mathrm{d}U^2 - U^2(\beta_3 - 2\mathbf{i}\varsigma) - 2U\alpha^{\overline{1}} - \epsilon U\zeta_1^2) \wedge \eta^3 \wedge \eta^{\overline{2}} \qquad \mathrm{mod} \ \{\eta^0, \eta^1, \eta^2\}.$$

With (3.16) in mind, we summarize

$$\left. \begin{array}{rcl} \mathrm{d}U^1 & \equiv & U^1(\beta_3 - 2\mathbf{i}\varrho) - 2U\alpha^1 \\[4pt] \mathrm{d}U & \equiv & U(\beta_3 - \mathbf{i}\varrho - \mathbf{i}\varsigma) + \epsilon U^1\alpha^{\overline{1}} - \epsilon U^2\alpha^1 \\[4pt] \mathrm{d}U^2 & \equiv & U^2(\beta_3 - 2\mathbf{i}\varsigma) + 2U\alpha^{\overline{1}} \end{array} \right\} \quad \mathrm{mod} \ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\}. \qquad (3.18)$$

Recall that the hypothesis of 2-nondegeneracy provides that for every local 1-adapted coframing θ, one of U, U^1, U^2 is nonvanishing at $\theta|_x$. We will show that this fact along with the differential equations (3.18) implies there is a coframe in the fiber over x where $U = 1$ and $U^2 = 0$ as follows.

Suppose that $U(\theta_{\mathbb{1}}|_x) = 0$. Let $X, Y \in \Gamma(TB_1)$ be the (vertical) vector fields dual to $\mathrm{Re}(\alpha^1)$ and $\mathrm{Im}(\alpha^1)$, respectively, with respect to the coframing of B_1 furnished by the real and imaginary parts of the tautological forms and the pseudoconnection forms. The fiber $(B_1)_x$ is foliated by flow curves of X and Y. For $\mathtt{t} \in \mathbb{R}$, take $c_X(\mathtt{t})$ and $c_Y(\mathtt{t})$ to be the flow curves in the fiber which go through $\theta_{\mathbb{1}}|_x \in (B_1)_x$ at time $\mathtt{t} = 0$. By (3.18), we calculate

$$\frac{d}{d\mathtt{t}}\Big|_{\mathtt{t}=0} U(c_X(\mathtt{t})) = \mathrm{d}U\left(\frac{d}{d\mathtt{t}}\Big|_{\mathtt{t}=0} c_X(\mathtt{t})\right)$$

$$= (U(\beta_3 - \mathbf{i}\varrho - \mathbf{i}\varsigma) + \epsilon U^1\alpha^{\overline{1}} - \epsilon U^2\alpha^1)\left(X|_{c_X(0)}\right)$$

$$= \epsilon(U^1 - U^2)(\theta_{\mathbb{1}}|_x),$$

and similarly,

$$\frac{d}{dt}\bigg|_{t=0} U(c_Y(t)) = -\epsilon i(U^1 + U^2)(\theta_{\mathbb{1}}|_x).$$

Since we have assumed $U(\theta_{\mathbb{1}}|_x) = 0$, one of these derivatives must be nonzero, and in particular U is not identically zero in the fiber over x. Thus, there is some $\widetilde{\theta}|_x \in (B_1)_x$ and a neighborhood around it where $U \neq 0$. In this neighborhood, we can define $\widetilde{X} = \frac{1}{2U}X$ and $\widetilde{Y} = -\frac{1}{2U}Y$, and their corresponding flow curves $\widetilde{c_X}(t)$ and $\widetilde{c_Y}(t)$ which go through $\widetilde{\theta}|_x$ when $t = 0$. For $j = 1, 2$, we use (3.18) again to calculate $\forall t \in \mathbb{R}$,

$$\frac{d}{dt}U^j(\widetilde{c_X}(t)) = (-1)^j, \qquad\qquad \frac{d}{dt}U^j(\widetilde{c_Y}(t)) = i,$$

whence $U^j(\widetilde{c_X}(t)) = (-1)^j t + U^j(\widetilde{\theta}|_x)$ and $U^j(\widetilde{c_Y}(t)) = it + U^j(\widetilde{\theta}|_x)$. As such, we can move along flow curves to a coframe where one of U^j vanishes, and by our full-rank assumption on the maps ad_K, we still have $U \neq 0$. From this coframe, we move along flow curves of vertical vector fields dual to $\mathrm{Re}(\beta_3)$ and $\mathrm{Im}(\beta_3)$ in order to rescale $U = 1$. Let us restrict to the level set

$$U = 1, \qquad\qquad U^2 = 0,$$

which defines a subbundle $\iota_2 : B_2 \hookrightarrow B_1$ of *2-adapted coframes*.

Note that we have not yet invoked the hypothesis that the cubic form is of conformal unitary type. Without this condition, $\iota_2^* U^1$ would be an invariant on B_2. However, by imposing this condition, it follows from (2.6) and (3.9) that we restrict to the case

$$U^1 = 0.$$

As such, sections of B_2 are local 1-adapted coframings θ as in (3.12), but which additionally satisfy

$$\mathrm{d}\begin{bmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \begin{bmatrix} i\theta^1 \wedge \theta^{\overline{1}} + \epsilon i\theta^2 \wedge \theta^{\overline{2}} \\ \epsilon\theta^3 \wedge \theta^{\overline{2}} \\ \theta^3 \wedge \theta^{\overline{1}} \\ 0 \end{bmatrix} \mod \begin{Bmatrix} \theta^0 \\ \theta^0, \theta^1, \theta^2 \\ \theta^0, \theta^1, \theta^2 \\ \theta^0, \theta^1, \theta^2, \theta^3 \end{Bmatrix}. \tag{3.19}$$

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Among such *2-adapted coframings* we fix a new $\theta_{\mathbb{1}}$ in order to locally trivialize B_2. We saw that B_1 was locally trivialized $B_1 \cong G_1 \times M$ by (3.4), where the subgroup $G_1 \subset G_0$ was defined by the added conditions (3.13). Now one calculates that a matrix in G_1 applied to the new $\theta_{\mathbb{1}}$ will preserve our latest normalization if and only if we additionally have

$$a_1^1 = b_3 a_{\overline{2}}^{\overline{2}}, \qquad a_2^1 = \epsilon b_3 a_{\overline{1}}^{\overline{2}}, \qquad a_2^2 = b_3 a_{\overline{1}}^{\overline{1}}, \qquad \epsilon a_1^2 = b_3 a_{\overline{2}}^{\overline{1}}.$$

Since the diagonal terms in the matrices are nonvanishing, these relations imply $a_2^1 = a_1^2 = 0$, while $b_3 \in \mathbb{C}$ is unimodular. Let $G_2 \subset G_1$ denote this reduced group of matrices, which is the structure group of B_2. If we let $\mathbf{e}$ denote the natural exponential, then we may parametrize G_2 by

$$\begin{bmatrix} t^2 & 0 & 0 & 0 \\ c^1 & t\mathbf{e}^{\mathbf{i}r} & 0 & 0 \\ c^2 & 0 & t\mathbf{e}^{\mathbf{i}s} & 0 \\ c^3 & b_1 & b_2 & \mathbf{e}^{\mathbf{i}(r+s)} \end{bmatrix} ; \qquad r, s, 0 \neq t \in \mathbb{R}; c^j, b_k \in \mathbb{C}. \tag{3.20}$$

By (3.18), we see that when restricted to B_2, we have

$$\beta_3 \equiv \mathbf{i}\varrho + \mathbf{i}\varsigma, \qquad \alpha^1 \equiv 0 \qquad \mathrm{mod} \ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\}. \tag{3.21}$$

Pulling back our coframing along the inclusion ι_2, we rename accordingly. First, some familiar names

$$\begin{bmatrix} \eta \\ \tau \\ \varrho \\ \varsigma \\ \gamma \\ \beta_1 \\ \beta_2 \end{bmatrix} := \iota_2^* \begin{bmatrix} \eta \\ \tau \\ \varrho \\ \varsigma \\ \gamma \\ \beta_1 \\ \beta_2 \end{bmatrix}.$$

The only new forms we must define are semibasic by (3.21), viz,

$$
\begin{bmatrix} \xi_2^1 \\[2mm] \xi_3^3 \end{bmatrix} := \iota_2^* \begin{bmatrix} -\alpha^1 \\[2mm] -\beta_3 + \mathbf{i}\varrho + \mathbf{i}\varsigma \end{bmatrix}.
$$

We will also preserve the names of the unknown apparent torsion forms on B_1, except to combine terms where appropriate:

$$
\begin{bmatrix} \xi_1^1 \\[2mm] \xi_2^2 \\[2mm] \xi_1^2 \end{bmatrix} := \iota_2^* \begin{bmatrix} \xi_1^1 \\[2mm] \xi_2^2 \\[2mm] \zeta_1^2 + \epsilon\alpha^{\overline{1}} \end{bmatrix}.
$$

Pulling back (3.17) along ι_2 yields new structure equations on B_2:

$$
\mathrm{d} \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = - \begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\varrho & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\varsigma & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 - \beta_1 & \mathbf{i}\gamma^1 - \beta_2 & \mathbf{i}\varrho + \mathbf{i}\varsigma \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} + \begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}\eta^2 \wedge \eta^{\overline{2}} \\ \epsilon\eta^3 \wedge \eta^{\overline{2}} + \xi_1^1 \wedge \eta^1 + \xi_2^1 \wedge \eta^2 \\ \eta^3 \wedge \eta^{\overline{1}} + \xi_1^2 \wedge \eta^1 + \xi_2^2 \wedge \eta^2 \\ \xi_3^3 \wedge \eta^3 \end{bmatrix},
$$

$$\tag{3.22}$$

where ξ_1^1, ξ_2^2 are still $\mathbb{R}$-valued, and by (3.16),(3.21), we can say

$$
\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2, \xi_3^3 \equiv 0 \mod \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\}. \tag{3.23}
$$

3.3   Absorption

This section is devoted to absorbing as much as we can of the apparent torsion from the ξ's in (3.22). It is easy to see that we can absorb any η^0 components of these forms into the γ's (using the β's to correct the equation for $\mathrm{d}\eta^3$ if necessary). As such, we suppress these components when we adduce (3.23) to expand $\xi_j^i = f_{jk}^i \eta^k + t_{j\overline{k}}^i \eta^{\overline{k}}$:

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$$\xi_1^1 = f_{11}^1\eta^1 + f_{12}^1\eta^2 + f_{13}^1\eta^3 + \overline{f}_{11}^1\eta^{\overline{1}} + \overline{f}_{12}^1\eta^{\overline{2}} + \overline{f}_{13}^1\eta^{\overline{3}},$$

$$\xi_2^2 = f_{21}^2\eta^1 + f_{22}^2\eta^2 + f_{23}^2\eta^3 + \overline{f}_{21}^2\eta^{\overline{1}} + \overline{f}_{22}^2\eta^{\overline{2}} + \overline{f}_{23}^2\eta^{\overline{3}},$$

$$\xi_2^1 = f_{21}^1\eta^1 + f_{22}^1\eta^2 + f_{23}^1\eta^3 + t_{2\overline{1}}^1\eta^{\overline{1}} + t_{2\overline{2}}^1\eta^{\overline{2}} + t_{2\overline{3}}^1\eta^{\overline{3}},$$

$$\xi_1^2 = f_{11}^2\eta^1 + f_{12}^2\eta^2 + f_{13}^2\eta^3 + t_{1\overline{1}}^2\eta^{\overline{1}} + t_{1\overline{2}}^2\eta^{\overline{2}} + t_{1\overline{3}}^2\eta^{\overline{3}},$$

$$\xi_3^3 = f_{31}^3\eta^1 + f_{32}^3\eta^2 + f_{33}^3\eta^3 + t_{3\overline{1}}^3\eta^{\overline{1}} + t_{3\overline{2}}^3\eta^{\overline{2}} + t_{3\overline{3}}^3\eta^{\overline{3}},$$

for some functions $f, t \in C^\infty(B_2, \mathbb{C})$. Because ξ_1^1 and ξ_2^2 are $\mathbb{R}$-valued, $t_{j\overline{k}}^j = \overline{f}_{jk}^j$ for $j = 1, 2$. Though these coefficients are unknown, we discover relationships between them by differentiating the structure equations. First differentiate $\mathbf{i}d\eta^0$ and reduce modulo η^0.

$$0 = d(\mathbf{i}d\eta^0)$$

$$\equiv -2\xi_1^1 \wedge \eta^1 \wedge \eta^{\overline{1}} - (\xi_2^1 + \epsilon\xi_1^{\overline{2}}) \wedge \eta^2 \wedge \eta^{\overline{1}} - (\epsilon\xi_1^2 + \xi_2^{\overline{1}}) \wedge \eta^1 \wedge \eta^{\overline{2}} - 2\epsilon\xi_2^2 \wedge \eta^2 \wedge \eta^{\overline{2}}$$

$$\equiv (2f_{12}^1 - f_{21}^1 - \epsilon\overline{t}_{1\overline{1}}^2)\eta^2 \wedge \eta^1 \wedge \eta^{\overline{1}} + (2\overline{f}_{12}^1 - \overline{f}_{21}^1 - \epsilon t_{1\overline{1}}^2)\eta^{\overline{2}} \wedge \eta^1 \wedge \eta^{\overline{1}} + 2f_{13}^1\eta^3 \wedge \eta^1 \wedge \eta^{\overline{1}}$$

$$+ (2\epsilon f_{21}^2 - \epsilon f_{12}^2 - \overline{t}_{2\overline{2}}^1)\eta^1 \wedge \eta^2 \wedge \eta^{\overline{2}} + (2\epsilon\overline{f}_{21}^2 - \epsilon\overline{f}_{12}^2 - t_{2\overline{2}}^1)\eta^{\overline{1}} \wedge \eta^2 \wedge \eta^{\overline{2}} + 2\epsilon f_{23}^2\eta^3 \wedge \eta^2 \wedge \eta^{\overline{2}}$$

$$+ (f_{23}^1 + \epsilon\overline{t}_{1\overline{3}}^2)\eta^3 \wedge \eta^2 \wedge \eta^{\overline{1}} + (\overline{f}_{23}^1 + \epsilon t_{1\overline{3}}^2)\eta^{\overline{3}} \wedge \eta^1 \wedge \eta^{\overline{2}} + 2\overline{f}_{13}^1\eta^{\overline{3}} \wedge \eta^1 \wedge \eta^{\overline{1}}$$

$$+ (\epsilon f_{13}^2 + \overline{t}_{2\overline{3}}^1)\eta^3 \wedge \eta^1 \wedge \eta^{\overline{2}} + (\epsilon\overline{f}_{13}^2 + t_{2\overline{3}}^1)\eta^{\overline{3}} \wedge \eta^2 \wedge \eta^{\overline{1}} + 2\epsilon\overline{f}_{23}^2\eta^{\overline{3}} \wedge \eta^2 \wedge \eta^{\overline{2}}$$

$$\mod \{\eta^0\}.$$

Coefficients of independent three-forms vanish independently, so this has revealed six distinct vanishing conditions and their complex conjugates. For example, we now know that $f_{13}^1 = f_{23}^2 = 0$. We will see that these six equations allow us to simplify our apparent torsion tensor via absorption, but first we find five more equations by differentiating $d\eta^1$ and $d\eta^2$ and reducing modulo η^0, η^1, η^2.

$$0 = d(d\eta^1)$$

$$\equiv \epsilon(\xi_3^3 + \xi_2^2 - \xi_1^1) \wedge \eta^3 \wedge \eta^{\overline{2}} + (\epsilon\xi_1^{\overline{2}} - \xi_2^1) \wedge \eta^3 \wedge \eta^{\overline{1}}$$

$$\equiv (\epsilon\overline{f}_{21}^2 - \epsilon\overline{f}_{12}^2 - \epsilon\overline{f}_{11}^1 + \epsilon t_{3\overline{1}}^3 + t_{2\overline{2}}^1)\eta^{\overline{1}} \wedge \eta^3 \wedge \eta^{\overline{2}} + \epsilon t_{3\overline{3}}^3\eta^{\overline{3}} \wedge \eta^3 \wedge \eta^{\overline{2}} + (\epsilon\overline{f}_{13}^2 - t_{2\overline{3}}^1)\eta^{\overline{3}} \wedge \eta^3 \wedge \eta^{\overline{1}}$$

$$\mod \{\eta^0, \eta^1, \eta^2\},$$

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and similarly,

$$0 = \mathrm{d}(\mathrm{d}\eta^2)$$

$$\equiv (\xi_3^3 + \xi_1^1 - \xi_2^2) \wedge \eta^3 \wedge \eta^{\overline{1}} + (\xi_{\overline{2}}^{\overline{1}} - \epsilon\xi_1^2) \wedge \eta^3 \wedge \eta^{\overline{2}}$$

$$\equiv (\overline{f}_{12}^1 - \overline{f}_{22}^2 - \overline{f}_{21}^1 + t_{3\overline{2}}^3 + \epsilon t_{1\overline{1}}^2)\eta^{\overline{2}} \wedge \eta^3 \wedge \eta^{\overline{1}} + t_{3\overline{3}}^3 \eta^{\overline{3}} \wedge \eta^3 \wedge \eta^{\overline{1}} + (\overline{f}_{23}^1 - \epsilon t_{1\overline{3}}^2)\eta^{\overline{3}} \wedge \eta^3 \wedge \eta^{\overline{2}}$$

$$\mathrm{mod}\ \{\eta^0, \eta^1, \eta^2\}.$$

In addition to concluding that

$$f_{13}^1 = f_{23}^2 = t_{3\overline{3}}^3 = 0,$$

we have eight vanishing conditions. The first four

$$0 = f_{23}^1 + \epsilon \overline{t}_{1\overline{3}}^2,$$

$$0 = \overline{f}_{23}^1 - \epsilon t_{1\overline{3}}^2,$$

$$0 = \epsilon f_{13}^2 + \overline{t}_{2\overline{3}}^1,$$

$$0 = \epsilon \overline{f}_{13}^2 - t_{2\overline{3}}^1,$$

imply

$$f_{23}^1 = f_{13}^2 = \overline{t}_{2\overline{3}}^1 = \overline{t}_{1\overline{3}}^2 = 0,$$

while the latter four

$$0 = 2\epsilon f_{21}^2 - \epsilon f_{12}^2 - \overline{t}_{2\overline{2}}^1,$$

$$0 = 2f_{12}^1 - f_{21}^1 - \epsilon \overline{t}_{1\overline{1}}^2,$$

$$0 = \epsilon \overline{f}_{21}^2 - \epsilon \overline{f}_{12}^2 - \epsilon \overline{f}_{11}^1 + \epsilon t_{3\overline{1}}^3 + t_{2\overline{2}}^1,$$

$$0 = \overline{f}_{12}^1 - \overline{f}_{22}^2 - \overline{f}_{21}^1 + t_{3\overline{2}}^3 + \epsilon t_{1\overline{1}}^2,$$

(3.24)

will be useful for absorbing the remaining terms. The structure equations (3.22) may now be expanded to read

$$
\mathrm{d}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = - \begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\varrho & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\varsigma & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 - \beta_1 & \mathbf{i}\gamma^1 - \beta_2 & \mathbf{i}\varrho + \mathbf{i}\varsigma \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix}
$$

(3.25)

$$
+ \begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon \mathbf{i}\eta^2 \wedge \eta^{\overline{2}} \\ \epsilon\eta^3 \wedge \eta^{\overline{2}} + (f^1_{12}\eta^2 + \overline{f}^1_{11}\eta^{\overline{1}} + \overline{f}^1_{12}\eta^{\overline{2}}) \wedge \eta^1 + (f^1_{21}\eta^1 + t^1_{2\overline{1}}\eta^{\overline{1}} + t^1_{2\overline{2}}\eta^{\overline{2}}) \wedge \eta^2 \\ \eta^3 \wedge \eta^{\overline{1}} + (f^2_{12}\eta^2 + t^2_{1\overline{1}}\eta^{\overline{1}} + t^2_{1\overline{2}}\eta^{\overline{2}}) \wedge \eta^1 + (f^2_{21}\eta^1 + \overline{f}^2_{21}\eta^{\overline{1}} + \overline{f}^2_{22}\eta^{\overline{2}}) \wedge \eta^2 \\ (f^3_{31}\eta^1 + f^3_{32}\eta^2 + t^3_{3\overline{1}}\eta^{\overline{1}} + t^3_{3\overline{2}}\eta^{\overline{2}}) \wedge \eta^3 \end{bmatrix}.
$$

We will simplify notation by focusing only on those two-forms which are involved in each step of the absorption. For example, in the structure equation for $\mathrm{d}\eta^3$, we have

$$
\mathrm{d}\eta^3 = \beta_1 \wedge \eta^1 + \beta_2 \wedge \eta^2 + f^3_{31}\eta^1 \wedge \eta^3 + f^3_{32}\eta^2 \wedge \eta^3 + \dots
$$
$$
= (\beta_1 - f^3_{31}\eta^3) \wedge \eta^1 + (\beta_2 - f^3_{32}\eta^3) \wedge \eta^2 + \dots
$$

so we let $\hat{\beta}_1 = \beta_1 - f^3_{31}\eta^3$ and $\hat{\beta}_2 = \beta_2 - f^3_{32}\eta^3$ to absorb these terms. Now that they are gone, we drop the hats off of β_1, β_2, as we will need to modify them again when considering other terms. Many of the remaining absorbable terms will be absorbed into the diagonal pseudoconnection forms $\mathbf{i}\varrho$ and $\mathbf{i}\varsigma$. Note that we can only alter them by purely imaginary, semibasic one-forms. Before proceeding, we state that the result of our absorption will be that the apparent torsion tensor in (3.25) will be changed to

$$
\begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon \mathbf{i}\eta^2 \wedge \eta^{\overline{2}} \\ \epsilon\eta^3 \wedge \eta^{\overline{2}} + \epsilon(t^1_{2\overline{2}}\eta^{\overline{1}} + t^2_{1\overline{1}}\eta^{\overline{2}}) \wedge \eta^1 + (t^1_{2\overline{1}}\eta^{\overline{1}} + t^1_{2\overline{2}}\eta^{\overline{2}}) \wedge \eta^2 \\ \eta^3 \wedge \eta^{\overline{1}} + (t^2_{1\overline{1}}\eta^{\overline{1}} + t^2_{1\overline{2}}\eta^{\overline{2}}) \wedge \eta^1 + \epsilon(t^1_{2\overline{2}}\eta^{\overline{1}} + t^2_{1\overline{1}}\eta^{\overline{2}}) \wedge \eta^2 \\ 0 \end{bmatrix}.
$$

(3.26)

We will arrive at (3.26) in two steps – one for each of the apparent torsion coefficients $t^3_{3\overline{1}}$ and

$t^3_{3\bar{2}}$ that currently remain in the equation for $d\eta^3$ in (3.25). First consider

$$d\eta^3 = \beta_1 \wedge \eta^1 - (\mathbf{i}\varrho + \mathbf{i}\varsigma) \wedge \eta^3 + t^3_{3\bar{1}}\eta^{\bar{1}} \wedge \eta^3 + \dots$$
$$= (\beta_1 - \bar{t}^3_{3\bar{1}}\eta^3) \wedge \eta^1 - (\mathbf{i}\varrho + \mathbf{i}\varsigma - t^3_{3\bar{1}}\eta^{\bar{1}} + \bar{t}^3_{3\bar{1}}\eta^1) \wedge \eta^3 + \dots$$

Let $\hat{\beta}_1 := \beta_1 - \bar{t}^3_{3\bar{1}}\eta^3$. Note that if we choose any imaginary form $\zeta \in \Omega^1(B_2, \mathbf{i}\mathbb{R})$, and define

$$\mathbf{i}\hat{\varrho} := \mathbf{i}\varrho - \tfrac{1}{2}(t^3_{3\bar{1}}\eta^{\bar{1}} - \bar{t}^3_{3\bar{1}}\eta^1) + \zeta, \qquad \mathbf{i}\hat{\varsigma} := \mathbf{i}\varsigma - \tfrac{1}{2}(t^3_{3\bar{1}}\eta^{\bar{1}} - \bar{t}^3_{3\bar{1}}\eta^1) - \zeta, \qquad (3.27)$$

then we have successfully absorbed the $t^3_{3\bar{1}}$ term in the expression for $d\eta^3$. We will choose ζ so that we also absorb terms in the expressions for $d\eta^1, d\eta^2$. Let

$$\zeta := -\tfrac{1}{2}\left(\bar{f}^1_{11} - \bar{f}^2_{12} + \bar{f}^2_{21} - \epsilon t^1_{2\bar{2}}\right)\eta^{\bar{1}} + \tfrac{1}{2}\left(f^1_{11} - f^2_{12} + f^2_{21} - \epsilon \bar{t}^1_{2\bar{2}}\right)\eta^1.$$

By the third equation in (3.24),

$$t^3_{3\bar{1}}\eta^{\bar{1}} - \bar{t}^3_{3\bar{1}}\eta^1 = \left(-\bar{f}^2_{21} + \bar{f}^2_{12} + \bar{f}^1_{11} - \epsilon t^1_{2\bar{2}}\right)\eta^{\bar{1}} - \left(-f^2_{21} + f^2_{12} + f^1_{11} - \epsilon \bar{t}^1_{2\bar{2}}\right)\eta^1,$$

so in (3.27) we have

$$\mathbf{i}\hat{\varrho} = \mathbf{i}\varrho - \bar{f}^1_{11}\eta^{\bar{1}} + \epsilon t^1_{2\bar{2}}\eta^{\bar{1}} + f^1_{11}\eta^1 - \epsilon \bar{t}^1_{2\bar{2}}\eta^1, \qquad (3.28)$$
$$\mathbf{i}\hat{\varsigma} = \mathbf{i}\varsigma + \bar{f}^2_{21}\eta^{\bar{1}} - \bar{f}^2_{12}\eta^{\bar{1}} - f^2_{21}\eta^1 + f^2_{12}\eta^1. \qquad (3.29)$$

Now (3.28) shows

$$d\eta^1 = -\mathbf{i}\varrho \wedge \eta^1 + \bar{f}^1_{11}\eta^{\bar{1}} \wedge \eta^1 + \dots$$
$$= -(\mathbf{i}\varrho - \bar{f}^1_{11}\eta^{\bar{1}} + \epsilon t^1_{2\bar{2}}\eta^{\bar{1}} + f^1_{11}\eta^1 - \epsilon \bar{t}^1_{2\bar{2}}\eta^1) \wedge \eta^1 + \epsilon t^1_{2\bar{2}}\eta^{\bar{1}} \wedge \eta^1 + \dots$$
$$= -\mathbf{i}\hat{\varrho} \wedge \eta^1 + \epsilon t^1_{2\bar{2}}\eta^{\bar{1}} \wedge \eta^1 + \dots$$

On the other hand, by the first equation in (3.24) we can write (3.29) as

$$\mathbf{i}\hat{\varsigma} = \mathbf{i}\varsigma - \bar{f}^2_{21}\eta^{\bar{1}} + (2\bar{f}^2_{21} - \bar{f}^2_{12})\eta^{\bar{1}} - f^2_{21}\eta^1 + f^2_{12}\eta^1$$

$$= \mathbf{i}\varsigma - \overline{f}^2_{21}\eta^{\overline{1}} + \epsilon t^1_{2\overline{2}}\eta^{\overline{1}} - f^2_{21}\eta^1 + f^2_{12}\eta^1,$$

which shows

$$\mathrm{d}\eta^2 = -\mathbf{i}\varsigma \wedge \eta^2 + f^2_{12}\eta^2 \wedge \eta^1 + f^2_{21}\eta^1 \wedge \eta^2 + \overline{f}^2_{21}\eta^{\overline{1}} \wedge \eta^2 + \ldots$$

$$= -(\mathbf{i}\varsigma - \overline{f}^2_{21}\eta^{\overline{1}} + \epsilon t^1_{2\overline{2}}\eta^{\overline{1}} - f^2_{21}\eta^1 + f^2_{12}\eta^1) \wedge \eta^2 + \epsilon t^1_{2\overline{2}}\eta^{\overline{1}} \wedge \eta^2 + \ldots$$

$$= -\mathbf{i}\hat{\varsigma} \wedge \eta^2 + \epsilon t^1_{2\overline{2}}\eta^{\overline{1}} \wedge \eta^2 + \ldots$$

This concludes the first step of the absorption, by which we modified (3.25) to yield

$$\mathrm{d}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = - \begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\hat{\varrho} & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\hat{\varsigma} & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 - \hat{\beta}_1 & \mathbf{i}\gamma^1 - \beta_2 & \mathbf{i}\hat{\varrho} + \mathbf{i}\hat{\varsigma} \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon \mathbf{i}\eta^2 \wedge \eta^{\overline{2}} \\ \epsilon\eta^3 \wedge \eta^{\overline{2}} + (f^1_{12}\eta^2 + \epsilon t^1_{2\overline{2}}\eta^{\overline{1}} + \overline{f}^1_{12}\eta^{\overline{2}}) \wedge \eta^1 + (f^1_{21}\eta^1 + t^1_{2\overline{1}}\eta^{\overline{1}} + t^1_{2\overline{2}}\eta^{\overline{2}}) \wedge \eta^2 \\ \eta^3 \wedge \eta^{\overline{1}} + (t^2_{1\overline{1}}\eta^{\overline{1}} + t^2_{1\overline{2}}\eta^{\overline{2}}) \wedge \eta^1 + (\epsilon t^1_{2\overline{2}}\eta^{\overline{1}} + \overline{f}^2_{22}\eta^{\overline{2}}) \wedge \eta^2 \\ t^3_{3\overline{2}}\eta^{\overline{2}} \wedge \eta^3 \end{bmatrix}.$$

We begin round two by dropping the hats off the pseudoconnection forms. Round two will proceed analogously to round one, only this time we will use the two remaining vanishing conditions; i.e., the second and the last equations of (3.24). We have

$$\mathrm{d}\eta^3 = \beta_2 \wedge \eta^2 - (\mathbf{i}\varrho + \mathbf{i}\varsigma) \wedge \eta^3 + t^3_{3\overline{2}}\eta^{\overline{2}} \wedge \eta^3 + \ldots$$

$$= (\beta_2 - \overline{t}^3_{3\overline{2}}\eta^3) \wedge \eta^2 - (\mathbf{i}\varrho + \mathbf{i}\varsigma - t^3_{3\overline{2}}\eta^{\overline{2}} + \overline{t}^3_{3\overline{2}}\eta^2) \wedge \eta^3 + \ldots$$

so let $\hat{\beta}_2 = \beta_2 - \overline{t}^3_{3\overline{2}}\eta^3$. We'll look for a new semibasic $\zeta \in \Omega^1(B_2, \mathbf{i}\mathbb{R})$ to write

$$\mathbf{i}\hat{\varrho} := \mathbf{i}\varrho - \tfrac{1}{2}(t^3_{3\overline{2}}\eta^{\overline{2}} - \overline{t}^3_{3\overline{2}}\eta^2) + \zeta, \qquad \mathbf{i}\hat{\varsigma} := \mathbf{i}\varsigma - \tfrac{1}{2}(t^3_{3\overline{2}}\eta^{\overline{2}} - \overline{t}^3_{3\overline{2}}\eta^2) - \zeta, \qquad (3.30)$$

and use the fact that the final equation in (3.24) implies

$$t_{3\bar{2}}^3 \eta^{\bar{2}} - \bar{t}_{3\bar{2}}^3 \eta^2 = \left(-\bar{f}_{12}^1 + \bar{f}_{22}^2 + \bar{f}_{21}^1 - \epsilon t_{1\bar{1}}^2\right)\eta^{\bar{2}} - \left(-f_{12}^1 + f_{22}^2 + f_{21}^1 - \epsilon \bar{t}_{1\bar{1}}^2\right)\eta^2.$$

This time, define

$$\zeta := \tfrac{1}{2}\left(\bar{f}_{12}^1 + \bar{f}_{22}^2 - \bar{f}_{21}^1 - \epsilon t_{1\bar{1}}^2\right)\eta^{\bar{2}} - \tfrac{1}{2}\left(f_{12}^1 + f_{22}^2 - f_{21}^1 - \epsilon \bar{t}_{1\bar{1}}^2\right)\eta^2,$$

so that (3.30) reads

$$\mathbf{i}\hat{\varsigma} = \mathbf{i}\varsigma - \bar{f}_{22}^2 \eta^{\bar{2}} + \epsilon t_{1\bar{1}}^2 \eta^{\bar{2}} + f_{22}^2 \eta^2 - \epsilon \bar{t}_{1\bar{1}}^2 \eta^2,$$

$$\mathbf{i}\hat{\varrho} = \mathbf{i}\varrho + \bar{f}_{12}^1 \eta^{\bar{2}} - \bar{f}_{21}^1 \eta^{\bar{2}} - f_{12}^1 \eta^2 + f_{21}^1 \eta^2$$

$$= \mathbf{i}\varrho - \bar{f}_{12}^1 \eta^{\bar{2}} + \epsilon t_{1\bar{1}}^2 \eta^{\bar{2}} - f_{12}^1 \eta^2 + f_{21}^1 \eta^2,$$

where the last equality follows from the second equation in (3.24). As promised, we now have

$$\mathrm{d}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = -\begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\hat{\varrho} & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\hat{\varsigma} & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 - \beta_1 & \mathbf{i}\gamma^1 - \hat{\beta}_2 & \mathbf{i}\hat{\varrho} + \mathbf{i}\hat{\varsigma} \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix}$$

(3.31)

$$+ \begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\bar{1}} + \epsilon \mathbf{i}\eta^2 \wedge \eta^{\bar{2}} \\ \epsilon\eta^3 \wedge \eta^{\bar{2}} + \epsilon(t_{2\bar{2}}^1 \eta^{\bar{1}} + t_{1\bar{1}}^2 \eta^{\bar{2}}) \wedge \eta^1 + (t_{2\bar{1}}^1 \eta^{\bar{1}} + t_{2\bar{2}}^1 \eta^{\bar{2}}) \wedge \eta^2 \\ \eta^3 \wedge \eta^{\bar{1}} + (t_{1\bar{1}}^2 \eta^{\bar{1}} + t_{1\bar{2}}^2 \eta^{\bar{2}}) \wedge \eta^1 + \epsilon(t_{2\bar{2}}^1 \eta^{\bar{1}} + t_{1\bar{1}}^2 \eta^{\bar{2}}) \wedge \eta^2 \\ 0 \end{bmatrix}.$$

3.4   Last Two Reductions

After removing the hats from our pseudoconnection forms, we normalize some of the remaining torsion coefficients and reduce the structure group as before. To see how these functions vary in

the fiber, we first differentiate $\mathrm{d}\eta^1$ and reduce modulo η^0, η^1, η^3.

$$0 = \mathrm{d}(\mathrm{d}\eta^1)$$
$$\equiv (\mathrm{d}t^1_{2\bar{1}} - t^1_{2\bar{1}}(\tau - 2\mathbf{i}\varrho + \mathbf{i}\varsigma)) \wedge \eta^{\bar{1}} \wedge \eta^2 + (\mathrm{d}t^1_{2\bar{2}} - t^1_{2\bar{2}}(\tau - \mathbf{i}\varrho) - \epsilon\beta_2) \wedge \eta^{\bar{2}} \wedge \eta^2 \qquad \mathrm{mod} \ \{\eta^0, \eta^1, \eta^3\}.$$

Now differentiate $\mathrm{d}\eta^2$ and reduce modulo η^0, η^2, η^3.

$$0 = \mathrm{d}(\mathrm{d}\eta^2)$$
$$\equiv (\mathrm{d}t^2_{1\bar{1}} - t^2_{1\bar{1}}(\tau - \mathbf{i}\varsigma) - \beta_1) \wedge \eta^{\bar{1}} \wedge \eta^1 + (\mathrm{d}t^2_{1\bar{2}} - t^2_{1\bar{2}}(\tau + \mathbf{i}\varrho - 2\mathbf{i}\varsigma)) \wedge \eta^{\bar{2}} \wedge \eta^1 \qquad \mathrm{mod} \ \{\eta^0, \eta^2, \eta^3\}.$$

The two identities

$$\left. \begin{array}{c} \mathrm{d}t^1_{2\bar{2}} \equiv t^1_{2\bar{2}}(\tau - \mathbf{i}\varrho) + \epsilon\beta_2 \\[2em] \mathrm{d}t^2_{1\bar{1}} \equiv t^2_{1\bar{1}}(\tau - \mathbf{i}\varsigma) + \beta_1 \end{array} \right\} \qquad \mathrm{mod} \ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\bar{1}}, \eta^{\bar{2}}\} \qquad (3.32)$$

imply that there is a subbundle $B_3 \subset B_2$ of *3-adapted coframes* on which

$$t^1_{2\bar{2}} = t^2_{1\bar{1}} = 0.$$

Observe how (3.32) shows that when restricted to B_3, we have

$$\beta_1, \beta_2 \equiv 0 \quad \mathrm{mod} \ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\bar{1}}, \eta^{\bar{2}}\}. \qquad (3.33)$$

We fix a 3-adapted coframing $\theta_{\mathbb{1}}$ in order to locally trivialize B_3. An explicit parametrization of the structure group $G_3 \subset G_2$ of B_3 is found by taking $g^{-1} \in C^\infty(B_2, G_2)$ to be the matrix in (3.20) and solving in coordinates the differential equations $\beta_1 = 0$ and $\beta_2 = 0$ from the identity

$$g^{-1}\mathrm{d}g = \begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\varrho & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\varsigma & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 - \beta_1 & \mathbf{i}\gamma^1 - \beta_2 & \mathbf{i}\varrho + \mathbf{i}\varsigma \end{bmatrix}.$$

The result of this calculation is that G_3 is comprised of those matrices in G_2 which satisfy $b_1 = \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}r}c^2$ and $b_2 = \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}s}c^1$ so that we locally have $B_3 \cong G_3 \times M$ where G_3 is parametrized by

$$
\begin{bmatrix}
t^2 & 0 & 0 & 0 \\
c^1 & t\mathbf{e}^{\mathbf{i}r} & 0 & 0 \\
c^2 & 0 & t\mathbf{e}^{\mathbf{i}s} & 0 \\
c^3 & \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}r}c^2 & \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}s}c^1 & \mathbf{e}^{\mathbf{i}(r+s)}
\end{bmatrix}
\quad ; \qquad
r, s, 0 \neq t \in \mathbb{R}; c^j \in \mathbb{C}.
\qquad (3.34)
$$

If $\iota_3 : B_3 \hookrightarrow B_2$ is the inclusion map, then we let

$$
F^1 := \iota_3^* t^1_{2\overline{1}}, \qquad\qquad\qquad F^2 := \iota_3^* t^2_{1\overline{2}}.
$$

Aside from this relabeling, we maintain the names of every one-form that we pull back along ι_3, so that the structure equations are the same except that β_1, β_2 are now semibasic. Thus, on B_3 we have

$$
\mathrm{d}
\begin{bmatrix}
\eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3
\end{bmatrix}
= -
\begin{bmatrix}
2\tau & 0 & 0 & 0 \\
\gamma^1 & \tau + \mathbf{i}\varrho & 0 & 0 \\
\gamma^2 & 0 & \tau + \mathbf{i}\varsigma & 0 \\
\gamma^3 & \mathbf{i}\gamma^2 & \mathbf{i}\gamma^1 & \mathbf{i}\varrho + \mathbf{i}\varsigma
\end{bmatrix}
\wedge
\begin{bmatrix}
\eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3
\end{bmatrix}
+
\begin{bmatrix}
\mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}\eta^2 \wedge \eta^{\overline{2}} \\
\epsilon\eta^3 \wedge \eta^{\overline{2}} + F^1\eta^{\overline{1}} \wedge \eta^2 \\
\eta^3 \wedge \eta^{\overline{1}} + F^2\eta^{\overline{2}} \wedge \eta^1 \\
\beta_1 \wedge \eta^1 + \beta_2 \wedge \eta^2
\end{bmatrix}.
\qquad (3.35)
$$

We use (3.33) to expand β_1 and β_2, implicitly using that we can absorb η^0 coefficients into γ^3.

$$
\beta_1 = f_{11}\eta^1 + t_{1\overline{1}}\eta^{\overline{1}} + f_{12}\eta^2 + t_{1\overline{2}}\eta^{\overline{2}} + f_{13}\eta^3, \qquad \beta_2 = f_{21}\eta^1 + t_{2\overline{1}}\eta^{\overline{1}} + f_{22}\eta^2 + t_{2\overline{2}}\eta^{\overline{2}} + f_{23}\eta^3,
$$

for some new functions $f, t \in C^\infty(B_3, \mathbb{C})$.

We now seek to normalize $t_{1\overline{1}}$ and $t_{2\overline{2}}$ to zero. This will require us to collect a few identities. First differentiate $\mathrm{d}\eta^0$.

$$
0 = \mathrm{d}(\mathrm{d}\eta^0)
$$
$$
= (-2\mathrm{d}\tau + \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} - \mathbf{i}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\mathbf{i}\gamma^{\overline{2}} \wedge \eta^2) \wedge \eta^0,
$$

whence

$$2\mathrm{d}\tau \equiv \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} - \mathbf{i}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon \mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon \mathbf{i}\gamma^{\overline{2}} \wedge \eta^2 \mod \{\eta^0\}. \tag{3.36}$$

Now differentiate $\mathrm{d}\eta^1$.

$$0 = \mathrm{d}(\mathrm{d}\eta^1)$$
$$= (-\mathrm{d}\gamma^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 + F^1\gamma^{\overline{1}} \wedge \eta^2 - F^1\gamma^2 \wedge \eta^{\overline{1}} + \epsilon\gamma^3 \wedge \eta^{\overline{2}}) \wedge \eta^0$$
$$+ (-\mathrm{d}\tau - \mathbf{i}\mathrm{d}\varrho - \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} + \epsilon \mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^1 F^2 \eta^{\overline{2}} \wedge \eta^{\overline{1}} + |F^1|^2\eta^{\overline{2}} \wedge \eta^2) \wedge \eta^1 \tag{3.37}$$
$$+ (\mathrm{d}F^1 - F^1(\tau - 2\mathbf{i}\varrho + \mathbf{i}\varsigma) + \epsilon\overline{F}^2\eta^3) \wedge \eta^{\overline{1}} \wedge \eta^2 + \epsilon\beta_1 \wedge \eta^1 \wedge \eta^{\overline{2}} + \epsilon\beta_2 \wedge \eta^2 \wedge \eta^{\overline{2}}.$$

If we reduce this modulo $\eta^0, \eta^1, \eta^{\overline{1}}$, we see that $f_{23} = 0$ in the expansion of β_2. Furthermore, if we reduce modulo η^1, η^2, then by the top line we conclude

$$\mathrm{d}\gamma^1 \equiv (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 - F^1\gamma^2 \wedge \eta^{\overline{1}} + \epsilon\gamma^3 \wedge \eta^{\overline{2}} \mod \{\eta^0, \eta^1, \eta^2\}. \tag{3.38}$$

Next, differentiate $\mathrm{d}\eta^2$.

$$0 = \mathrm{d}(\mathrm{d}\eta^2)$$
$$= (-\mathrm{d}\gamma^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 + F^2\gamma^{\overline{2}} \wedge \eta^1 - F^2\gamma^1 \wedge \eta^{\overline{2}} + \gamma^3 \wedge \eta^{\overline{1}}) \wedge \eta^0$$
$$+ (-\mathrm{d}\tau - \mathbf{i}\mathrm{d}\varsigma - \epsilon \mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^2 F^1 \eta^{\overline{1}} \wedge \eta^{\overline{2}} + |F^2|^2\eta^{\overline{1}} \wedge \eta^1) \wedge \eta^2 \tag{3.39}$$
$$+ (\mathrm{d}F^2 - F^2(\tau + \mathbf{i}\varrho - 2\mathbf{i}\varsigma) + \overline{F}^1\eta^3) \wedge \eta^{\overline{2}} \wedge \eta^1 + \beta_1 \wedge \eta^1 \wedge \eta^{\overline{1}} + \beta_2 \wedge \eta^2 \wedge \eta^{\overline{1}}.$$

Reducing modulo $\eta^0, \eta^2, \eta^{\overline{2}}$ shows $f_{13} = 0$ in the expansion of β_1. Reducing mod η^1, η^2 then gives

$$\mathrm{d}\gamma^2 \equiv (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 - F^2\gamma^1 \wedge \eta^{\overline{2}} + \gamma^3 \wedge \eta^{\overline{1}} \mod \{\eta^0, \eta^1, \eta^2\}. \tag{3.40}$$

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Finally, we differentiate $\mathrm{d}\eta^3$.

$$
\begin{aligned}
0 &= \mathrm{d}(\mathrm{d}\eta^3) \\
&= -(\mathrm{d}\gamma^3 + \gamma^3 \wedge (2\tau - \mathbf{i}\varrho - \mathbf{i}\varsigma) + \gamma^1 \wedge \beta_1 + \gamma^2 \wedge \beta_2) \wedge \eta^0 \\
&\quad - \mathbf{i}(\mathrm{d}\gamma^2 + \gamma^2 \wedge (\tau - \mathbf{i}\varsigma) - F^2\gamma^1 \wedge \eta^{\overline{2}} + \gamma^3 \wedge \eta^{\overline{1}}) \wedge \eta^1 \\
&\quad - \mathbf{i}(\mathrm{d}\gamma^1 + \gamma^1 \wedge (\tau - \mathbf{i}\varrho) - F^1\gamma^2 \wedge \eta^{\overline{1}} + \epsilon\gamma^3 \wedge \eta^{\overline{2}}) \wedge \eta^2 \\
&\quad + (-\mathbf{i}\mathrm{d}\varrho - \mathbf{i}\mathrm{d}\varsigma - \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} - \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} + \epsilon\beta_1 \wedge \eta^{\overline{2}} + \beta_2 \wedge \eta^{\overline{1}}) \wedge \eta^3 \\
&\quad + (\mathrm{d}\beta_1 - (\tau - \mathbf{i}\varsigma) \wedge \beta_1 - F^2\beta_2 \wedge \eta^{\overline{2}}) \wedge \eta^1 + (\mathrm{d}\beta_2 - (\tau - \mathbf{i}\varrho) \wedge \beta_2 - F^1\beta_1 \wedge \eta^{\overline{1}}) \wedge \eta^2.
\end{aligned} \tag{3.41}
$$

For later use, we note that by reducing modulo η^0, η^1, η^2, we get

$$
\mathbf{i}\mathrm{d}\varrho + \mathbf{i}\mathrm{d}\varsigma \equiv -\epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} - \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} + \epsilon\beta_1 \wedge \eta^{\overline{2}} + \beta_2 \wedge \eta^{\overline{1}} \quad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3\}. \tag{3.42}
$$

Returning to the unreduced equation (3.41), if we reduce modulo η^0, η^1, η^3, plug in the identity for $\mathrm{d}\gamma^1$ from (3.38), and expand β_1 and β_2, then we have

$$
\begin{aligned}
0 &\equiv (\mathrm{d}t_{2\overline{1}} - 2t_{2\overline{1}}(\tau - \mathbf{i}\varrho) + 2\mathbf{i}F^1\gamma^2) \wedge \eta^{\overline{1}} \wedge \eta^2 - F^1 t_{1\overline{2}}\eta^{\overline{2}} \wedge \eta^{\overline{1}} \wedge \eta^2 \\
&\quad + (\mathrm{d}t_{2\overline{2}} - t_{2\overline{2}}(2\tau - \mathbf{i}\varrho - \mathbf{i}\varsigma) - \epsilon 2\mathbf{i}\gamma^3) \wedge \eta^{\overline{2}} \wedge \eta^2 \qquad\qquad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^3\}.
\end{aligned}
$$

If we instead reduce modulo η^0, η^2, η^3 and plug in $\mathrm{d}\gamma^2$ from (3.40), we see

$$
\begin{aligned}
0 &\equiv (\mathrm{d}t_{1\overline{1}} - t_{1\overline{1}}(2\tau - \mathbf{i}\varrho - \mathbf{i}\varsigma) - 2\mathbf{i}\gamma^3) \wedge \eta^{\overline{1}} \wedge \eta^1 - F^2 t_{2\overline{1}}\eta^{\overline{1}} \wedge \eta^{\overline{2}} \wedge \eta^1 \\
&\quad + (\mathrm{d}t_{1\overline{2}} - 2t_{1\overline{2}}(\tau - \mathbf{i}\varsigma) + 2\mathbf{i}F^2\gamma^1) \wedge \eta^{\overline{2}} \wedge \eta^1 \qquad\qquad \mathrm{mod}\ \{\eta^0, \eta^2, \eta^3\}.
\end{aligned}
$$

The two together show

$$
\left.
\begin{aligned}
\mathrm{d}t_{2\overline{2}} &\equiv t_{2\overline{2}}(2\tau - \mathbf{i}\varrho - \mathbf{i}\varsigma) + \epsilon 2\mathbf{i}\gamma^3 \\
\mathrm{d}t_{1\overline{1}} &\equiv t_{1\overline{1}}(2\tau - \mathbf{i}\varrho - \mathbf{i}\varsigma) + 2\mathbf{i}\gamma^3
\end{aligned}
\right\} \quad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}\}. \tag{3.43}
$$

These imply that we can find a subbundle where one of $t_{1\overline{1}}, t_{2\overline{2}}$ vanishes identically, but it is not yet clear that there are any coframings on which both vanish. To show this, we revisit the

equations (3.37),(3.39). For the former, we wedge the right side of the equation with η^2.

$$0 = (\mathrm{d}^2\eta^1) \wedge \eta^2$$

$$= (-\mathrm{d}\gamma^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 - F^1\gamma^2 \wedge \eta^{\overline{1}} + \epsilon\gamma^3 \wedge \eta^{\overline{2}}) \wedge \eta^0 \wedge \eta^2$$

$$+ (-\mathrm{d}\tau - \mathbf{i}\mathrm{d}\varrho - \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\eta^{\overline{3}} \wedge \eta^3) \wedge \eta^1 \wedge \eta^2$$

$$+ F^1 F^2 \eta^{\overline{2}} \wedge \eta^{\overline{1}} \wedge \eta^1 \wedge \eta^2 + \epsilon t_{1\overline{1}}\eta^{\overline{1}} \wedge \eta^1 \wedge \eta^{\overline{2}} \wedge \eta^2.$$

Similarly, wedge the right side of the identity for $\mathrm{d}(\mathrm{d}\eta^2)$ with η^1.

$$0 = (\mathrm{d}^2\eta^2) \wedge \eta^1$$

$$= (-\mathrm{d}\gamma^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 - F^2\gamma^1 \wedge \eta^{\overline{2}} + \gamma^3 \wedge \eta^{\overline{1}}) \wedge \eta^0 \wedge \eta^1$$

$$+ (-\mathrm{d}\tau - \mathbf{i}\mathrm{d}\varsigma - \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} + \epsilon\eta^{\overline{3}} \wedge \eta^3) \wedge \eta^2 \wedge \eta^1$$

$$+ F^2 F^1 \eta^{\overline{1}} \wedge \eta^{\overline{2}} \wedge \eta^2 \wedge \eta^1 + t_{2\overline{2}}\eta^{\overline{2}} \wedge \eta^2 \wedge \eta^{\overline{1}} \wedge \eta^1.$$

Now subtract the latter from the former, reduce modulo η^0, η^3, and plug in $2\mathrm{d}\tau$ and $\mathbf{i}\mathrm{d}\varrho + \mathbf{i}\mathrm{d}\varsigma$ from (3.36) and (3.42).

$$0 = (\mathrm{d}^2\eta^1) \wedge \eta^2 - (\mathrm{d}^2\eta^2) \wedge \eta^1$$

$$\equiv -(2\mathrm{d}\tau + \mathbf{i}\mathrm{d}\varrho + \mathbf{i}\mathrm{d}\varsigma) \wedge \eta^1 \wedge \eta^2 + (\epsilon t_{1\overline{1}} - t_{2\overline{2}})\eta^{\overline{2}} \wedge \eta^2 \wedge \eta^{\overline{1}} \wedge \eta^1 \qquad \mathrm{mod}\ \{\eta^0, \eta^3\}$$

$$\equiv 2(\epsilon t_{1\overline{1}} - t_{2\overline{2}})\eta^{\overline{2}} \wedge \eta^2 \wedge \eta^{\overline{1}} \wedge \eta^1 \qquad \mathrm{mod}\ \{\eta^0, \eta^3\}.$$

Thus we see that $\epsilon t_{1\overline{1}} = t_{2\overline{2}}$, and by (3.43) there exists a subbundle $B_4 \subset B_3$ of *4-adapted coframes* on which $t_{1\overline{1}} = t_{2\overline{2}} = 0$. We also see from (3.43) that when restricted to B_4,

$$\gamma^3 \equiv 0 \quad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}\}. \tag{3.44}$$

Fix a new 4-adapted coframing $\theta_{\overline{1}}$ in order to locally trivialize B_4. As with G_3, we seek a parametrization of the structure group $G_4 \subset G_3$ of B_4 by taking $g^{-1} \in C^\infty(B_3, G_3)$ to be the

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matrix (3.34) and solving the differential equation $\gamma^3 = 0$ in

$$
g^{-1}\mathrm{d}g = \begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\varrho & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\varsigma & 0 \\ \gamma^3 & \mathbf{i}\gamma^2 & \mathbf{i}\gamma^1 & \mathbf{i}\varrho + \mathbf{i}\varsigma \end{bmatrix}.
$$

The result is that we locally have $B_4 \cong G_4 \times M$ where G_4 is all matrices of the form

$$
\begin{bmatrix} t^2 & 0 & 0 & 0 \\ c^1 & t\mathbf{e}^{\mathbf{i}r} & 0 & 0 \\ c^2 & 0 & t\mathbf{e}^{\mathbf{i}s} & 0 \\ \frac{\mathbf{i}}{t^2}c^1c^2 & \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}r}c^2 & \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}s}c^1 & \mathbf{e}^{\mathbf{i}(r+s)} \end{bmatrix} ; \qquad r, s, 0 \neq t \in \mathbb{R}; c^1, c^2 \in \mathbb{C}. \tag{3.45}
$$

Pulling back along $\iota_4 : B_4 \hookrightarrow B_3$, we keep the names of all the forms, and relabel

$$
T^3 := \iota_4^*(f_{21} - f_{12}), \qquad\qquad F_1^3 := \iota_4^* t_{1\overline{2}}, \qquad\qquad F_2^3 := \iota_4^* t_{2\overline{1}},
$$

so that the structure equations (3.35) pull back to

$$
\mathrm{d}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = -\begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\varrho & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\varsigma & 0 \\ 0 & \mathbf{i}\gamma^2 & \mathbf{i}\gamma^1 & \mathbf{i}\varrho + \mathbf{i}\varsigma \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}\eta^2 \wedge \eta^{\overline{2}} \\ \epsilon\eta^3 \wedge \eta^{\overline{2}} + F^1\eta^{\overline{1}} \wedge \eta^2 \\ \eta^3 \wedge \eta^{\overline{1}} + F^2\eta^{\overline{2}} \wedge \eta^1 \\ -\gamma^3 \wedge \eta^0 + T^3\eta^1 \wedge \eta^2 + F_1^3\eta^{\overline{2}} \wedge \eta^1 + F_2^3\eta^{\overline{1}} \wedge \eta^2 \end{bmatrix}. \tag{3.46}
$$

We absorb the real part of T^3 as follows. As in §3.3, we focus only on the relevant two-forms.

$$
\mathrm{d}\eta^3 = -\mathbf{i}\gamma^2 \wedge \eta^1 - \mathbf{i}\gamma^1 \wedge \eta^2 - (\mathbf{i}\varrho + \mathbf{i}\varsigma) \wedge \eta^3 + T^3\eta^1 \wedge \eta^2 + \ldots
$$

$$
= -\mathbf{i}(\gamma^2 - \mathbf{i}\tfrac{1}{2}\mathrm{Re}T^3\eta^2) \wedge \eta^1 - \mathbf{i}(\gamma^1 + \mathbf{i}\tfrac{1}{2}\mathrm{Re}T^3\eta^1) \wedge \eta^2
$$

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$$- (\mathbf{i}\varrho + \mathbf{i}\tfrac{1}{2}\mathrm{Re}T^3\eta^0 + \mathbf{i}\varsigma - \mathbf{i}\tfrac{1}{2}\mathrm{Re}T^3\eta^0) \wedge \eta^3 + \mathbf{i}\mathrm{Im}T^3\eta^1 \wedge \eta^2 + \ldots$$

so let

$$\mathbf{i}\hat{\varrho} := \mathbf{i}\varrho + \mathbf{i}\tfrac{1}{2}\mathrm{Re}T^3\eta^0, \quad \mathbf{i}\hat{\varsigma} := \mathbf{i}\varsigma - \mathbf{i}\tfrac{1}{2}\mathrm{Re}T^3\eta^0, \quad \hat{\gamma}^1 := \gamma^1 + \mathbf{i}\tfrac{1}{2}\mathrm{Re}T^3\eta^1, \quad \hat{\gamma}^2 := \gamma^2 - \mathbf{i}\tfrac{1}{2}\mathrm{Re}T^3\eta^2,$$

and note that these choices leave the structure equations for $\mathrm{d}\eta^1, \mathrm{d}\eta^2$ unaltered. We drop the hats as we prepare to absorb new torsion introduced by the pullback along ι_4 of γ^3. According to (3.44), we expand

$$\gamma^3 = -f_0^3\eta^0 - f_1^3\eta^1 - T_{\overline{1}}^3\eta^{\overline{1}} - f_2^3\eta^2 - T_{\overline{2}}^3\eta^{\overline{2}} - f_3^3\eta^3,$$

for some functions $f, T \in C^\infty(B_4, \mathbb{C})$. We absorb the f_1^3 and f_2^3 terms via

$$\mathbf{i}\hat{\gamma}^2 := \mathbf{i}\gamma^2 - f_1^3\eta^0, \qquad\qquad\qquad \mathbf{i}\hat{\gamma}^1 := \mathbf{i}\gamma^1 - f_2^3\eta^0.$$

Now drop the hats for one final absorption – the imaginary part of f_3^3 – which will proceed in a similar manner to how we treated the real part of T^3 above. Notably, we modify forms so that the equations for $\mathrm{d}\eta^1, \mathrm{d}\eta^2$ remain unaffected. We have

$$\begin{aligned}
\mathrm{d}\eta^3 &= -\mathbf{i}\gamma^2 \wedge \eta^1 - \mathbf{i}\gamma^1 \wedge \eta^2 - (\mathbf{i}\varrho + \mathbf{i}\varsigma) \wedge \eta^3 + f_3^3\eta^3 \wedge \eta^0 + \ldots \\
&= -\mathbf{i}(\gamma^2 + \mathbf{i}\tfrac{1}{2}\mathrm{Im}(f_3^3)\eta^2) \wedge \eta^1 - \mathbf{i}(\gamma^1 + \mathbf{i}\tfrac{1}{2}\mathrm{Im}(f_3^3)\eta^1) \wedge \eta^2 \\
&\quad - (\mathbf{i}\varrho + \mathbf{i}\varsigma + \mathbf{i}\mathrm{Im}(f_3^3)\eta^0) \wedge \eta^3 + \mathrm{Re}(f_3^3)\eta^3 \wedge \eta^0 \ldots,
\end{aligned}$$

so we define

$$\mathbf{i}\hat{\varrho} := \mathbf{i}\varrho + \mathbf{i}\tfrac{1}{2}\mathrm{Im}(f_3^3)\eta^0, \quad \mathbf{i}\hat{\varsigma} := \mathbf{i}\varsigma + \mathbf{i}\tfrac{1}{2}\mathrm{Im}(f_3^3)\eta^0, \quad \hat{\gamma}^1 := \gamma^1 + \mathbf{i}\tfrac{1}{2}\mathrm{Im}(f_3^3)\eta^1, \quad \hat{\gamma}^2 := \gamma^2 + \mathbf{i}\tfrac{1}{2}\mathrm{Im}(f_3^3)\eta^2.$$

Let us drop the hats and rename

$$f^3 := \mathrm{Re}(f_3^3), \qquad\qquad\qquad \mathbf{i}t^3 := \mathbf{i}\mathrm{Im}T^3.$$

By arranging for these torsion coefficients to be purely real and imaginary, we have exhausted the ambiguity in the pseudoconnection forms $\gamma^1, \gamma^2, \mathbf{i}\varrho, \mathbf{i}\varsigma \in \Omega^1(B_4, \mathbb{C})$ which is associated with Lie-algebra compatible additions of semibasic, $\mathbf{i}\mathbb{R}$-valued forms to $\mathbf{i}\varrho$ and $\mathbf{i}\varsigma$. In particular, $\mathbf{i}\varrho$ and $\mathbf{i}\varsigma$ are now completely and intrinsically determined by our choices of torsion normalization, manifested in the structure equations

$$
\mathrm{d}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = -\begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\varrho & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\varsigma & 0 \\ 0 & \mathbf{i}\gamma^2 & \mathbf{i}\gamma^1 & \mathbf{i}\varrho + \mathbf{i}\varsigma \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon \mathbf{i}\eta^2 \wedge \eta^{\overline{2}} \\ \epsilon\eta^3 \wedge \eta^{\overline{2}} + F^1\eta^{\overline{1}} \wedge \eta^2 \\ \eta^3 \wedge \eta^{\overline{1}} + F^2\eta^{\overline{2}} \wedge \eta^1 \\ f^3\eta^3 \wedge \eta^0 + \mathbf{i}t^3\eta^1 \wedge \eta^2 + T^3_{\overline{1}}\eta^{\overline{1}} \wedge \eta^0 + T^3_{\overline{2}}\eta^{\overline{2}} \wedge \eta^0 + F^3_1\eta^{\overline{2}} \wedge \eta^1 + F^3_2\eta^{\overline{1}} \wedge \eta^2 \end{bmatrix}.
$$

(3.47)

In contrast to $\mathbf{i}\varrho$ and $\mathbf{i}\varsigma$, the pseudoconnection forms τ, γ^1, and γ^2 are not uniquely determined by the structure equations (3.47), as they are only determined up to permissible additions of semibasic, $\mathbb{R}$-valued one-forms to τ. Specifically, these structure equations are unaltered if we replace

$$
\begin{bmatrix} \hat{\tau} \\ \hat{\gamma}^1 \\ \hat{\gamma}^2 \end{bmatrix} := \begin{bmatrix} \tau \\ \gamma^1 \\ \gamma^2 \end{bmatrix} + \begin{bmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \end{bmatrix} ; \ y \in C^\infty(B_4, \mathbb{R}).
$$

(3.48)

The new variable y fully parameterizes the remaining ambiguity in our pseudoconnection forms; i.e., adding any other combination of semibasic forms to τ, γ^1, γ^2 will not preserve the structure equations.

3.5   Prolongation

The collection of all choices (3.48) of $\hat{\tau}, \hat{\gamma}^1, \hat{\gamma}^2$ preserving (3.47) defines an affine, real line bundle $\hat{\pi} : B_4^{(1)} \to B_4$ with y as a fiber coordinate. $B_4^{(1)}$ is the *prolongation* of our G_4-structure $\underline{\pi} : B_4 \to M$, and may be interpreted as the bundle of coframes on B_4 which are adapted to the structure

equations, so that we are essentially starting over the method of equivalence. We commit our usual notational abuse of recycling names as we recursively define the following global, tautological one-forms on $B_4^{(1)}$.

$$
\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \varrho \\ \varsigma \\ \tau \\ \gamma^1 \\ \gamma^2 \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \hat{\pi}^* \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \varrho \\ \varsigma \\ \tau \\ \gamma^1 \\ \gamma^2 \end{bmatrix}. \tag{3.49}
$$

These four $\mathbb{R}$-valued forms, along with the real and imaginary parts of these five $\mathbb{C}$-valued forms, are one real dimension shy of a full, global coframing of $B_4^{(1)}$. As usual, we find the missing one-form by differentiating the tautological forms and normalizing torsion until the resulting pseudoconnection form is uniquely (hence, globally) defined. From (3.49) we see that if we maintain the names of our torsion coefficients after pulling back along $\hat{\pi}$, the structure equations (3.47) still hold on $B_4^{(1)}$:

$$
\begin{aligned}
d\eta^0 &= -2\tau \wedge \eta^0 + \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon \mathbf{i}\eta^2 \wedge \eta^{\overline{2}}, \\
d\eta^1 &= -\gamma^1 \wedge \eta^0 - (\tau + \mathbf{i}\varrho) \wedge \eta^1 + \epsilon\eta^3 \wedge \eta^{\overline{2}} + F^1\eta^{\overline{1}} \wedge \eta^2, \\
d\eta^2 &= -\gamma^2 \wedge \eta^0 - (\tau + \mathbf{i}\varsigma) \wedge \eta^2 + \eta^3 \wedge \eta^{\overline{1}} + F^2\eta^{\overline{2}} \wedge \eta^1, \\
d\eta^3 &= -\mathbf{i}\gamma^2 \wedge \eta^1 - \mathbf{i}\gamma^1 \wedge \eta^2 - (\mathbf{i}\varrho + \mathbf{i}\varsigma) \wedge \eta^3 + f^3\eta^3 \wedge \eta^0 + \mathbf{i}t^3\eta^1 \wedge \eta^2 \\
&\quad + T_{\overline{1}}^3\eta^{\overline{1}} \wedge \eta^0 + T_{\overline{2}}^3\eta^{\overline{2}} \wedge \eta^0 + F_1^3\eta^{\overline{2}} \wedge \eta^1 + F_2^3\eta^{\overline{1}} \wedge \eta^2.
\end{aligned} \tag{3.50}
$$

For the remaining tautological forms, we have in analogy with (3.8),

$$
\mathrm{d}
\begin{bmatrix}
\mathbf{i}\varrho \\
\mathbf{i}\varsigma \\
\tau \\
\gamma^1 \\
\gamma^2
\end{bmatrix}
= -
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \psi & 0 & 0 \\
0 & 0 & 0 & \psi & 0 \\
0 & 0 & 0 & 0 & \psi
\end{bmatrix}
\wedge
\begin{bmatrix}
0 \\
0 \\
\eta^0 \\
\eta^1 \\
\eta^2
\end{bmatrix}
+
\begin{bmatrix}
\Xi^\varrho \\
\Xi^\varsigma \\
\Xi^\tau \\
\Xi^1 \\
\Xi^2
\end{bmatrix} ,
\tag{3.51}
$$

where $\psi \in \Omega^1(B_4^{(1)})$ is our new pseudoconnection form and the $\Xi \in \Omega^2(B_4^{(1)}, \mathbb{C})$ are $\hat{\pi}$-semibasic, apparent torsion two-forms. As always, we discover explicit expressions for our Ξ's by differentiating the known structure equations (3.50). Differentiating the equation for $\mathrm{d}\eta^0$ yields something familiar:

$$
0 = \mathrm{d}(\mathrm{d}\eta^0)
$$
$$
= (-2\mathrm{d}\tau + \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} - \mathbf{i}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\mathbf{i}\gamma^{\overline{2}} \wedge \eta^2) \wedge \eta^0,
$$

whence we conclude

$$
2\mathrm{d}\tau = \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} - \mathbf{i}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\mathbf{i}\gamma^{\overline{2}} \wedge \eta^2 + 2\zeta_0 \wedge \eta^0,
\tag{3.52}
$$

for some $\mathbb{R}$-valued $\zeta_0 \in \Omega^1(B_4^{(1)})$. Using the equation for $\mathrm{d}\eta^1$, we find

$$
0 = \mathrm{d}(\mathrm{d}\eta^1)
$$
$$
= (-\mathrm{d}\gamma^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 + F^1\gamma^{\overline{1}} \wedge \eta^2 - F^1\gamma^2 \wedge \eta^{\overline{1}} - \epsilon T_{\overline{1}}^3 \eta^{\overline{1}} \wedge \eta^{\overline{2}} - \epsilon f^3 \eta^3 \wedge \eta^{\overline{2}}) \wedge \eta^0
$$
$$
+ (-\mathrm{d}\tau - \mathbf{i}\mathrm{d}\varrho - \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\eta^{\overline{3}} \wedge \eta^3 + \epsilon\mathbf{i}t^3\eta^2 \wedge \eta^{\overline{2}} + F^1 F^2 \eta^{\overline{2}} \wedge \eta^{\overline{1}} + |F^1|^2 \eta^{\overline{2}} \wedge \eta^2) \wedge \eta^1
$$
$$
+ (\mathrm{d}F^1 - F^1(\tau - 2\mathbf{i}\varrho + \mathbf{i}\varsigma) + \epsilon\overline{F}^2 \eta^3 + \epsilon F_2^3 \eta^{\overline{2}}) \wedge \eta^{\overline{1}} \wedge \eta^2,
$$

which by Cartan's lemma yields

$$
\begin{bmatrix}
-\mathrm{d}\gamma^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 + F^1\gamma^{\overline{1}} \wedge \eta^2 - F^1\gamma^2 \wedge \eta^{\overline{1}} - \epsilon T_1^3\eta^{\overline{1}} \wedge \eta^{\overline{2}} - \epsilon f^3\eta^3 \wedge \eta^{\overline{2}} \\
-\mathrm{d}\tau - \mathbf{i}\mathrm{d}\varrho - \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\eta^{\overline{3}} \wedge \eta^3 + \epsilon\mathbf{i}t^3\eta^2 \wedge \eta^{\overline{2}} + F^1 F^2\eta^{\overline{2}} \wedge \eta^{\overline{1}} + |F^1|^2\eta^{\overline{2}} \wedge \eta^2 \\
(\mathrm{d}F^1 - F^1(\tau - 2\mathbf{i}\varrho + \mathbf{i}\varsigma) + \epsilon\overline{F}^2\eta^3 + \epsilon F_2^3\eta^{\overline{2}}) \wedge \eta^{\overline{1}}
\end{bmatrix}
$$

$$
= -
\begin{bmatrix}
\zeta_0^1 & \zeta_1^1 & \xi_2^1 \\
\zeta_1^1 & \zeta_1^\varrho & \xi_2^\varrho \\
\xi_2^1 & \xi_2^\varrho & \zeta^1
\end{bmatrix}
\wedge
\begin{bmatrix}
\eta^0 \\
\eta^1 \\
\eta^2
\end{bmatrix},
$$
(3.53)

for some $\xi, \zeta \in \Omega^1(B_4^{(1)}, \mathbb{C})$. Plugging this back into the same equation $0 = \mathrm{d}(\mathrm{d}\eta^1)$ reduced by $\eta^{\overline{1}}$ shows

$$
0 \equiv \xi_2^1 \wedge \eta^2 \wedge \eta^0 + \xi_2^\varrho \wedge \eta^2 \wedge \eta^1 \qquad \mathrm{mod}\ \{\eta^{\overline{1}}\}
$$
$$
\Rightarrow 0 \equiv \xi_2^1, \xi_2^\varrho \qquad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^{\overline{1}}\}.
$$
(3.54)

Moving on to $\mathrm{d}\eta^2$,

$$
0 = \mathrm{d}(\mathrm{d}\eta^2)
$$
$$
= (-\mathrm{d}\gamma^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 - F^2\gamma^1 \wedge \eta^{\overline{2}} + F^2\gamma^{\overline{2}} \wedge \eta^1 - T_2^3\eta^{\overline{2}} \wedge \eta^{\overline{1}} - f^3\eta^3 \wedge \eta^{\overline{1}}) \wedge \eta^0
$$
$$
+ (-\mathrm{d}\tau - \mathbf{i}\mathrm{d}\varsigma + \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} - \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\eta^{\overline{3}} \wedge \eta^3 - \mathbf{i}t^3\eta^1 \wedge \eta^{\overline{1}} + F^2 F^1\eta^{\overline{1}} \wedge \eta^{\overline{2}} + |F^2|^2\eta^{\overline{1}} \wedge \eta^1) \wedge \eta^2
$$
$$
+ (\mathrm{d}F^2 - F^2(\tau + \mathbf{i}\varrho - 2\mathbf{i}\varsigma) + \overline{F}^1\eta^3 + F_1^3\eta^{\overline{1}}) \wedge \eta^{\overline{2}} \wedge \eta^1.
$$

By the same argument,

$$
\begin{bmatrix}
-\mathrm{d}\gamma^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 - F^2\gamma^1 \wedge \eta^{\overline{2}} + F^2\gamma^{\overline{2}} \wedge \eta^1 - T_2^3\eta^{\overline{2}} \wedge \eta^{\overline{1}} - f^3\eta^3 \wedge \eta^{\overline{1}} \\
-\mathrm{d}\tau - \mathbf{i}\mathrm{d}\varsigma + \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} - \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\eta^{\overline{3}} \wedge \eta^3 - \mathbf{i}t^3\eta^1 \wedge \eta^{\overline{1}} + F^2 F^1\eta^{\overline{1}} \wedge \eta^{\overline{2}} + |F^2|^2\eta^{\overline{1}} \wedge \eta^1 \\
(\mathrm{d}F^2 - F^2(\tau + \mathbf{i}\varrho - 2\mathbf{i}\varsigma) + \overline{F}^1\eta^3 + F_1^3\eta^{\overline{1}}) \wedge \eta^{\overline{2}}
\end{bmatrix}
$$

$$
= -
\begin{bmatrix}
\zeta_0^2 & \zeta_2^2 & \xi_1^2 \\
\zeta_2^2 & \zeta_2^\varsigma & \xi_1^\varsigma \\
\xi_1^2 & \xi_1^\varsigma & \zeta^2
\end{bmatrix}
\wedge
\begin{bmatrix}
\eta^0 \\
\eta^2 \\
\eta^1
\end{bmatrix},
$$
(3.55)

for more, yet-unknown $\xi, \zeta \in \Omega^1(B_4^{(1)}, \mathbb{C})$ which satisfy

$$0 \equiv \xi_1^2 \wedge \eta^1 \wedge \eta^0 + \xi_1^\varsigma \wedge \eta^1 \wedge \eta^2 \qquad \mod \{\eta^{\overline{2}}\}$$

$$\Rightarrow 0 \equiv \xi_1^2, \xi_1^\varsigma \qquad \mod \{\eta^0, \eta^1, \eta^2, \eta^{\overline{2}}\}. \tag{3.56}$$

From (3.52),(3.53), and (3.55) we have gleaned

$$d\tau = \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \zeta_0 \wedge \eta^0,$$

$$\mathbf{i}d\varrho = -\tfrac{3\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} + \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + \epsilon \mathbf{i}t^3\eta^2 \wedge \eta^{\overline{2}}$$
$$+ F^1 F^2 \eta^{\overline{2}} \wedge \eta^{\overline{1}} + |F^1|^2 \eta^{\overline{2}} \wedge \eta^2 + (\zeta_1^1 - \zeta_0) \wedge \eta^0 + \zeta_1^\varrho \wedge \eta^1 + \xi_2^\varrho \wedge \eta^2,$$

$$\mathbf{i}d\varsigma = \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} + \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 - \mathbf{i}t^3\eta^1 \wedge \eta^{\overline{1}}$$
$$+ F^2 F^1 \eta^{\overline{1}} \wedge \eta^{\overline{2}} + |F^2|^2 \eta^{\overline{1}} \wedge \eta^1 + (\zeta_2^2 - \zeta_0) \wedge \eta^0 + \zeta_2^\varsigma \wedge \eta^2 + \xi_1^\varsigma \wedge \eta^1, \tag{3.57}$$

$$d\gamma^1 = (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 + F^1\gamma^{\overline{1}} \wedge \eta^2 - F^1\gamma^2 \wedge \eta^{\overline{1}} - \epsilon T_1^3 \eta^{\overline{1}} \wedge \eta^{\overline{2}} - \epsilon f^3 \eta^3 \wedge \eta^{\overline{2}}$$
$$+ \zeta_0^1 \wedge \eta^0 + \zeta_1^1 \wedge \eta^1 + \xi_2^1 \wedge \eta^2,$$

$$d\gamma^2 = (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 - F^2\gamma^1 \wedge \eta^{\overline{2}} + F^2\gamma^{\overline{2}} \wedge \eta^1 - T_2^3 \eta^{\overline{2}} \wedge \eta^{\overline{1}} - f^3 \eta^3 \wedge \eta^{\overline{1}}$$
$$+ \zeta_0^2 \wedge \eta^0 + \zeta_2^2 \wedge \eta^2 + \xi_1^2 \wedge \eta^1.$$

We learn a bit more about the ξ's and ζ's by differentiating the final equation from (3.50).

$$0 = d(d\eta^3)$$
$$= \mathbf{i}(-d\gamma^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 + F^2\gamma^1 \wedge \eta^{\overline{2}} + T_2^3 \eta^{\overline{2}} \wedge \eta^{\overline{1}}) \wedge \eta^1$$
$$+ \mathbf{i}(-d\gamma^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 + F^1\gamma^2 \wedge \eta^{\overline{1}} + \epsilon T_1^3 \eta^{\overline{1}} \wedge \eta^{\overline{2}}) \wedge \eta^2$$
$$+ (-\mathbf{i}d\varrho - \mathbf{i}d\varsigma - \mathbf{i}\gamma^1 \wedge \eta^{\overline{1}} - \epsilon\mathbf{i}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\mathbf{i}(t^3 + f^3)\eta^{\overline{2}} \wedge \eta^2 + \mathbf{i}(t^3 - f^3)\eta^1 \wedge \eta^{\overline{1}}) \wedge \eta^3$$
$$+ (F_2^3\gamma^{\overline{1}} + \mathbf{i}(t^3 - f^3)\gamma^1 + \epsilon T_1^3 \eta^{\overline{3}}) \wedge \eta^2 \wedge \eta^0$$
$$+ (F_1^3\gamma^{\overline{2}} - \mathbf{i}(t^3 + f^3)\gamma^2 + T_2^3 \eta^{\overline{3}}) \wedge \eta^1 \wedge \eta^0$$

$$+ (\mathrm{d}T_{\bar{1}}^3 - T_{\bar{1}}^3(3\tau - 2\mathbf{i}\varrho - \mathbf{i}\varsigma) - F_2^3\gamma^2 + (T_{\bar{2}}^3\overline{F}^2 - f^3F_2^3)\eta^2) \wedge \eta^{\bar{1}} \wedge \eta^0$$

$$+ (\mathrm{d}T_{\bar{2}}^3 - T_{\bar{2}}^3(3\tau - \mathbf{i}\varrho - 2\mathbf{i}\varsigma) - F_1^3\gamma^1 + (T_{\bar{1}}^3\overline{F}^1 - f^3F_1^3)\eta^1) \wedge \eta^{\bar{2}} \wedge \eta^0$$

$$+ (\mathrm{d}F_1^3 - 2F_1^3(\tau - \mathbf{i}\varsigma) - F_2^3\overline{F}^1\eta^2 - F_2^3F^2\eta^{\bar{1}}) \wedge \eta^{\bar{2}} \wedge \eta^1$$

$$+ (\mathrm{d}F_2^3 - 2F_2^3(\tau - \mathbf{i}\varrho) - F_1^3\overline{F}^2\eta^1 - F_1^3F^1\eta^{\bar{2}}) \wedge \eta^{\bar{1}} \wedge \eta^2$$

$$+ \mathbf{i}(\mathrm{d}t^3 - 2t^3\tau + f^3t^3\eta^0) \wedge \eta^1 \wedge \eta^2 + (\mathrm{d}f^3 - 2f^3\tau) \wedge \eta^3 \wedge \eta^0.$$

After plugging in (3.57), this becomes

$$0 = (\zeta_1^\varrho + \xi_1^\varsigma + 2\mathbf{i}\gamma^{\bar{1}} + (|F^2|^2 + 2\mathbf{i}(t^3 - f^3))\eta^{\bar{1}}) \wedge \eta^3 \wedge \eta^1$$

$$+ (\xi_2^\varrho + \zeta_2^\varsigma + \epsilon 2\mathbf{i}\gamma^{\bar{2}} + (|F^1|^2 - \epsilon 2\mathbf{i}(t^3 + f^3))\eta^{\bar{2}}) \wedge \eta^3 \wedge \eta^2$$

$$+ (\mathbf{i}\zeta_0^1 + F_2^3\gamma^{\bar{1}} + \mathbf{i}(t^3 - f^3)\gamma^1 + \epsilon T_{\bar{1}}^3\eta^{\bar{3}}) \wedge \eta^2 \wedge \eta^0$$

$$+ (\mathbf{i}\zeta_0^2 + F_1^3\gamma^{\bar{2}} - \mathbf{i}(t^3 + f^3)\gamma^2 + T_{\bar{2}}^3\eta^{\bar{3}}) \wedge \eta^1 \wedge \eta^0$$

$$+ (\mathrm{d}T_{\bar{1}}^3 - T_{\bar{1}}^3(3\tau - 2\mathbf{i}\varrho - \mathbf{i}\varsigma) - F_2^3\gamma^2 + (T_{\bar{2}}^3\overline{F}^2 - f^3F_2^3)\eta^2) \wedge \eta^{\bar{1}} \wedge \eta^0 \qquad (3.58)$$

$$+ (\mathrm{d}T_{\bar{2}}^3 - T_{\bar{2}}^3(3\tau - \mathbf{i}\varrho - 2\mathbf{i}\varsigma) - F_1^3\gamma^1 + (T_{\bar{1}}^3\overline{F}^1 - f^3F_1^3)\eta^1) \wedge \eta^{\bar{2}} \wedge \eta^0$$

$$+ (\mathrm{d}F_1^3 - 2F_1^3(\tau - \mathbf{i}\varsigma) + 2\mathbf{i}F^2\gamma^1 - F_2^3\overline{F}^1\eta^2 - (F_2^3F^2 + 2\mathbf{i}T_{\bar{2}}^3)\eta^{\bar{1}}) \wedge \eta^{\bar{2}} \wedge \eta^1$$

$$+ (\mathrm{d}F_2^3 - 2F_2^3(\tau - \mathbf{i}\varrho) + 2\mathbf{i}F^1\gamma^2 - F_1^3\overline{F}^2\eta^1 - (F_1^3F^1 + \epsilon 2\mathbf{i}T_{\bar{1}}^3)\eta^{\bar{2}}) \wedge \eta^{\bar{1}} \wedge \eta^2$$

$$+ \mathbf{i}(\mathrm{d}t^3 - 2t^3\tau + f^3t^3\eta^0 - \zeta_1^1 + \zeta_2^2) \wedge \eta^1 \wedge \eta^2 + (\mathrm{d}f^3 - 2f^3\tau + \zeta_1^1 + \zeta_2^2 - 2\zeta_0) \wedge \eta^3 \wedge \eta^0.$$

For later use, we observe that if we reduce by $\{\eta^0, \eta^3, \eta^{\bar{1}}, \eta^{\bar{2}}\}$ or $\{\eta^1, \eta^2, \eta^{\bar{1}}, \eta^{\bar{2}}\}$, respectively, then we can say

$$\left.\begin{array}{l} 0 \equiv \mathrm{d}t^3 - 2t^3\tau + f^3t^3\eta^0 - \zeta_1^1 + \zeta_2^2 \\[6pt] 0 \equiv \mathrm{d}f^3 - 2f^3\tau + \zeta_1^1 + \zeta_2^2 - 2\zeta_0 \end{array}\right\} \quad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\bar{1}}, \eta^{\bar{2}}\}. \qquad (3.59)$$

Now we return to the unreduced equation (3.58). With (3.54) and (3.56) in mind, we see that

reduction modulo $\{\eta^0, \eta^1, \eta^{\overline{1}}\}$, $\{\eta^0, \eta^2, \eta^{\overline{2}}\}$, $\{\eta^1, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}\}$, $\{\eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}\}$, respectively yields

$$\zeta_2^\varsigma \equiv -\epsilon 2\mathbf{i}\gamma^{\overline{2}} - (|F^1|^2 - \epsilon 2\mathbf{i}(t^3 + f^3))\eta^{\overline{2}} \qquad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^{\overline{1}}, \eta^2, \eta^3\},$$

$$\zeta_1^\varrho \equiv -2\mathbf{i}\gamma^{\overline{1}} - (|F^2|^2 + 2\mathbf{i}(t^3 - f^3))\eta^{\overline{1}} \qquad \mathrm{mod}\ \{\eta^0, \eta^2, \eta^{\overline{2}}, \eta^1, \eta^3\},$$

$$\zeta_0^1 \equiv \mathbf{i}F_2^3\gamma^{\overline{1}} - (t^3 - f^3)\gamma^1 + \epsilon\mathbf{i}T_{\overline{1}}^3\eta^{\overline{3}} \qquad \mathrm{mod}\ \{\eta^1, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^2, \eta^0\},$$

$$\zeta_0^2 \equiv \mathbf{i}F_1^3\gamma^{\overline{2}} + (t^3 + f^3)\gamma^2 + \mathbf{i}T_{\overline{2}}^3\eta^{\overline{3}} \qquad \mathrm{mod}\ \{\eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^1, \eta^0\}.$$

(3.60)

Thus, if we define

$$\xi_2^\varsigma := \zeta_2^\varsigma + \epsilon 2\mathbf{i}\gamma^{\overline{2}} + (|F^1|^2 - \epsilon 2\mathbf{i}(t^3 + f^3))\eta^{\overline{2}},$$

$$\xi_1^\varrho := \zeta_1^\varrho + 2\mathbf{i}\gamma^{\overline{1}} + (|F^2|^2 + 2\mathbf{i}(t^3 - f^3))\eta^{\overline{1}},$$

$$\xi_0^1 := \zeta_0^1 - \mathbf{i}F_2^3\gamma^{\overline{1}} + (t^3 - f^3)\gamma^1 - \epsilon\mathbf{i}T_{\overline{1}}^3\eta^{\overline{3}},$$

$$\xi_0^2 := \zeta_0^2 - \mathbf{i}F_1^3\gamma^{\overline{2}} - (t^3 + f^3)\gamma^2 - \mathbf{i}T_{\overline{2}}^3\eta^{\overline{3}},$$

$$\xi_0 := \zeta_0 + \psi,$$

$$\xi_1 := \zeta_1^1 + \psi,$$

$$\xi_2 := \zeta_2^2 + \psi,$$

then we are left with an expression for each of the Ξ's in the structure equations (3.51) of $B_4^{(1)}$:

$$\Xi^\tau = \tfrac{i}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{i}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{i}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{i}{2}\gamma^{\overline{2}} \wedge \eta^2 + \xi_0 \wedge \eta^0,$$

$$\Xi^\varrho = -\tfrac{3i}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{3i}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{i}{2}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\tfrac{i}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^1 F^2 \eta^{\overline{2}} \wedge \eta^{\overline{1}}$$
$$+ (|F^1|^2 - \epsilon i t^3)\eta^{\overline{2}} \wedge \eta^2 - (|F^2|^2 + 2i(t^3 - f^3))\eta^{\overline{1}} \wedge \eta^1 + (\xi_1^1 - \xi_0) \wedge \eta^0 + \xi_1^\varrho \wedge \eta^1 + \xi_2^\varrho \wedge \eta^2,$$

$$\Xi^\varsigma = \tfrac{i}{2}\gamma^1 \wedge \eta^{\overline{1}} + \tfrac{i}{2}\gamma^{\overline{1}} \wedge \eta^1 - \epsilon\tfrac{3i}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{3i}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^2 F^1 \eta^{\overline{1}} \wedge \eta^{\overline{2}}$$
$$+ (|F^2|^2 + i t^3)\eta^{\overline{1}} \wedge \eta^1 - (|F^1|^2 - \epsilon 2i(f^3 + t^3))\eta^{\overline{2}} \wedge \eta^2 + (\xi_2^2 - \xi_0) \wedge \eta^0 + \xi_1^\varsigma \wedge \eta^1 + \xi_2^\varsigma \wedge \eta^2,$$

$$\Xi^1 = (\tau - i\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 + i F_2^3 \gamma^{\overline{1}} \wedge \eta^0 + F^1 \gamma^{\overline{1}} \wedge \eta^2 - F^1 \gamma^2 \wedge \eta^{\overline{1}} + (t^3 - f^3)\gamma^1 \wedge \eta^0$$
$$- \epsilon f^3 \eta^3 \wedge \eta^{\overline{2}} - \epsilon T_{\overline{1}}^3 \eta^{\overline{1}} \wedge \eta^2 + \epsilon i T_{\overline{1}}^3 \eta^{\overline{3}} \wedge \eta^0 + \xi_0^1 \wedge \eta^0 + \xi_1^1 \wedge \eta^1 + \xi_2^1 \wedge \eta^2,$$

$$\Xi^2 = (\tau - i\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 + i F_1^3 \gamma^{\overline{2}} \wedge \eta^0 - F^2 \gamma^1 \wedge \eta^{\overline{2}} + F^2 \gamma^{\overline{2}} \wedge \eta^1 + (t^3 + f^3)\gamma^2 \wedge \eta^0$$
$$- f^3 \eta^3 \wedge \eta^{\overline{1}} - T_{\overline{2}}^3 \eta^{\overline{2}} \wedge \eta^1 + i T_{\overline{2}}^3 \eta^{\overline{3}} \wedge \eta^0 + \xi_0^2 \wedge \eta^0 + \xi_2^2 \wedge \eta^2 + \xi_1^2 \wedge \eta^1,$$
$$(3.61)$$

where, by (3.54),(3.56), and (3.60), we now have

$$0 \equiv \begin{cases} \xi_2^1, \xi_2^\varrho & \mod \{\eta^0, \eta^1, \eta^2, \eta^{\overline{1}}\}, \\ \xi_1^2, \xi_1^\varsigma & \mod \{\eta^0, \eta^1, \eta^2, \eta^{\overline{2}}\}, \\ \xi_1^\varrho & \mod \{\eta^0, \eta^2, \eta^{\overline{2}}, \eta^1, \eta^3\}, \\ \xi_2^\varsigma & \mod \{\eta^0, \eta^1, \eta^{\overline{1}}, \eta^2, \eta^3\}, \\ \xi_0^1, \xi_0^2 & \mod \{\eta^0, \eta^1, \eta^{\overline{1}}, \eta^2, \eta^{\overline{2}}, \eta^3\}. \end{cases} \qquad (3.62)$$

Using the fact that $\mathbf{id}\varrho$ is $i\mathbb{R}$-valued, we can write

$$0 = \Xi^\varrho + \overline{\Xi}^\varrho$$
$$= (\xi_1^1 + \xi_{\overline{1}}^{\overline{1}} - 2\xi_0) \wedge \eta^0 + \xi_1^\varrho \wedge \eta^1 + \xi_{\overline{1}}^{\overline{\varrho}} \wedge \eta^{\overline{1}} + \epsilon 2i t^3 \eta^2 \wedge \eta^{\overline{2}} + 4i(t^3 - f^3)\eta^1 \wedge \eta^{\overline{1}} \qquad (3.63)$$
$$+ (\xi_2^\varrho - \overline{F}^1 \overline{F}^2 \eta^1) \wedge \eta^2 + (\xi_{\overline{2}}^{\overline{\varrho}} - F^1 F^2 \eta^{\overline{1}}) \wedge \eta^{\overline{2}},$$

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which along with (3.54) shows that $t^3 = f^3 = 0$. Plugging these zeros into (3.59) yields

$$\left. \begin{array}{l} 0 \equiv -\xi_1^1 + \xi_2^2 \\[1mm] 0 \equiv \xi_1^1 + \xi_2^2 - 2\xi_0 \end{array} \right\} \quad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}\},$$

so in particular,

$$\left. \begin{array}{l} \xi^1 := \xi_1^1 - \xi_0 \\[1mm] \xi^2 := \xi_2^2 - \xi_0 \end{array} \right\} \equiv 0 \quad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}\}. \tag{3.64}$$

We know that ξ_0 is $\mathbb{R}$-valued, so we can replace ψ with $\hat{\psi} = \psi - \xi_0$, which has the effect of removing the ξ_0 term in the equation for $\mathrm{d}\tau$ and replacing ξ_i^i with $\xi^i := \xi_i^i - \xi_0$ $(i = 1, 2)$ in the equation for $\mathrm{d}\gamma^i$. We therefore update our structure equations

$$\mathrm{d}\tau = -\hat{\psi} \wedge \eta^0 + \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2,$$

$$\mathbf{i}\mathrm{d}\varrho = -\tfrac{3\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{3\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^1 F^2 \eta^{\overline{2}} \wedge \eta^{\overline{1}}$$
$$+ |F^1|^2 \eta^{\overline{2}} \wedge \eta^2 - |F^2|^2 \eta^{\overline{1}} \wedge \eta^1 + \xi^1 \wedge \eta^0 + \xi_1^\varrho \wedge \eta^1 + \xi_2^\varrho \wedge \eta^2,$$

$$\mathbf{i}\mathrm{d}\varsigma = \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} + \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^2 F^1 \eta^{\overline{1}} \wedge \eta^{\overline{2}}$$
$$+ |F^2|^2 \eta^{\overline{1}} \wedge \eta^1 - |F^1|^2 \eta^{\overline{2}} \wedge \eta^2 + \xi^2 \wedge \eta^0 + \xi_1^\varsigma \wedge \eta^1 + \xi_2^\varsigma \wedge \eta^2, \tag{3.65}$$

$$\mathrm{d}\gamma^1 = -\hat{\psi} \wedge \eta^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 + \mathbf{i}F_2^3 \gamma^{\overline{1}} \wedge \eta^0 + F^1 \gamma^{\overline{1}} \wedge \eta^2 - F^1 \gamma^2 \wedge \eta^{\overline{1}}$$
$$- \epsilon T_{\overline{1}}^3 \eta^{\overline{1}} \wedge \eta^2 + \epsilon\mathbf{i}T_{\overline{1}}^3 \eta^{\overline{3}} \wedge \eta^0 + \xi_0^1 \wedge \eta^0 + \xi^1 \wedge \eta^1 + \xi_2^1 \wedge \eta^2,$$

$$\mathrm{d}\gamma^2 = -\hat{\psi} \wedge \eta^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 + \mathbf{i}F_1^3 \gamma^{\overline{2}} \wedge \eta^0 - F^2 \gamma^1 \wedge \eta^{\overline{2}} + F^2 \gamma^{\overline{2}} \wedge \eta^1$$
$$- T_{\overline{2}}^3 \eta^{\overline{2}} \wedge \eta^{\overline{1}} + \mathbf{i}T_{\overline{2}}^3 \eta^{\overline{3}} \wedge \eta^0 + \xi_0^2 \wedge \eta^0 + \xi_1^2 \wedge \eta^1 + \xi^2 \wedge \eta^2,$$

where by (3.62) and (3.64) we can say

$$0 \equiv \begin{cases} \xi_2^1, \xi_2^\varrho & \mod \{\eta^0, \eta^1, \eta^2, \eta^{\overline{1}}\}, \\ \xi_1^2, \xi_1^\varsigma & \mod \{\eta^0, \eta^1, \eta^2, \eta^{\overline{2}}\}, \\ \xi_1^\varrho & \mod \{\eta^0, \eta^2, \eta^{\overline{2}}, \eta^1, \eta^3\}, \\ \xi_2^\varsigma & \mod \{\eta^0, \eta^1, \eta^{\overline{1}}, \eta^2, \eta^3\}, \\ \xi_0^1, \xi_0^2, \xi^1, \xi^2 & \mod \{\eta^0, \eta^1, \eta^{\overline{1}}, \eta^2, \eta^{\overline{2}}, \eta^3\}. \end{cases}$$

By collecting coefficients of redundant two-forms and suppressing forms which are only wedged against themselves in all of the equations, we may more specifically assume

$$0 \equiv \begin{cases} \xi_2^1, \xi_2^\varrho & \mod \{\eta^0, \eta^1, \eta^{\overline{1}}\}, \\ \xi_1^2, \xi_1^\varsigma & \mod \{\eta^0, \eta^2, \eta^{\overline{2}}\}, \\ \xi_1^\varrho & \mod \{\eta^0, \eta^{\overline{2}}, \eta^3\}, \\ \xi_2^\varsigma & \mod \{\eta^0, \eta^{\overline{1}}, \eta^3\} \\ \xi_0^1 & \mod \{\eta^1, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^3\}, \\ \xi_0^2 & \mod \{\eta^{\overline{1}}, \eta^2, \eta^{\overline{2}}, \eta^3\}, \\ \xi^1, \xi^2 & \mod \{\eta^{\overline{1}}, \eta^{\overline{2}}, \eta^3\}. \end{cases}$$

Let us therefore expand

$$\xi_0^1 = P_{01}^1 \eta^1 + P_{0\overline{1}}^1 \eta^{\overline{1}} + P_{0\overline{2}}^1 \eta^{\overline{2}} + P_{03}^1 \eta^3, \qquad \xi_0^2 = P_{0\overline{1}}^2 \eta^{\overline{1}} + P_{02}^2 \eta^2 + P_{0\overline{2}}^2 \eta^{\overline{2}} + P_{03}^2 \eta^3,$$
$$\xi_2^1 = P_{20}^1 \eta^0 + P_{21}^1 \eta^1 + P_{2\overline{1}}^1 \eta^{\overline{1}}, \qquad \xi_1^2 = P_{10}^2 \eta^0 + P_{12}^2 \eta^2 + P_{1\overline{2}}^2 \eta^{\overline{2}},$$
$$\xi^1 = Q_{\overline{1}}^1 \eta^{\overline{1}} + Q_{\overline{2}}^1 \eta^{\overline{2}} + Q_3^1 \eta^3, \qquad \xi^2 = Q_{\overline{1}}^2 \eta^{\overline{1}} + Q_{\overline{2}}^2 \eta^{\overline{2}} + Q_3^2 \eta^3,$$
$$\xi_1^\varrho = R_{10} \eta^0 + R_{1\overline{2}} \eta^{\overline{2}} + R_{13} \eta^3, \qquad \xi_2^\varrho = R_{20} \eta^0 + R_{21} \eta^1 + R_{2\overline{1}} \eta^{\overline{1}},$$
$$\xi_1^\varsigma = S_{10} \eta^0 + S_{12} \eta^2 + S_{1\overline{2}} \eta^{\overline{2}}, \qquad \xi_2^\varsigma = S_{20} \eta^0 + S_{2\overline{1}} \eta^{\overline{1}} + S_{23} \eta^3,$$

for some functions $P, Q, R, S \in C^\infty(B_4^{(1)}, \mathbb{C})$. With these in hand, we return to our argument about the imaginary value of $\mathbf{i} d\varrho$ from (3.63).

$$0 = \Xi^\varrho + \overline{\Xi}^\varrho$$
$$= (\xi^1 + \xi^{\overline{1}}) \wedge \eta^0 + \xi_1^\varrho \wedge \eta^1 + \xi_{\overline{1}}^{\overline{\varrho}} \wedge \eta^{\overline{1}} + (\xi_2^\varrho - \overline{F}^1 \overline{F}^2 \eta^1) \wedge \eta^2 + (\xi_{\overline{2}}^{\overline{\varrho}} - F^1 F^2 \eta^{\overline{1}}) \wedge \eta^{\overline{2}}$$
$$= (Q_{\overline{1}}^1 \eta^{\overline{1}} + Q_{\overline{2}}^1 \eta^{\overline{2}} + Q_3^1 \eta^3 + \overline{Q}_{\overline{1}}^1 \eta^1 + \overline{Q}_{\overline{2}}^1 \eta^2 + \overline{Q}_3^1 \eta^{\overline{3}}) \wedge \eta^0 + (R_{10} \eta^0 + R_{1\overline{2}} \eta^{\overline{2}} + R_{13} \eta^3) \wedge \eta^1$$

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$$+ (\overline{R}_{10}\eta^0 + \overline{R}_{1\overline{2}}\eta^2 + \overline{R}_{13}\eta^{\overline{3}}) \wedge \eta^{\overline{1}} + (R_{20}\eta^0 + R_{21}\eta^1 + R_{2\overline{1}}\eta^{\overline{1}} - \overline{F}^1\overline{F}^2\eta^1) \wedge \eta^2$$

$$+ (\overline{R}_{20}\eta^0 + \overline{R}_{21}\eta^{\overline{1}} + \overline{R}_{2\overline{1}}\eta^1 - F^1F^2\eta^{\overline{1}}) \wedge \eta^{\overline{2}}.$$

Thus we see that

$$Q_3^1 = R_{13} = 0, \qquad R_{21} = \overline{F}^1\overline{F}^2, \qquad R_{10} = \overline{Q}_{\overline{1}}^1, \qquad R_{20} = \overline{Q}_{\overline{2}}^1, \qquad R_{1\overline{2}} = \overline{R}_{2\overline{1}}.$$

Similarly, $\mathbf{i}\varsigma$ is $\mathbf{i}\mathbb{R}$-valued, and we have

$$0 = \Xi^\varsigma + \overline{\Xi}^\varsigma$$

$$= (\xi^2 + \xi^{\overline{2}}) \wedge \eta^0 + (\xi_1^\varsigma - \overline{F}^2\overline{F}^1\eta^2) \wedge \eta^1 + (\xi_{\overline{1}}^{\overline{\varsigma}} - F^2F^1\eta^{\overline{2}}) \wedge \eta^{\overline{1}} + \xi_2^\varsigma \wedge \eta^2 + \xi_{\overline{2}}^{\overline{\varsigma}} \wedge \eta^{\overline{2}}$$

$$= (Q_{\overline{1}}^2\eta^{\overline{1}} + Q_{\overline{2}}^2\eta^{\overline{2}} + Q_3^2\eta^3 + \overline{Q}_{\overline{1}}^2\eta^1 + \overline{Q}_{\overline{2}}^2\eta^2 + \overline{Q}_3^2\eta^{\overline{3}}) \wedge \eta^0 + (S_{10}\eta^0 + S_{12}\eta^2 + S_{1\overline{2}}\eta^{\overline{2}} - \overline{F}^2\overline{F}^1\eta^2) \wedge \eta^1$$

$$+ (\overline{S}_{10}\eta^0 + \overline{S}_{12}\eta^{\overline{2}} + \overline{S}_{1\overline{2}}\eta^2 - F^2F^1\eta^{\overline{2}}) \wedge \eta^{\overline{1}} + (S_{20}\eta^0 + S_{2\overline{1}}\eta^{\overline{1}} + S_{23}\eta^3) \wedge \eta^2$$

$$+ (\overline{S}_{20}\eta^0 + \overline{S}_{2\overline{1}}\eta^1 + \overline{S}_{23}\eta^{\overline{3}}) \wedge \eta^{\overline{2}},$$

whence

$$Q_3^2 = S_{23} = 0, \qquad S_{12} = \overline{F}^1\overline{F}^2, \qquad S_{10} = \overline{Q}_{\overline{1}}^2, \qquad S_{20} = \overline{Q}_{\overline{2}}^2, \qquad S_{1\overline{2}} = \overline{S}_{2\overline{1}}.$$

We reveal a few more relations by revisiting our original structure equations.

$$0 \equiv \mathrm{d}^2\eta^1 \qquad\qquad\qquad \mathrm{mod}\ \{\eta^{\overline{1}}\}$$

$$\equiv \xi_2^1 \wedge \eta^2 \wedge \eta^0 + \xi_2^\varrho \wedge \eta^2 \wedge \eta^1 \qquad\qquad\qquad \mathrm{mod}\ \{\eta^{\overline{1}}\}$$

$$\equiv P_{21}^1\eta^1 \wedge \eta^2 \wedge \eta^0 + \overline{Q}_{\overline{2}}^1\eta^0 \wedge \eta^2 \wedge \eta^1 \qquad\qquad\qquad \mathrm{mod}\ \{\eta^{\overline{1}}\}$$

$$\Rightarrow P_{21}^1 = \overline{Q}_{\overline{2}}^1.$$

Similarly,

$$0 \equiv \mathrm{d}^2\eta^2 \qquad\qquad\qquad \mathrm{mod}\ \{\eta^{\overline{2}}\}$$

$$\equiv \xi_1^2 \wedge \eta^1 \wedge \eta^0 + \xi_1^\varsigma \wedge \eta^1 \wedge \eta^2 \qquad\qquad\qquad \mathrm{mod}\ \{\eta^{\overline{2}}\}$$

$$\equiv P^2_{12}\eta^2 \wedge \eta^1 \wedge \eta^0 + \overline{Q}^2_{\overline{1}}\eta^0 \wedge \eta^1 \wedge \eta^2 \qquad\qquad \mathrm{mod}\ \{\eta^{\overline{2}}\}$$

$$\Rightarrow P^2_{12} = \overline{Q}^2_{\overline{1}}.$$

And finally,

$$0 \equiv \mathrm{d}^2\eta^3 \qquad\qquad \mathrm{mod}\ \{\eta^{\overline{1}}, \eta^{\overline{2}}\}$$

$$\equiv (\xi^\varrho_1 + \xi^\varsigma_1) \wedge \eta^3 \wedge \eta^1 + (\xi^\varrho_2 + \xi^\varsigma_2) \wedge \eta^3 \wedge \eta^2 + \mathbf{i}\xi^1_0 \wedge \eta^2 \wedge \eta^0 + \mathbf{i}\xi^2_0 \wedge \eta^1 \wedge \eta^0$$

$$+ \mathbf{i}(-\xi^1 + \xi^2) \wedge \eta^1 \wedge \eta^2 + (\xi^1 + \xi^2) \wedge \eta^3 \wedge \eta^0 \qquad\qquad \mathrm{mod}\ \{\eta^{\overline{1}}, \eta^{\overline{2}}\}$$

$$\equiv (\overline{Q}^1_{\overline{1}} + \overline{Q}^2_{\overline{1}})\eta^0 \wedge \eta^3 \wedge \eta^1 + (\overline{Q}^1_{\overline{2}} + \overline{Q}^2_{\overline{2}})\eta^0 \wedge \eta^3 \wedge \eta^2$$

$$+ \mathbf{i}(P^1_{01}\eta^1 + P^1_{03}\eta^3) \wedge \eta^2 \wedge \eta^0 + \mathbf{i}(P^2_{02}\eta^2 + P^2_{03}\eta^3) \wedge \eta^1 \wedge \eta^0 \qquad\qquad \mathrm{mod}\ \{\eta^{\overline{1}}, \eta^{\overline{2}}\}$$

$$\Rightarrow P^1_{03} = \mathbf{i}(\overline{Q}^1_{\overline{2}} + \overline{Q}^2_{\overline{2}}), \qquad P^2_{03} = \mathbf{i}(\overline{Q}^1_{\overline{1}} + \overline{Q}^2_{\overline{1}}), \qquad P^1_{01} = P^2_{02}.$$

We give preference to the Q's in our notation, so we can rename the only remaining $R := R_{1\overline{2}}$ and $S := S_{1\overline{2}}$. We also rename $P_0 := P^1_{01} = P^2_{02}$ to emphasize that the equations for $\mathrm{d}\gamma^1$ and $\mathrm{d}\gamma^2$ have

this term in common. Dropping the hat off of ψ in (3.65), we summarize our results so far.

$$\mathrm{d}\tau = -\psi \wedge \eta^0 + \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2,$$

$$\begin{aligned}
\mathbf{i}\mathrm{d}\varrho = {}& -\tfrac{3\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{3\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^1 F^2 \eta^{\overline{2}} \wedge \eta^{\overline{1}} + \overline{F}^1\overline{F}^2 \eta^1 \wedge \eta^2 \\
& + |F^1|^2 \eta^{\overline{2}} \wedge \eta^2 - |F^2|^2 \eta^{\overline{1}} \wedge \eta^1 + (Q^1_{\overline{1}}\eta^{\overline{1}} - \overline{Q}^1_{\overline{1}}\eta^1 + Q^1_{\overline{2}}\eta^{\overline{2}} - \overline{Q}^1_{\overline{2}}\eta^2) \wedge \eta^0 \\
& + R\eta^{\overline{2}} \wedge \eta^1 + \overline{R}\eta^{\overline{1}} \wedge \eta^2,
\end{aligned}$$

$$\begin{aligned}
\mathbf{i}\mathrm{d}\varsigma = {}& \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} + \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^1 F^2 \eta^{\overline{1}} \wedge \eta^{\overline{2}} + \overline{F}^1\overline{F}^2 \eta^2 \wedge \eta^1 \\
& + |F^2|^2 \eta^{\overline{1}} \wedge \eta^1 - |F^1|^2 \eta^{\overline{2}} \wedge \eta^2 + (Q^2_{\overline{1}}\eta^{\overline{1}} - \overline{Q}^2_{\overline{1}}\eta^1 + Q^2_{\overline{2}}\eta^{\overline{2}} - \overline{Q}^2_{\overline{2}}\eta^2) \wedge \eta^0 \\
& + S\eta^{\overline{2}} \wedge \eta^1 + \overline{S}\eta^{\overline{1}} \wedge \eta^2,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\gamma^1 = {}& -\psi \wedge \eta^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 + \mathbf{i}F^3_2\gamma^{\overline{1}} \wedge \eta^0 + F^1\gamma^{\overline{1}} \wedge \eta^2 - F^1\gamma^2 \wedge \eta^{\overline{1}} \\
& - \epsilon T^3_{\overline{1}}\eta^{\overline{1}} \wedge \eta^{\overline{2}} + \epsilon\mathbf{i}T^3_{\overline{1}}\eta^{\overline{3}} \wedge \eta^0 + (P_0\eta^1 + P^1_{0\overline{1}}\eta^{\overline{1}} + P^1_{0\overline{2}}\eta^{\overline{2}} + \mathbf{i}(\overline{Q}^1_{\overline{2}} + \overline{Q}^2_{\overline{2}})\eta^3) \wedge \eta^0 \\
& + (Q^1_{\overline{1}}\eta^{\overline{1}} + Q^1_{\overline{2}}\eta^{\overline{2}} - \overline{Q}^1_{\overline{2}}\eta^2) \wedge \eta^1 + (P^1_{20}\eta^0 + P^1_{2\overline{1}}\eta^{\overline{1}}) \wedge \eta^2,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\gamma^2 = {}& -\psi \wedge \eta^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 + \mathbf{i}F^3_1\gamma^{\overline{2}} \wedge \eta^0 - F^2\gamma^1 \wedge \eta^{\overline{2}} + F^2\gamma^{\overline{2}} \wedge \eta^1 \\
& - T^3_{\overline{2}}\eta^{\overline{2}} \wedge \eta^{\overline{1}} + \mathbf{i}T^3_{\overline{2}}\eta^{\overline{3}} \wedge \eta^0 + (P^2_{0\overline{1}}\eta^{\overline{1}} + P_0\eta^2 + P^2_{0\overline{2}}\eta^{\overline{2}} + \mathbf{i}(\overline{Q}^1_{\overline{1}} + \overline{Q}^2_{\overline{1}})\eta^3) \wedge \eta^0 \\
& + (Q^2_{\overline{1}}\eta^{\overline{1}} - \overline{Q}^2_{\overline{1}}\eta^1 + Q^2_{\overline{2}}\eta^{\overline{2}}) \wedge \eta^2 + (P^2_{10}\eta^0 + P^2_{1\overline{2}}\eta^{\overline{2}}) \wedge \eta^1.
\end{aligned}$$

$$(3.66)$$

By replacing $\hat{\psi} := \psi + \tfrac{1}{2}(P_0 + \overline{P}_0)\eta^0$, we absorb the real part of P_0 in the equations for $\mathrm{d}\gamma^1$ and $\mathrm{d}\gamma^2$ without affecting the equation for $\mathrm{d}\tau$. After this absorption (and dropping the hat), ψ is uniquely and globally determined, and we may replace P_0 in our equations with $\mathbf{i}p_0$ where $p_0 \in C^\infty(B^{(1)}_4)$ is the $\mathbb{R}$-valued $-\tfrac{\mathbf{i}}{2}(P_0 - \overline{P}_0)$.

Note that our equations are now free of any unknown one-forms, which is just in time for us to introduce the last one we will need. It shows up in the equation for $\mathrm{d}\psi$, which we obtain by differentiating $2\mathrm{d}\tau$.

$$0 = \mathrm{d}(2\mathrm{d}\tau)$$

$$= \Big(-2\mathrm{d}\psi - 4\psi \wedge \tau + 2\mathbf{i}\gamma^1 \wedge \gamma^{\overline{1}} + \epsilon 2\mathbf{i}\gamma^2 \wedge \gamma^{\overline{2}} \dots$$

$$+ \mathbf{i}(P_{0\overline{2}}^1 - \epsilon P_{0\overline{1}}^2)\eta^{\overline{1}} \wedge \eta^2 + \mathbf{i}(\overline{P}_{0\overline{2}}^1 - \epsilon \overline{P}_{0\overline{1}}^2)\eta^2 \wedge \eta^1 + \mathbf{i}(P_{20}^1 + \epsilon\overline{P}_{10}^2)\eta^2 \wedge \eta^{\overline{1}} + \mathbf{i}(\epsilon P_{10}^2 + \overline{P}_{20}^1)\eta^1 \wedge \eta^{\overline{2}} \dots$$

$$+ \epsilon(Q_{\overline{1}}^1 + Q_{\overline{1}}^2)\eta^{\overline{3}} \wedge \eta^2 + \epsilon(\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2)\eta^3 \wedge \eta^{\overline{2}} + (\overline{Q}_{\overline{2}}^1 + \overline{Q}_{\overline{2}}^2)\eta^3 \wedge \eta^{\overline{1}} + (Q_{\overline{2}}^1 + Q_{\overline{2}}^2)\eta^{\overline{3}} \wedge \eta^1 \dots$$

$$+ \overline{F}_2^3\gamma^1 \wedge \eta^1 + F_2^3\gamma^{\overline{1}} \wedge \eta^{\overline{1}} + \epsilon\overline{F}_1^3\gamma^2 \wedge \eta^2 + \epsilon F_1^3\gamma^{\overline{2}} \wedge \eta^{\overline{2}} \dots$$

$$+ \epsilon\overline{T}_{\overline{1}}^3\eta^3 \wedge \eta^1 + \epsilon T_{\overline{1}}^3\eta^{\overline{3}} \wedge \eta^{\overline{1}} + \epsilon\overline{T}_{\overline{2}}^3\eta^3 \wedge \eta^2 + \epsilon T_{\overline{2}}^3\eta^{\overline{3}} \wedge \eta^{\overline{2}} \Big) \wedge \eta^0.$$

Thus, for some $\mathbb{R}$-valued $\zeta \in \Omega^1(B_4^{(1)})$, we have a final structure equation

$$\mathrm{d}\psi = -2\psi \wedge \tau + \mathbf{i}\gamma^1 \wedge \gamma^{\overline{1}} + \epsilon\mathbf{i}\gamma^2 \wedge \gamma^{\overline{2}} + \zeta \wedge \eta^0 + \tfrac{\mathbf{i}}{2}(P_{0\overline{2}}^1 - \epsilon P_{0\overline{1}}^2)\eta^{\overline{1}} \wedge \eta^{\overline{2}}$$

$$+ \tfrac{\mathbf{i}}{2}(\overline{P}_{0\overline{2}}^1 - \epsilon\overline{P}_{0\overline{1}}^2)\eta^2 \wedge \eta^1 + \tfrac{\mathbf{i}}{2}(P_{20}^1 + \epsilon\overline{P}_{10}^2)\eta^2 \wedge \eta^{\overline{1}} + \tfrac{\mathbf{i}}{2}(\epsilon P_{10}^2 + \overline{P}_{20}^1)\eta^1 \wedge \eta^{\overline{2}}$$

$$+ \epsilon\tfrac{1}{2}(Q_{\overline{1}}^1 + Q_{\overline{1}}^2)\eta^{\overline{3}} \wedge \eta^2 + \epsilon\tfrac{1}{2}(\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2)\eta^3 \wedge \eta^{\overline{2}} + \tfrac{1}{2}(\overline{Q}_{\overline{2}}^1 + \overline{Q}_{\overline{2}}^2)\eta^3 \wedge \eta^{\overline{1}} + \tfrac{1}{2}(Q_{\overline{2}}^1 + Q_{\overline{2}}^2)\eta^{\overline{3}} \wedge \eta^1$$

$$+ \tfrac{1}{2}\overline{F}_2^3\gamma^1 \wedge \eta^1 + \tfrac{1}{2}F_2^3\gamma^{\overline{1}} \wedge \eta^{\overline{1}} + \epsilon\tfrac{1}{2}\overline{F}_1^3\gamma^2 \wedge \eta^2 + \epsilon\tfrac{1}{2}F_1^3\gamma^{\overline{2}} \wedge \eta^{\overline{2}}$$

$$+ \epsilon\tfrac{1}{2}\overline{T}_{\overline{1}}^3\eta^3 \wedge \eta^1 + \epsilon\tfrac{1}{2}T_{\overline{1}}^3\eta^{\overline{3}} \wedge \eta^{\overline{1}} + \epsilon\tfrac{1}{2}\overline{T}_{\overline{2}}^3\eta^3 \wedge \eta^2 + \epsilon\tfrac{1}{2}T_{\overline{2}}^3\eta^{\overline{3}} \wedge \eta^{\overline{2}}.$$

$$(3.67)$$

In order to expand ζ, we first revisit

$$0 = \mathrm{d}^2\eta^3$$

$$\equiv (\mathrm{d}F_2^3 - 2F_2^3(\tau - \mathbf{i}\varrho) + 2\mathbf{i}F^1\gamma^2 + (\mathbf{i}(Q_{\overline{1}}^1 - Q_{\overline{1}}^2) - F_1^3\overline{F}^2)\eta^1) \wedge \eta^{\overline{1}} \wedge \eta^2 \qquad \mathrm{mod}\ \{\eta^0, \eta^3, \eta^{\overline{2}}\},$$

which implies

$$\mathrm{d}F_2^3 \equiv 2F_2^3(\tau - \mathbf{i}\varrho) - 2\mathbf{i}F^1\gamma^2 - (\mathbf{i}(Q_{\overline{1}}^1 - Q_{\overline{1}}^2) - F_1^3\overline{F}^2)\eta^1 \qquad \mathrm{mod}\ \{\eta^0, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}\}. \qquad (3.68)$$

Now differentiate $\mathrm{d}\gamma^1$ and reduce by all of the η's except η^0, η^1.

$$0 = \mathrm{d}^2\gamma^1$$

$$\equiv -\zeta \wedge \eta^0 \wedge \eta^1 + \mathbf{i}(\mathrm{d}F_2^3 - 2F_2^3(\tau - \mathbf{i}\varrho) + 2\mathbf{i}F^1\gamma^2) \wedge \gamma^{\overline{1}} \wedge \eta^0$$

$$+ (\mathbf{i}\mathrm{d}p_0 - 4\mathbf{i}p_0\tau + Q_{\overline{1}}^1\gamma^{\overline{1}} + \overline{Q}_{\overline{1}}^1\gamma^1 + \overline{Q}_{\overline{2}}^2\gamma^2 + (Q_{\overline{2}}^1 - \mathbf{i}\overline{F}^1F_2^3)\gamma^{\overline{2}}) \wedge \eta^1 \wedge \eta^0 \qquad \mathrm{mod}\ \{\eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\}.$$

Plugging in (3.68) then yields

$$d^2\gamma^1 \equiv (-\mathbf{i}\mathrm{d}p_0 + 4\mathbf{i}p_0\tau - \zeta - \overline{Q}^1_{\overline{1}}\gamma^1 - (Q^2_{\overline{1}} - \mathbf{i}F^3_1\overline{F}^2)\gamma^{\overline{1}} - \overline{Q}^2_{\overline{2}}\gamma^2 - (Q^1_{\overline{2}} - \mathbf{i}\overline{F}^1 F^3_2)\gamma^{\overline{2}}) \wedge \eta^0 \wedge \eta^1$$
$$\mathrm{mod}\ \{\eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\},$$

$$\Rightarrow d^2\gamma^{\overline{1}} \equiv (\mathbf{i}\mathrm{d}p_0 - 4\mathbf{i}p_0\tau - \zeta - Q^1_{\overline{1}}\gamma^{\overline{1}} - (\overline{Q}^2_{\overline{1}} + \mathbf{i}\overline{F}^3_1 F^2)\gamma^1 - Q^2_{\overline{2}}\gamma^{\overline{2}} - (\overline{Q}^1_{\overline{2}} + \mathbf{i}F^1\overline{F}^3_2)\gamma^2) \wedge \eta^0 \wedge \eta^{\overline{1}}$$
$$\mathrm{mod}\ \{\eta^1, \eta^2, \eta^3, \eta^{\overline{2}}, \eta^{\overline{3}}\},$$

where we have used the fact that ζ and p_0 are $\mathbb{R}$-valued. We exploit this further to calculate

$$0 \equiv d^2\gamma^1 \wedge \eta^{\overline{1}} - d^2\gamma^{\overline{1}} \wedge \eta^1 \qquad\qquad \mathrm{mod}\ \{\eta^2, \eta^3, \eta^{\overline{2}}, \eta^{\overline{3}}\}$$
$$\equiv \Big(- 2\zeta - (\overline{Q}^1_{\overline{1}} + \overline{Q}^2_{\overline{1}} + \mathbf{i}\overline{F}^3_1 F^2)\gamma^1 - (\overline{Q}^2_{\overline{2}} + \overline{Q}^1_{\overline{2}} + \mathbf{i}F^1\overline{F}^3_2)\gamma^2 \dots$$
$$- (Q^1_{\overline{1}} + Q^2_{\overline{1}} - \mathbf{i}F^3_1\overline{F}^2)\gamma^{\overline{1}} - (Q^2_{\overline{2}} + Q^1_{\overline{2}} - \mathbf{i}\overline{F}^1 F^3_2)\gamma^{\overline{2}}\Big) \wedge \eta^0 \wedge \eta^1 \wedge \eta^{\overline{1}}$$
$$\mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\},$$

by which we find

$$\zeta \equiv -\tfrac{1}{2}(\overline{Q}^1_{\overline{1}} + \overline{Q}^2_{\overline{1}} + \mathbf{i}\overline{F}^3_1 F^2)\gamma^1 - \tfrac{1}{2}(\overline{Q}^2_{\overline{2}} + \overline{Q}^1_{\overline{2}} + \mathbf{i}F^1\overline{F}^3_2)\gamma^2$$
$$- \tfrac{1}{2}(Q^1_{\overline{1}} + Q^2_{\overline{1}} - \mathbf{i}F^3_1\overline{F}^2)\gamma^{\overline{1}} - \tfrac{1}{2}(Q^2_{\overline{2}} + Q^1_{\overline{2}} - \mathbf{i}\overline{F}^1 F^3_2)\gamma^{\overline{2}} \qquad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\}.$$
$$(3.69)$$

Thus, if we define $\xi \in \Omega^1(B_4^{(1)})$ to be

$$\xi := \zeta + \tfrac{1}{2}(\overline{Q}^1_{\overline{1}} + \overline{Q}^2_{\overline{1}} + \mathbf{i}\overline{F}^3_1 F^2)\gamma^1 + \tfrac{1}{2}(\overline{Q}^2_{\overline{2}} + \overline{Q}^1_{\overline{2}} + \mathbf{i}F^1\overline{F}^3_2)\gamma^2$$
$$+ \tfrac{1}{2}(Q^1_{\overline{1}} + Q^2_{\overline{1}} - \mathbf{i}F^3_1\overline{F}^2)\gamma^{\overline{1}} + \tfrac{1}{2}(Q^2_{\overline{2}} + Q^1_{\overline{2}} - \mathbf{i}\overline{F}^1 F^3_2)\gamma^{\overline{2}},$$

then by (3.69) we know

$$\xi \equiv 0 \quad \mathrm{mod}\ \{\eta^0, \eta^1, \eta^2, \eta^3, \eta^{\overline{1}}, \eta^{\overline{2}}, \eta^{\overline{3}}\},$$

which along with the fact that ξ is $\mathbb{R}$-valued (and wedged against η^0) means we can expand

$$\xi = O_1\eta^1 + \overline{O}_1\eta^{\overline{1}} + O_2\eta^2 + \overline{O}_2\eta^{\overline{2}} + O_3\eta^3 + \overline{O}_3\eta^{\overline{3}}, \qquad\qquad (3.70)$$

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for some $O \in C^\infty(B_4^{(1)}, \mathbb{C})$. We incorporate the expressions (3.69) and (3.70) into our equation (3.67) for $\mathrm{d}\psi$, which we append to our list of completely determined structure equations

$$\mathrm{d}\eta^0 = -2\tau \wedge \eta^0 + \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}\eta^2 \wedge \eta^{\overline{2}},$$

$$\mathrm{d}\eta^1 = -\gamma^1 \wedge \eta^0 - (\tau + \mathbf{i}\varrho) \wedge \eta^1 + \epsilon\eta^3 \wedge \eta^{\overline{2}} + F^1\eta^{\overline{1}} \wedge \eta^2,$$

$$\mathrm{d}\eta^2 = -\gamma^2 \wedge \eta^0 - (\tau + \mathbf{i}\varsigma) \wedge \eta^2 + \eta^3 \wedge \eta^{\overline{1}} + F^2\eta^{\overline{2}} \wedge \eta^1,$$

$$\mathrm{d}\eta^3 = -\mathbf{i}\gamma^2 \wedge \eta^1 - \mathbf{i}\gamma^1 \wedge \eta^2 - (\mathbf{i}\varrho + \mathbf{i}\varsigma) \wedge \eta^3 + T_{\overline{1}}^3\eta^{\overline{1}} \wedge \eta^0 + T_{\overline{2}}^3\eta^{\overline{2}} \wedge \eta^0 + F_1^3\eta^{\overline{2}} \wedge \eta^1 + F_2^3\eta^{\overline{1}} \wedge \eta^2,$$

$$\mathrm{d}\tau = -\psi \wedge \eta^0 + \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2,$$

$$\mathbf{i}\mathrm{d}\varrho = -\tfrac{3\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{3\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^1F^2\eta^{\overline{2}} \wedge \eta^{\overline{1}} + \overline{F}^1\overline{F}^2\eta^1 \wedge \eta^2$$

$$+ |F^1|^2\eta^{\overline{2}} \wedge \eta^2 - |F^2|^2\eta^{\overline{1}} \wedge \eta^1 + (Q_{\overline{1}}^1\eta^{\overline{1}} - \overline{Q}_{\overline{1}}^1\eta^1 + Q_{\overline{2}}^1\eta^{\overline{2}} - \overline{Q}_{\overline{2}}^1\eta^2) \wedge \eta^0$$

$$+ R\eta^{\overline{2}} \wedge \eta^1 + \overline{R}\eta^{\overline{1}} \wedge \eta^2,$$

$$\mathbf{i}\mathrm{d}\varsigma = \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} + \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3 + F^1F^2\eta^{\overline{1}} \wedge \eta^{\overline{2}} + \overline{F}^1\overline{F}^2\eta^2 \wedge \eta^1$$

$$+ |F^2|^2\eta^{\overline{1}} \wedge \eta^1 - |F^1|^2\eta^{\overline{2}} \wedge \eta^2 + (Q_{\overline{1}}^2\eta^{\overline{1}} - \overline{Q}_{\overline{1}}^2\eta^1 + Q_{\overline{2}}^2\eta^{\overline{2}} - \overline{Q}_{\overline{2}}^2\eta^2) \wedge \eta^0$$

$$+ S\eta^{\overline{2}} \wedge \eta^1 + \overline{S}\eta^{\overline{1}} \wedge \eta^2,$$

$$\mathrm{d}\gamma^1 = -\psi \wedge \eta^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3 + \mathbf{i}F_2^3\gamma^{\overline{1}} \wedge \eta^0 + F^1\gamma^{\overline{1}} \wedge \eta^2 - F^1\gamma^2 \wedge \eta^{\overline{1}}$$

$$- \epsilon T_{\overline{1}}^3\eta^{\overline{1}} \wedge \eta^2 + \epsilon\mathbf{i}T_{\overline{1}}^3\eta^{\overline{3}} \wedge \eta^0 + (\mathbf{i}p_0\eta^1 + P_{0\overline{1}}^1\eta^{\overline{1}} + P_{0\overline{2}}^1\eta^{\overline{2}} + \mathbf{i}(\overline{Q}_{\overline{2}}^1 + \overline{Q}_{\overline{2}}^2)\eta^3) \wedge \eta^0$$

$$+ (Q_{\overline{1}}^1\eta^{\overline{1}} + Q_{\overline{2}}^1\eta^{\overline{2}} - \overline{Q}_{\overline{2}}^1\eta^2) \wedge \eta^1 + (P_{20}^1\eta^0 + P_{2\overline{1}}^1\eta^{\overline{1}}) \wedge \eta^2,$$

$$\mathrm{d}\gamma^2 = -\psi \wedge \eta^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3 + \mathbf{i}F_1^3\gamma^{\overline{2}} \wedge \eta^0 - F^2\gamma^1 \wedge \eta^{\overline{2}} + F^2\gamma^{\overline{2}} \wedge \eta^1$$

$$- T_{\overline{2}}^3\eta^{\overline{2}} \wedge \eta^1 + \mathbf{i}T_{\overline{2}}^3\eta^{\overline{3}} \wedge \eta^0 + (P_{0\overline{1}}^2\eta^{\overline{1}} + \mathbf{i}p_0\eta^2 + P_{0\overline{2}}^2\eta^{\overline{2}} + \mathbf{i}(\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2)\eta^3) \wedge \eta^0$$

$$+ (Q_{\overline{1}}^2\eta^{\overline{1}} - \overline{Q}_{\overline{1}}^2\eta^1 + Q_{\overline{2}}^2\eta^{\overline{2}}) \wedge \eta^2 + (P_{10}^2\eta^0 + P_{1\overline{2}}^2\eta^{\overline{2}}) \wedge \eta^1,$$

$$\mathrm{d}\psi = -2\psi \wedge \tau + \mathbf{i}\gamma^1 \wedge \gamma^{\overline{1}} + \epsilon\mathbf{i}\gamma^2 \wedge \gamma^{\overline{2}} + (O_1\eta^1 + \overline{O}_1\eta^{\overline{1}} + O_2\eta^2 + \overline{O}_2\eta^{\overline{2}} + O_3\eta^3 + \overline{O}_3\eta^{\overline{3}}) \wedge \eta^0$$

$$- \tfrac{1}{2}(\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2 + \mathbf{i}\overline{F}_1^3F^2)\gamma^1 \wedge \eta^0 - \tfrac{1}{2}(\overline{Q}_{\overline{2}}^2 + \overline{Q}_{\overline{2}}^1 + \mathbf{i}F^1\overline{F}_2^3)\gamma^2 \wedge \eta^0$$

$$- \tfrac{1}{2}(Q_{\overline{1}}^1 + Q_{\overline{1}}^2 - \mathbf{i}F_1^3\overline{F}^2)\gamma^{\overline{1}} \wedge \eta^0 - \tfrac{1}{2}(Q_{\overline{2}}^2 + Q_{\overline{2}}^1 - \mathbf{i}\overline{F}^1F_2^3)\gamma^{\overline{2}} \wedge \eta^0 + \tfrac{\mathbf{i}}{2}(P_{0\overline{2}}^1 - \epsilon P_{0\overline{1}}^2)\eta^{\overline{1}} \wedge \eta^{\overline{2}}$$

$$+ \tfrac{\mathbf{i}}{2}(\overline{P}_{0\overline{2}}^1 - \epsilon\overline{P}_{0\overline{1}}^2)\eta^2 \wedge \eta^1 + \tfrac{\mathbf{i}}{2}(P_{20}^1 + \epsilon\overline{P}_{10}^2)\eta^2 \wedge \eta^{\overline{1}} + \tfrac{\mathbf{i}}{2}(\epsilon P_{10}^2 + \overline{P}_{20}^1)\eta^1 \wedge \eta^{\overline{2}}$$

$$+ \epsilon\tfrac{1}{2}(Q_{\overline{1}}^1 + Q_{\overline{1}}^2)\eta^{\overline{3}} \wedge \eta^2 + \epsilon\tfrac{1}{2}(\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2)\eta^3 \wedge \eta^{\overline{2}} + \tfrac{1}{2}(\overline{Q}_{\overline{2}}^1 + \overline{Q}_{\overline{2}}^2)\eta^3 \wedge \eta^{\overline{1}} + \tfrac{1}{2}(Q_{\overline{2}}^1 + Q_{\overline{2}}^2)\eta^{\overline{3}} \wedge \eta^1$$

$$+ \tfrac{1}{2}\overline{F}_2^3\gamma^1 \wedge \eta^1 + \tfrac{1}{2}F_2^3\gamma^{\overline{1}} \wedge \eta^{\overline{1}} + \epsilon\tfrac{1}{2}\overline{F}_1^3\gamma^2 \wedge \eta^2 + \epsilon\tfrac{1}{2}F_1^3\gamma^{\overline{2}} \wedge \eta^{\overline{2}}$$

$$+ \epsilon\tfrac{1}{2}\overline{T}_{\overline{1}}^3\eta^3 \wedge \eta^1 + \epsilon\tfrac{1}{2}T_{\overline{1}}^3\eta^{\overline{3}} \wedge \eta^{\overline{1}} + \epsilon\tfrac{1}{2}\overline{T}_{\overline{2}}^3\eta^3 \wedge \eta^2 + \epsilon\tfrac{1}{2}T_{\overline{2}}^3\eta^{\overline{3}} \wedge \eta^{\overline{2}}.$$

$$(3.71)$$

Let $\pi := \underline{\pi} \circ \hat{\pi}$ so we have the bundle $\pi : B_4^{(1)} \to M$. At this point, the coframing of $B_4^{(1)}$ given by the five $\mathbb{R}$-valued forms $\eta^0, \tau, \varrho, \varsigma, \psi$ and the real and imaginary parts of the five $\mathbb{C}$-valued forms $\eta^1, \eta^2, \eta^3, \gamma^1, \gamma^2$ is uniquely and globally determined by the structure equations (3.71). Thus, this coframing constitutes a solution in the sense of E. Cartan to the equivalence problem for 7-dimensional, 2-nondegenerate CR manifolds whose cubic form is of conformal unitary type.

4. THE PARALLELISM

4.1 Homogeneous Model

Consider $\mathbb{C}^4$ with its standard basis $\underline{\mathbf{v}} = (\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4)$ of column vectors and corresponding complex, linear coordinates z^1, z^2, z^3, z^4. A basis $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ of column vectors for $\mathbb{C}^4$ will be called an *oriented frame* if

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \wedge \mathbf{v}_4 = \underline{\mathbf{v}}_1 \wedge \underline{\mathbf{v}}_2 \wedge \underline{\mathbf{v}}_3 \wedge \underline{\mathbf{v}}_4. \tag{4.1}$$

Let $B_{\mathbb{C}}^{(1)}$ denote the set of oriented frames, and observe that fixing an identity element $\underline{\mathbf{v}}$ determines an isomorphism $B_{\mathbb{C}}^{(1)} \cong SL_4\mathbb{C}$ whereby the oriented frame $\mathbf{v}$ is identified with the 4×4 matrix $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$. If $Gr(2,4) \subset \mathbb{P}(\Lambda^2\mathbb{C}^4)$ denotes the Grassmannian manifold of 2-planes in $\mathbb{C}^4$, then $B_{\mathbb{C}}^{(1)}$ fibers over $Gr(2,4)$ via the projection map

$$\pi(\mathbf{v}) = [\![\mathbf{v}_1 \wedge \mathbf{v}_2]\!],$$

where the bold brackets denote the projective equivalence class à la Plücker embedding. This fibration exhibits $Gr(2,4)$ as the homogeneous quotient of $SL_4\mathbb{C}$ by the parabolic subgroup $P \subset SL_4\mathbb{C}$ represented as all matrices of the form

$$P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix},$$

i.e., the stabilizer subgroup of the plane spanned by $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$.

Let $\epsilon, \delta_\epsilon$ be as in §2.4, and introduce a Hermitian inner product h of signature $(2 + \delta_\epsilon, 2 - \delta_\epsilon)$ on $\mathbb{C}^4$ given in our linear coordinates by

$$h(z, w) = z^1\overline{w}^4 + z^4\overline{w}^1 - \epsilon z^2\overline{w}^2 + z^3\overline{w}^3.$$

Now $SU_\star := SU(2 + \delta_\epsilon, 2 - \delta_\epsilon) \subset SL_4\mathbb{C}$ denotes the subgroup $\{A \in SL_4\mathbb{C} \mid h(Az, Aw) =$

$h(z, w) \; \forall z, w \in \mathbb{C}^4\}$, and $Gr(2, 4)$ decomposes into $SU_\star$ orbits as follows. Let $\Pi \in Gr(2, 4)$. In the $SU(2, 2)$ case, $h|_\Pi$ has one of the signatures $(2, 0), (0, 2), (1, 1), (1, 0), (0, 1), (0, 0)$. In the $SU(3, 1)$ case, $h|_\Pi$ has one of the signatures $(2, 0), (1, 1), (1, 0)$. In both cases, we let $M_\star$ denote $SU_\star \cdot [\![\underline{v}_1 \wedge \underline{v}_2]\!]$, which is an orbit of codimension-one in $Gr(2, 4)$ where $h|_\Pi$ has signature $(1, 0)$.

An oriented frame $\mathbf{v} \in B_{\mathbb{C}}^{(1)}$ will be called a *Hermitian frame* if

$$[h(\mathbf{v}_i, \mathbf{v}_j)]_{i,j=1}^4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4.2}$$

In particular, $\underline{v}$ is a Hermitian frame. Let $B^{(1)} \subset B_{\mathbb{C}}^{(1)}$ be the subset of Hermitian frames, and note that fixing $\underline{v}$ once again determines an isomorphism $B^{(1)} \cong SU_\star$ in the same manner as before. The most general transformation of $\underline{v}$ which preserves the 2-plane $[\![\underline{v}_1 \wedge \underline{v}_2]\!] \in Gr(2, 4)$ and yields a new Hermitian frame $\mathbf{v}$ is given by

$$\mathbf{v}_1 = \tfrac{1}{t} \mathbf{e}^{\mathrm{i}/4(-r+s)} \underline{v}_1,$$

$$\mathbf{v}_2 = c^2 \mathbf{e}^{-\mathrm{i}/4(r+3s)} \underline{v}_1 + \mathbf{e}^{-\mathrm{i}/4(r+3s)} \underline{v}_2,$$

$$\mathbf{v}_3 = -\bar{c}^1 \mathbf{e}^{\mathrm{i}/4(3r+s)} \underline{v}_1 + \mathbf{e}^{\mathrm{i}/4(3r+s)} \underline{v}_3,$$

$$\mathbf{v}_4 = t\mathbf{e}^{\mathrm{i}/4(-r+s)} (\mathrm{i}y - \tfrac{1}{2}(|c^1|^2 - \epsilon|c^2|^2)) \underline{v}_1 + \epsilon \bar{c}^2 t \mathbf{e}^{\mathrm{i}/4(-r+s)} \underline{v}_2 + c^1 t \mathbf{e}^{\mathrm{i}/4(-r+s)} \underline{v}_3 + t\mathbf{e}^{\mathrm{i}/4(-r+s)} \underline{v}_4,$$

for $r, s, t, y \in \mathbb{R}$ ($t \neq 0$) and $c^1, c^2 \in \mathbb{C}$. Thus we see that the eight-dimensional Lie group $P_\star := P \cap SU_\star$ is parametrized by

$$\begin{bmatrix} \tfrac{1}{t}\mathbf{e}^{\mathrm{i}/4(-r+s)} & c^2\mathbf{e}^{-\mathrm{i}/4(r+3s)} & -\bar{c}^1\mathbf{e}^{\mathrm{i}/4(3r+s)} & t\mathbf{e}^{\mathrm{i}/4(-r+s)}(\mathrm{i}y - \tfrac{1}{2}(|c^1|^2 - \epsilon|c^2|^2)) \\ 0 & \mathbf{e}^{-\mathrm{i}/4(r+3s)} & 0 & \epsilon\bar{c}^2 t\mathbf{e}^{\mathrm{i}/4(-r+s)} \\ 0 & 0 & \mathbf{e}^{\mathrm{i}/4(3r+s)} & c^1 t\mathbf{e}^{\mathrm{i}/4(-r+s)} \\ 0 & 0 & 0 & t\mathbf{e}^{\mathrm{i}/4(-r+s)} \end{bmatrix}. \tag{4.3}$$

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The restriction of the projection π to $B^{(1)}$ now determines a fibration over our model space $M_\star$ by which we realize $M_\star$ as the homogeneous quotient $SU_\star/P_\star$. Observe that our parametrization of $P_\star$ may be decomposed into the product $P_\star = P_\star^2 P_\star^1 P_\star^0$ where the factors are matrices of the form

$$
P_\star^2 = \begin{bmatrix} 1 & 0 & 0 & \mathbf{i}y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad
P_\star^1 = \begin{bmatrix} 1 & c^2 & -\bar{c}^1 & -\frac{1}{2}(|c^1|^2 - \epsilon|c^2|^2) \\ 0 & 1 & 0 & \epsilon\bar{c}^2 \\ 0 & 0 & 1 & c^1 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
P_\star^0 = \begin{bmatrix} \frac{1}{t}\mathbf{e}^{\mathbf{i}/4(-r+s)} & 0 & 0 & 0 \\ 0 & \mathbf{e}^{-\mathbf{i}/4(r+3s)} & 0 & 0 \\ 0 & 0 & \mathbf{e}^{\mathbf{i}/4(3r+s)} & 0 \\ 0 & 0 & 0 & t\mathbf{e}^{\mathbf{i}/4(-r+s)} \end{bmatrix},
$$

(4.4)

with matrix entries as above. Each of $P_\star^0, P_\star^2$, and the product $P_\star^2 P_\star^1$ define subgroups of $SU_\star$, and there is a corresponding tower of fibrations

$$
\begin{array}{ccc}
P_\star^2 & \longrightarrow & SU_\star \\
& & \downarrow \\
(P_\star^2 P_\star^1)/P_\star^2 & \longrightarrow & SU_\star/P_\star^2 \\
& & \downarrow \\
P_\star^0 & \longrightarrow & SU_\star/(P_\star^2 P_\star^1) \\
& & \downarrow \\
& & SU_\star/P_\star
\end{array}
. \qquad (4.5)
$$

The four vector-valued functions $B^{(1)} \to \mathbb{C}^4$ given by $\mathbf{v} \mapsto \mathbf{v}_j$ ($1 \le j \le 4$) may be differentiated to obtain one-forms $\omega_j^i \in \Omega^1(B^{(1)}, \mathbb{C})$ which we express by

$$
\mathrm{d}\mathbf{v}_j = \mathbf{v}_i \omega_j^i,
$$

so that $\omega := [\omega_j^i]$ is the Maurer-Cartan form of $SU_\star$. Differentiating (4.1) will show that $\mathrm{trace}(\omega) =$

0, while differentiating (4.2) reveals

$$
\begin{bmatrix}
\omega_{\bar{1}}^{\bar{4}} & -\epsilon\omega_{\bar{1}}^{\bar{2}} & \omega_{\bar{1}}^{\bar{3}} & \omega_{\bar{1}}^{\bar{1}} \\[1em]
\omega_{\bar{2}}^{\bar{4}} & -\epsilon\omega_{\bar{2}}^{\bar{2}} & \omega_{\bar{2}}^{\bar{3}} & \omega_{\bar{2}}^{\bar{1}} \\[1em]
\omega_{\bar{3}}^{\bar{4}} & -\epsilon\omega_{\bar{3}}^{\bar{2}} & \omega_{\bar{3}}^{\bar{3}} & \omega_{\bar{3}}^{\bar{1}} \\[1em]
\omega_{\bar{4}}^{\bar{4}} & -\epsilon\omega_{\bar{4}}^{\bar{2}} & \omega_{\bar{4}}^{\bar{3}} & \omega_{\bar{4}}^{\bar{1}}
\end{bmatrix}
+
\begin{bmatrix}
\omega_1^4 & \omega_2^4 & \omega_3^4 & \omega_4^4 \\[1em]
-\epsilon\omega_1^2 & -\epsilon\omega_2^2 & -\epsilon\omega_3^2 & -\epsilon\omega_4^2 \\[1em]
\omega_1^3 & \omega_2^3 & \omega_3^3 & \omega_4^3 \\[1em]
\omega_1^1 & \omega_2^1 & \omega_3^1 & \omega_4^1
\end{bmatrix}
= 0,
$$

which is simply to say that ω takes values in the Lie algebra $\mathfrak{su}_\star$ of $SU_\star$. These conditions show that if we let

$$
\eta^0 := -\mathrm{Im}(\omega_1^4), \qquad \eta^1 := \omega_1^3, \qquad\qquad \eta^2 := \omega_2^4, \qquad \eta^3 := \omega_2^3, \qquad \tau := \mathrm{Re}(\omega_1^1),
$$

$$
\mathbf{i}\varrho := \tfrac{1}{2}(3\omega_3^3 + \omega_2^2), \quad \mathbf{i}\varsigma := -\tfrac{1}{2}(3\omega_2^2 + \omega_3^3), \qquad \mathbf{i}\gamma^1 := \omega_4^3, \qquad -\mathbf{i}\gamma^2 := \omega_2^1, \qquad \psi := -\mathrm{Im}(\omega_4^1),
$$

then we can write

$$
\omega =
\begin{bmatrix}
-\tau - \mathbf{i}\tfrac{1}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma & -\mathbf{i}\gamma^2 & -\mathbf{i}\overline{\gamma^1} & -\mathbf{i}\psi \\[1.2em]
-\epsilon\overline{\eta^2} & -\mathbf{i}\tfrac{1}{4}\varrho - \mathbf{i}\tfrac{3}{4}\varsigma & \epsilon\overline{\eta^3} & -\epsilon\mathbf{i}\overline{\gamma^2} \\[1.2em]
\eta^1 & \eta^3 & \mathbf{i}\tfrac{3}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma & \mathbf{i}\gamma^1 \\[1.2em]
-\mathbf{i}\eta^0 & \eta^2 & \overline{\eta^1} & \tau - \mathbf{i}\tfrac{1}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma
\end{bmatrix},
\qquad (4.6)
$$

and the $SU_\star$ Maurer-Cartan equations $d\omega + \omega \wedge \omega = 0$ read

$$d\eta^0 = -2\tau \wedge \eta^0 + \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}\eta^2 \wedge \eta^{\overline{2}},$$

$$d\eta^1 = -\gamma^1 \wedge \eta^0 - (\tau + \mathbf{i}\varrho) \wedge \eta^1 + \epsilon\eta^3 \wedge \eta^{\overline{2}},$$

$$d\eta^2 = -\gamma^2 \wedge \eta^0 - (\tau + \mathbf{i}\varsigma) \wedge \eta^2 + \eta^3 \wedge \eta^{\overline{1}},$$

$$d\eta^3 = -\mathbf{i}\gamma^2 \wedge \eta^1 - \mathbf{i}\gamma^1 \wedge \eta^2 - (\mathbf{i}\varrho + \mathbf{i}\varsigma) \wedge \eta^3,$$

$$d\tau = -\psi \wedge \eta^0 + \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2,$$

$$\mathbf{i}d\varrho = -\tfrac{3\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{3\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \epsilon\tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} + \epsilon\tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3,$$

$$\mathbf{i}d\varsigma = \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} + \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \epsilon\tfrac{3\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 + \epsilon\eta^{\overline{3}} \wedge \eta^3,$$

$$d\gamma^1 = -\psi \wedge \eta^1 + (\tau - \mathbf{i}\varrho) \wedge \gamma^1 - \epsilon\gamma^{\overline{2}} \wedge \eta^3,$$

$$d\gamma^2 = -\psi \wedge \eta^2 + (\tau - \mathbf{i}\varsigma) \wedge \gamma^2 - \gamma^{\overline{1}} \wedge \eta^3,$$

$$d\psi = -2\psi \wedge \tau + \mathbf{i}\gamma^1 \wedge \gamma^{\overline{1}} + \epsilon\mathbf{i}\gamma^2 \wedge \gamma^{\overline{2}}.$$

(4.7)

Observe that the equations (4.7) show

$$d(\psi - 2\tau + \eta^0) = (\psi - 2\tau + \eta^0) \wedge (\eta^0 - \psi) + \mathbf{i}(\gamma^1 - \eta^1) \wedge (\gamma^{\overline{1}} - \eta^{\overline{1}}) + \epsilon\mathbf{i}(\gamma^2 - \eta^2) \wedge (\gamma^{\overline{2}} - \eta^{\overline{2}}),$$

$$d(\gamma^1 - \eta^1) = -(\psi - 2\tau + \eta^0) \wedge \eta^1 + (\gamma^1 - \eta^1) \wedge \eta^0 + (\tau - \mathbf{i}\varrho) \wedge (\gamma^1 - \eta^1) - \epsilon(\gamma^{\overline{2}} - \eta^{\overline{2}}) \wedge \eta^3,$$

$$d(\gamma^2 - \eta^2) = -(\psi - 2\tau + \eta^0) \wedge \eta^2 + (\gamma^2 - \eta^2) \wedge \eta^0 + (\tau - \mathbf{i}\varsigma) \wedge (\gamma^2 - \eta^2) - (\gamma^{\overline{1}} - \eta^{\overline{1}}) \wedge \eta^3,$$

which proves that the Pfaffian system $\mathcal{I} := \{\psi - 2\tau + \eta^0, \gamma^1 - \eta^1, \gamma^2 - \eta^2, \gamma^{\overline{1}} - \eta^{\overline{1}}, \gamma^{\overline{2}} - \eta^{\overline{2}}\}$ on $B^{(1)}$ is Frobenius. We let $B_{\mathcal{I}}$ denote the maximal integral manifold of $\mathcal{I}$ that contains $\underline{v}$, with $\iota : B_{\mathcal{I}} \hookrightarrow B^{(1)}$ as the inclusion. Then $\omega \in \Omega^1(B^{(1)}, \mathfrak{su}_\star)$ pulls back to

$$\iota^*\omega = \iota^* \begin{bmatrix} -\tau - \mathbf{i}\tfrac{1}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma & -\mathbf{i}\eta^2 & -\mathbf{i}\eta^{\overline{1}} & -\mathbf{i}(2\tau - \eta^0) \\[1.5em] -\epsilon\eta^{\overline{2}} & -\mathbf{i}\tfrac{1}{4}\varrho - \mathbf{i}\tfrac{3}{4}\varsigma & \epsilon\eta^{\overline{3}} & -\epsilon\mathbf{i}\eta^2 \\[1.5em] \eta^1 & \eta^3 & \mathbf{i}\tfrac{3}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma & \mathbf{i}\eta^1 \\[1.5em] -\mathbf{i}\eta^0 & \eta^2 & \eta^{\overline{1}} & \tau - \mathbf{i}\tfrac{1}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma \end{bmatrix} \in \Omega^1(B_{\mathcal{I}}, \mathfrak{su}_\star),$$

and in particular on $B_{\mathcal{I}}$ we have

$$\iota^* \mathrm{d}\omega + \iota^* \omega \wedge \iota^* \omega = 0. \tag{4.8}$$

Moreover, when restricted to the fibers of $\pi|_{B_{\mathcal{I}}} : B_{\mathcal{I}} \to M_\star$ (where the pullbacks of the η's vanish), (4.8) is exactly the Maurer-Cartan equations of the abelian subgroup $P_\star^0 \subset SU_\star$. By a theorem of E. Cartan ([IL03, Thm 1.6.10]), there exist local lifts $B_{\mathcal{I}} \to SU_\star$ by which the fibers of $B_{\mathcal{I}}$ are diffeomorphic to $P_\star^0$, and the fibration

$$
\begin{array}{ccc}
P_\star^0 & \longrightarrow & B_{\mathcal{I}} \\
& & \downarrow \\
& & M_\star
\end{array}
$$

corresponds to the lowest level of the tower (4.5).

Using our identifications $B^{(1)} \cong SU_\star$ and $B_{\mathcal{I}} \cong SU_\star/(P_\star^2 P_\star^1)$, we see that $B^{(1)}$ fibers over $B_{\mathcal{I}}$ as the $P_\star^2 P_\star^1$-orbits of Hermitian frames in $B_{\mathcal{I}}$. We therefore identify an intermediate bundle $B \cong SU_\star/P_\star^2$ as the $(P_\star^2 P_\star^1)/P_\star^2$-orbits ($P_\star^2$ is normal in $P_\star^2 P_\star^1$). The significance of B is that it corresponds to the bundle B_4 constructed in §3 when $M = M_\star$.

4.2   Bianchi Identities, Fundamental Invariants

We return to the bundle $\pi : B_4^{(1)} \to M$ as in §3. The coframing constructed therein is interpreted as a parallelism $\omega \in \Omega^1(B_4^{(1)}, \mathfrak{su}_\star)$ by writing ω as in (4.6). The structure equations (3.71) on $B_4^{(1)}$ are now summarized

$$\mathrm{d}\omega = -\omega \wedge \omega + C$$

where the curvature tensor $C \in \Omega^2(B_4^{(1)}, \mathfrak{su}_\star)$ may be written

$$
C = \begin{bmatrix}
C_1^1 & -\mathbf{i}C_2^1 & -\mathbf{i}\overline{C}_4^3 & -\mathbf{i}C_4^1 \\
-\epsilon\overline{F}^2\eta^2 \wedge \eta^{\overline{1}} & C_2^2 & \epsilon\overline{C}_2^3 & -\epsilon\mathbf{i}\overline{C}_2^1 \\
F^1\eta^{\overline{1}} \wedge \eta^2 & C_2^3 & C_3^3 & \mathbf{i}C_4^3 \\
0 & F^2\eta^{\overline{2}} \wedge \eta^1 & \overline{F}^1\eta^1 \wedge \eta^{\overline{2}} & C_1^1
\end{bmatrix}, \tag{4.9}
$$

for $C_j^i \in \Omega^2(B_4^{(1)}, \mathbb{C})$ given by

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$$C_2^3 = T_{\overline{1}}^3 \eta^{\overline{1}} \wedge \eta^0 + T_{\overline{2}}^3 \eta^{\overline{2}} \wedge \eta^0 + F_1^3 \eta^{\overline{2}} \wedge \eta^1 + F_2^3 \eta^{\overline{1}} \wedge \eta^2,$$

$$C_1^1 = \tfrac{1}{4}(Q_{\overline{1}}^1 - Q_{\overline{1}}^2)\eta^0 \wedge \eta^{\overline{1}} + \tfrac{1}{4}(Q_{\overline{2}}^1 - Q_{\overline{2}}^2)\eta^0 \wedge \eta^{\overline{2}} + \tfrac{1}{4}(\overline{Q}_{\overline{1}}^1 - \overline{Q}_{\overline{1}}^2)\eta^1 \wedge \eta^0 + \tfrac{1}{4}(\overline{Q}_{\overline{2}}^1 - \overline{Q}_{\overline{2}}^2)\eta^2 \wedge \eta^0$$
$$+ \tfrac{1}{2}F^1 F^2 \eta^{\overline{1}} \wedge \eta^{\overline{2}} - \tfrac{1}{2}\overline{F}^1 \overline{F}^2 \eta^1 \wedge \eta^2 + \tfrac{1}{2}|F^1|^2 \eta^2 \wedge \eta^{\overline{2}} - \tfrac{1}{2}|F^2|^2 \eta^1 \wedge \eta^{\overline{1}}$$
$$+ \tfrac{1}{4}(R - S)\eta^1 \wedge \eta^{\overline{2}} + \tfrac{1}{4}(\overline{R} - \overline{S})\eta^2 \wedge \eta^{\overline{1}},$$

$$C_2^2 = \tfrac{1}{4}(Q_{\overline{1}}^1 + 3Q_{\overline{1}}^2)\eta^0 \wedge \eta^{\overline{1}} + \tfrac{1}{4}(Q_{\overline{2}}^1 + 3Q_{\overline{2}}^2)\eta^0 \wedge \eta^{\overline{2}} + \tfrac{1}{4}(\overline{Q}_{\overline{1}}^1 + 3\overline{Q}_{\overline{1}}^2)\eta^1 \wedge \eta^0 + \tfrac{1}{4}(\overline{Q}_{\overline{2}}^1 + 3\overline{Q}_{\overline{2}}^2)\eta^2 \wedge \eta^0$$
$$- \tfrac{1}{2}F^1 F^2 \eta^{\overline{1}} \wedge \eta^{\overline{2}} + \tfrac{1}{2}\overline{F}^1 \overline{F}^2 \eta^1 \wedge \eta^2 - \tfrac{1}{2}|F^1|^2 \eta^2 \wedge \eta^{\overline{2}} + \tfrac{1}{2}|F^2|^2 \eta^1 \wedge \eta^{\overline{1}}$$
$$+ \tfrac{1}{4}(R + 3S)\eta^1 \wedge \eta^{\overline{2}} + \tfrac{1}{4}(\overline{R} + 3\overline{S})\eta^2 \wedge \eta^{\overline{1}},$$

$$C_3^3 = -\tfrac{1}{4}(3Q_{\overline{1}}^1 + Q_{\overline{1}}^2)\eta^0 \wedge \eta^{\overline{1}} - \tfrac{1}{4}(3Q_{\overline{2}}^1 + Q_{\overline{2}}^2)\eta^0 \wedge \eta^{\overline{2}} + \tfrac{1}{4}(3\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2)\eta^0 \wedge \eta^1 + \tfrac{1}{4}(3\overline{Q}_{\overline{2}}^1 + \overline{Q}_{\overline{2}}^2)\eta^0 \wedge \eta^2$$
$$- \tfrac{1}{2}F^1 F^2 \eta^{\overline{1}} \wedge \eta^{\overline{2}} + \tfrac{1}{2}\overline{F}^1 \overline{F}^2 \eta^1 \wedge \eta^2 - \tfrac{1}{2}|F^1|^2 \eta^2 \wedge \eta^{\overline{2}} + \tfrac{1}{2}|F^2|^2 \eta^1 \wedge \eta^{\overline{1}}$$
$$- \tfrac{1}{4}(3R + S)\eta^1 \wedge \eta^{\overline{2}} - \tfrac{1}{4}(3\overline{R} + \overline{S})\eta^2 \wedge \eta^{\overline{1}},$$

$$C_2^1 = \mathbf{i}F_1^3 \gamma^{\overline{2}} \wedge \eta^0 - F^2 \gamma^1 \wedge \eta^{\overline{2}} + F^2 \gamma^{\overline{2}} \wedge \eta^1 - T_{\overline{2}}^3 \eta^{\overline{2}} \wedge \eta^{\overline{1}} + \mathbf{i}T_{\overline{2}}^3 \eta^{\overline{3}} \wedge \eta^0$$
$$+ (P_{0\overline{1}}^2 \eta^{\overline{1}} + \mathbf{i}p_0 \eta^2 + P_{0\overline{2}}^2 \eta^{\overline{2}} + \mathbf{i}(\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2)\eta^3) \wedge \eta^0 + (P_{10}^2 \eta^0 + P_{1\overline{2}}^2 \eta^{\overline{2}}) \wedge \eta^1$$
$$+ (Q_{\overline{1}}^2 \eta^{\overline{1}} - \overline{Q}_{\overline{1}}^2 \eta^1 + Q_{\overline{2}}^2 \eta^{\overline{2}}) \wedge \eta^2,$$

$$C_4^3 = \mathbf{i}F_2^3 \gamma^{\overline{1}} \wedge \eta^0 + F^1 \gamma^{\overline{1}} \wedge \eta^2 - F^1 \gamma^2 \wedge \eta^{\overline{1}} - \epsilon T_{\overline{1}}^3 \eta^{\overline{1}} \wedge \eta^{\overline{2}} + \epsilon \mathbf{i}T_{\overline{1}}^3 \eta^{\overline{3}} \wedge \eta^0$$
$$+ (\mathbf{i}p_0 \eta^1 + P_{0\overline{1}}^1 \eta^{\overline{1}} + P_{0\overline{2}}^1 \eta^{\overline{2}} + \mathbf{i}(\overline{Q}_{\overline{2}}^1 + \overline{Q}_{\overline{2}}^2)\eta^3) \wedge \eta^0 + (P_{20}^1 \eta^0 + P_{2\overline{1}}^1 \eta^{\overline{1}}) \wedge \eta^2$$
$$+ (Q_{\overline{1}}^1 \eta^{\overline{1}} + Q_{\overline{2}}^1 \eta^{\overline{2}} - \overline{Q}_{\overline{2}}^1 \eta^2) \wedge \eta^1,$$

$$C_4^1 = (O_1 \eta^1 + \overline{O}_1 \eta^{\overline{1}} + O_2 \eta^2 + \overline{O}_2 \eta^{\overline{2}} + O_3 \eta^3 + \overline{O}_3 \eta^{\overline{3}}) \wedge \eta^0 - \tfrac{1}{2}(\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2 + \mathbf{i}\overline{F}_1^3 F^2)\gamma^1 \wedge \eta^0$$
$$- \tfrac{1}{2}(\overline{Q}_{\overline{2}}^2 + \overline{Q}_{\overline{2}}^1 + \mathbf{i}F^1 \overline{F}_2^3)\gamma^2 \wedge \eta^0 - \tfrac{1}{2}(Q_{\overline{1}}^1 + Q_{\overline{1}}^2 - \mathbf{i}F_1^3 \overline{F}^2)\gamma^{\overline{1}} \wedge \eta^0 - \tfrac{1}{2}(Q_{\overline{2}}^2 + Q_{\overline{2}}^1 - \mathbf{i}\overline{F}^1 F_2^3)\gamma^{\overline{2}} \wedge \eta^0$$
$$+ \tfrac{\mathbf{i}}{2}(P_{0\overline{2}}^1 - \epsilon P_{0\overline{1}}^2)\eta^{\overline{1}} \wedge \eta^{\overline{2}} + \tfrac{\mathbf{i}}{2}(\overline{P}_{0\overline{2}}^1 - \epsilon \overline{P}_{0\overline{1}}^2)\eta^2 \wedge \eta^1 + \tfrac{\mathbf{i}}{2}(P_{20}^1 + \epsilon \overline{P}_{10}^2)\eta^2 \wedge \eta^{\overline{1}}$$

$$+ \tfrac{\mathbf{i}}{2}(\epsilon P_{10}^2 + \overline{P}_{20}^1)\eta^1 \wedge \eta^{\overline{2}} + \epsilon\tfrac{1}{2}(Q_{\overline{1}}^1 + Q_{\overline{1}}^2)\eta^{\overline{3}} \wedge \eta^2 + \epsilon\tfrac{1}{2}(\overline{Q}_{\overline{1}}^1 + \overline{Q}_{\overline{1}}^2)\eta^3 \wedge \eta^{\overline{2}} + \tfrac{1}{2}(\overline{Q}_{\overline{2}}^1 + \overline{Q}_{\overline{2}}^2)\eta^3 \wedge \eta^{\overline{1}}$$

$$+ \tfrac{1}{2}(Q_{\overline{2}}^1 + Q_{\overline{2}}^2)\eta^{\overline{3}} \wedge \eta^1 + \tfrac{1}{2}\overline{F}_2^3\gamma^1 \wedge \eta^1 + \tfrac{1}{2}F_2^3\gamma^{\overline{1}} \wedge \eta^{\overline{1}} + \epsilon\tfrac{1}{2}\overline{F}_1^3\gamma^2 \wedge \eta^2 + \epsilon\tfrac{1}{2}F_1^3\gamma^{\overline{2}} \wedge \eta^{\overline{2}}$$

$$+ \epsilon\tfrac{1}{2}\overline{T}_{\overline{1}}^3\eta^3 \wedge \eta^1 + \epsilon\tfrac{1}{2}T_{\overline{1}}^3\eta^{\overline{3}} \wedge \eta^{\overline{1}} + \epsilon\tfrac{1}{2}\overline{T}_{\overline{2}}^3\eta^3 \wedge \eta^2 + \epsilon\tfrac{1}{2}T_{\overline{2}}^3\eta^{\overline{3}} \wedge \eta^{\overline{2}}.$$

The coefficients which appear at lowest order are F^1, F^2. We find how they vary on $B_4^{(1)}$ by differentiating the structure equations

$$0 = \mathrm{d}(\mathrm{d}\eta^1)$$
$$= (\mathrm{d}F^1 - F^1(\tau - 2\mathbf{i}\varrho + \mathbf{i}\varsigma) + \epsilon\overline{F}^2\eta^3 + \epsilon F_2^3\eta^{\overline{2}} - \overline{R}\eta^1 - P_{2\overline{1}}^1\eta^0) \wedge \eta^{\overline{1}} \wedge \eta^2,$$

and similarly,

$$0 = \mathrm{d}(\mathrm{d}\eta^2)$$
$$= (\mathrm{d}F^2 - F^2(\tau + \mathbf{i}\varrho - 2\mathbf{i}\varsigma) + \overline{F}^1\eta^3 + F_1^3\eta^{\overline{1}} - S\eta^2 - P_{1\overline{2}}^2\eta^0) \wedge \eta^{\overline{2}} \wedge \eta^1.$$

Therefore, for some functions $f_{\overline{1}}^1, f_2^1, f_1^2, f_{\overline{2}}^2 \in C^\infty(B_4^{(1)}, \mathbb{C})$ we can write

$$\mathrm{d}F^1 = F^1(\tau - 2\mathbf{i}\varrho + \mathbf{i}\varsigma) - \epsilon\overline{F}^2\eta^3 - \epsilon F_2^3\eta^{\overline{2}} + \overline{R}\eta^1 + P_{2\overline{1}}^1\eta^0 + f_{\overline{1}}^1\eta^{\overline{1}} + f_2^1\eta^2,$$
$$\mathrm{d}F^2 = F^2(\tau + \mathbf{i}\varrho - 2\mathbf{i}\varsigma) - \overline{F}^1\eta^3 - F_1^3\eta^{\overline{1}} + S\eta^2 + P_{1\overline{2}}^2\eta^0 + f_1^2\eta^1 + f_{\overline{2}}^2\eta^{\overline{2}}. \tag{4.10}$$

Recall ([IL03, Prop B.3.3]) that a form $\alpha \in \Omega^\bullet(B_4^{(1)}, \mathbb{C})$ is π-basic if and only if α and $\mathrm{d}\alpha$ are π-semibasic. We consider the $\mathbb{R}$-valued semibasic forms

$$|F^1|^2\eta^0, \qquad\qquad\qquad |F^2|^2\eta^0, \tag{4.11}$$

and use (4.10) to calculate

$$\mathrm{d}(|F^1|^2\eta^0) = -(\overline{F}^1\overline{R} + \overline{f}_{\overline{1}}^1 F^1)\eta^0 \wedge \eta^1 - (F^1 R + f_{\overline{1}}^1\overline{F}^1)\eta^0 \wedge \eta^{\overline{1}} + \mathbf{i}|F^1|^2\eta^1 \wedge \eta^{\overline{1}} + \epsilon\overline{F}^1\overline{F}^2\eta^0 \wedge \eta^3$$
$$- (\overline{F}^1 f_2^1 - \epsilon\overline{F}_2^3 F^1)\eta^0 \wedge \eta^2 - (F^1\overline{f}_2^1 - \epsilon F_2^3\overline{F}^1)\eta^0 \wedge \eta^{\overline{2}} + \epsilon\mathbf{i}|F^1|^2\eta^2 \wedge \eta^{\overline{2}} + \epsilon F^1 F^2\eta^0 \wedge \eta^{\overline{3}},$$
$$\mathrm{d}(|F^2|^2\eta^0) = -(\overline{F}^2\overline{S} + \overline{f}_{\overline{1}}^2 F^2)\eta^0 \wedge \eta^2 - (F^2 S + f_{\overline{2}}^2\overline{F}^2)\eta^0 \wedge \eta^{\overline{2}} + \mathbf{i}|F^2|^2\eta^1 \wedge \eta^{\overline{1}} + \epsilon\overline{F}^1\overline{F}^2\eta^0 \wedge \eta^3$$
$$- (\overline{F}^2 f_1^2 - \epsilon\overline{F}_1^3 F^2)\eta^0 \wedge \eta^1 - (F^2\overline{f}_1^2 - \epsilon F_1^3\overline{F}^2)\eta^0 \wedge \eta^{\overline{1}} + \epsilon\mathbf{i}|F^2|^2\eta^2 \wedge \eta^{\overline{2}} + \epsilon F^1 F^2\eta^0 \wedge \eta^{\overline{3}}.$$

These are semibasic as well, so we've shown that the one-forms (4.11) on $B_4^{(1)}$ are the π-pullbacks of well-defined invariants on M.

Let us make a few more observations about the equations (4.10). First, they show that if F^1 or F^2 is locally constant on $B_4^{(1)}$, then they must locally vanish. Second, we see that if either of F^1, F^2 vanishes identically, the other must as well. By the same token, we will have

$$F_1^3 = F_2^3 = R = S = P_{2\bar{1}}^1 = P_{1\bar{2}}^2 = 0 \tag{4.12}$$

in this case. In fact, if either of $F^1, F^2 = 0$, we will show that every coefficient function in the curvature tensor C must vanish too. This will follow by differentiating more of the structure equations. We revisit

$$0 = \mathrm{d}^2\eta^3$$
$$= (\mathrm{d}T_{\bar{1}}^3 - T_{\bar{1}}^3(3\tau - 2\mathbf{i}\varrho - \mathbf{i}\varsigma) - F_2^3\gamma^2 - (Q_{\bar{1}}^1 + Q_{\bar{1}}^2)\eta^3 - \mathbf{i}P_{0\bar{1}}^2\eta^1) \wedge \eta^{\bar{1}} \wedge \eta^0$$
$$+ (\mathrm{d}T_{\bar{2}}^3 - T_{\bar{2}}^3(3\tau - \mathbf{i}\varrho - 2\mathbf{i}\varsigma) - F_1^3\gamma^1 - (Q_{\bar{2}}^1 + Q_{\bar{2}}^2)\eta^3 - \mathbf{i}P_{0\bar{2}}^1\eta^2) \wedge \eta^{\bar{2}} \wedge \eta^0$$
$$+ (\mathrm{d}F_1^3 - 2F_1^3(\tau - \mathbf{i}\varsigma) + 2\mathbf{i}F^2\gamma^1 - (R+S)\eta^3 + (\mathbf{i}(Q_{\bar{2}}^2 - Q_{\bar{2}}^1) - F_2^3\overline{F}^1)\eta^2 - (F_2^3F^2 + 2\mathbf{i}T_{\bar{2}}^3)\eta^{\bar{1}}) \wedge \eta^{\bar{2}} \wedge \eta^1$$
$$+ (\mathrm{d}F_2^3 - 2F_2^3(\tau - \mathbf{i}\varrho) + 2\mathbf{i}F^1\gamma^2 - (\overline{R}+\overline{S})\eta^3 + (\mathbf{i}(Q_{\bar{1}}^1 - Q_{\bar{1}}^2) - F_1^3\overline{F}^2)\eta^1 - (F_1^3F^1 + \epsilon 2\mathbf{i}T_{\bar{1}}^3)\eta^{\bar{2}}) \wedge \eta^{\bar{1}} \wedge \eta^2$$
$$+ (T_{\bar{2}}^3\overline{F}^2 - \mathbf{i}P_{0\bar{1}}^1)\eta^2 \wedge \eta^{\bar{1}} \wedge \eta^0 + (T_{\bar{1}}^3\overline{F}^1 - \mathbf{i}P_{0\bar{2}}^2)\eta^1 \wedge \eta^{\bar{2}} \wedge \eta^0. \tag{4.13}$$

Reducing (4.13) by η^0 and plugging in $F_1^3 = F_2^3 = 0$ implies $T_{\bar{1}}^3 = T_{\bar{2}}^3 = 0$ and $Q_{\bar{1}}^1 = Q_{\bar{1}}^2$, $Q_{\bar{2}}^2 = Q_{\bar{2}}^1$. Then, returning to the unreduced (4.13) and setting $T_{\bar{1}}^3 = T_{\bar{2}}^3 = 0$ will show

$$T_{\bar{1}}^3 = T_{\bar{2}}^3 = Q_{\bar{1}}^1 = Q_{\bar{2}}^1 = Q_{\bar{1}}^2 = Q_{\bar{2}}^2 = P_{0\bar{1}}^1 = P_{0\bar{2}}^1 = P_{0\bar{1}}^2 = P_{0\bar{2}}^2 = 0. \tag{4.14}$$

We assume that we have (4.12) and (4.14) as we now differentiate $\mathbf{i}\mathrm{d}\varrho$ and $\mathbf{i}\mathrm{d}\varsigma$;

$$0 = \mathrm{d}(\mathbf{i}\mathrm{d}\varrho)$$
$$= -3p_0\eta^0 \wedge \eta^1 \wedge \eta^{\bar{1}} + \epsilon p_0\eta^0 \wedge \eta^2 \wedge \eta^{\bar{2}}$$
$$+ \tfrac{\mathbf{i}}{2}(\epsilon P_{10}^2 + 3\overline{P}_{20}^1)\eta^0 \wedge \eta^1 \wedge \eta^{\bar{2}} - \tfrac{\mathbf{i}}{2}(\epsilon\overline{P}_{10}^2 + 3P_{20}^1)\eta^0 \wedge \eta^2 \wedge \eta^{\bar{1}},$$

and

$$0 = \mathrm{d}(\mathbf{id}\varsigma)$$

$$= p_0 \eta^0 \wedge \eta^1 \wedge \eta^{\overline{1}} - \epsilon 3 p_0 \eta^0 \wedge \eta^2 \wedge \eta^{\overline{2}}$$

$$- \tfrac{\mathbf{i}}{2}(\epsilon 3 P_{10}^2 + \overline{P}_{20}^1) \eta^0 \wedge \eta^1 \wedge \eta^{\overline{2}} + \tfrac{\mathbf{i}}{2}(\epsilon 3 \overline{P}_{10}^2 + P_{20}^1) \eta^0 \wedge \eta^2 \wedge \eta^{\overline{1}},$$

which together demonstrate

$$p_0 = P_{20}^1 = P_{10}^2 = 0. \tag{4.15}$$

Finally, we simply state that differentiating $\mathrm{d}\gamma^1$ and $\mathrm{d}\gamma^2$ will now show

$$O_1 = O_2 = O_3 = 0. \tag{4.16}$$

By (4.12),(4.14),(4.15), and (4.16), we have shown that $C = 0$ when one of (4.11) vanishes. In this case, the structure equations of M are exactly the Maurer-Cartan equations (4.7), and M is locally CR-equivalent to the homogeneous model $M_\star$.

4.3   Equivariance

Let us establish some general definitions which we will use to interpret the bundles $\hat{\pi} : B_4^{(1)} \to B_4$ and $\pi : B_4^{(1)} \to M$ constructed in §3. A reference for this material is [ČS09]. Let G be a Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ a Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, and $\exp : \mathfrak{h} \to H$ the exponential map. For each $g \in G$, G acts on itself isomorphically by conjugation $a \mapsto gag^{-1}$ $\forall a \in G$, which induces the adjoint representation $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ acting automorphically on $\mathfrak{g}$. By restriction of this adjoint action, $\mathfrak{g}$ is a representation of H as well.

Suppose we have a manifold M and a principal bundle $\pi : \mathcal{B} \to M$ with structure group H. For $h \in H$, we let $R_h : \mathcal{B} \to \mathcal{B}$ denote the right principal action of h on the fibers of $\mathcal{B}$. In particular, the vertical bundle $\ker \pi_* \subset T\mathcal{B}$ is trivialized by fundamental vector fields ζ_X associated to $X \in \mathfrak{h}$, where the value at $u \in \mathcal{B}$ of ζ_X is $\frac{d}{dt}\big|_{t=0} R_{\exp(tX)}(u)$. The bundle $\pi : \mathcal{B} \to M$ defines a *Cartan geometry* of type (G, H) if it admits a *Cartan connection*:

Definition 4.1 *A Cartan connection is a $\mathfrak{g}$-valued one form $\omega \in \Omega^1(\mathcal{B}, \mathfrak{g})$ which satisfies:*

- $\omega : T_u \mathcal{B} \to \mathfrak{g}$ *is a linear isomorphism for every $u \in \mathcal{B}$,*

- $\omega(\zeta_X) = X$ *for every* $X \in \mathfrak{h}$,

- $R_h^* \omega = Ad_{h^{-1}} \circ \omega$ *for every* $h \in H$.

The purpose of this section is to prove the following

Proposition 4.2 *For* $\mathcal{B} = B_4^{(1)}$ *and* $G = SU_\star$, *the bundles* $\hat{\pi} : B_4^{(1)} \to B_4$ *and* $\pi : B_4^{(1)} \to M$ *are principal bundles with structure groups isomorphic to* $H = P_\star^2$ *and* $H = P_\star$, *respectively – c.f.* §4.1. *The* $\mathfrak{su}_\star$-*valued parallelism* ω *constructed in the previous section defines a Cartan connection for the former bundle, but not the latter.*

By construction, ω satisfies the first property of a Cartan connection, and the fundamental vector fields are spanned by vertical vector fields dual to the pseudoconnection forms that are vertical for $\hat{\pi}$ or π, so it remains to determine if ω satisfies the final, equivariancy condition. In the process, we confirm the first statement of the proposition when we realize a local trivialization of the bundle $\pi : B_4^{(1)} \to M$ via those of the bundles $\hat{\pi} : B_4^{(1)} \to B_4$ and $\underline{\pi} : B_4 \to M$.

Let $\mathfrak{g}_4$ be the Lie algebra of G_4. We know that $G_4 \subset GL(V)$, so $\mathfrak{g}_4 \subset V \otimes V^*$ and we can define $\mathfrak{g}_4^{(1)}$ to be the kernel in $\mathfrak{g}_4 \otimes V^*$ of the skew-symmetrization map $V \otimes V^* \otimes V^* \to V \otimes \Lambda^2 V^*$. This abelian group parameterizes the ambiguity in the pseudoconnection forms on B_4 (c.f. [BGG03, §3.1.2]). In particular, if we write $\underline{\eta} \in \Omega^1(B_4, V)$ for the tautological form on B_4 and use underlines to indicate a coframing of B_4 which satisfies the structure equations (3.47), we have a local trivialization $B_4^{(1)} \cong \mathfrak{g}_4^{(1)} \times B_4$ as all coframings of B_4 which satisfy the structure equations:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & y & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\underline{\eta}^0 \\
\underline{\eta}^1 \\
\underline{\eta}^2 \\
\underline{\eta}^3 \\
\underline{\varrho} \\
\underline{\varsigma} \\
\underline{\tau} \\
\underline{\gamma}^1 \\
\underline{\gamma}^2
\end{bmatrix}.
\tag{4.17}
$$

We abbreviate the coframing (4.17) by $\eta_y \in B_4^{(1)}$, and we let η_+ denote the column vector (3.49)

of tautological forms on $B_4^{(1)}$. With this notation we can concisely say

$$\eta_+ = \hat{\pi}^* \eta_y.$$

For fixed $\check{y} \in \mathbb{R}$, let $\check{g} \in \mathfrak{g}_4^{(1)}$ be the group element represented by the matrix (4.17) where the fiber coordinate $y \in C^\infty(B_4^{(1)})$ equals $\check{y}$. The right principal $\mathfrak{g}_4^{(1)}$-action $R_{\check{g}} : B_4^{(1)} \to B_4^{(1)}$ is simply given by matrix multiplication

$$R_{\check{g}} : \eta_y \mapsto \check{g}^{-1} \eta_y = \eta_{y-\check{y}}.$$

Thus, the pullback $R_{\check{g}}^* : T_{\eta_{y-\check{y}}}^* B_4^{(1)} \to T_{\eta_y}^* B_4^{(1)}$ of the tautological forms along this principal action is also given by matrix multiplication

$$R_{\check{g}}^* \eta_+ = \check{g}^{-1} \eta_+.$$

More explicitly,

$$R_{\check{g}}^* \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \varrho \\ \varsigma \\ \tau \\ \gamma^1 \\ \gamma^2 \end{bmatrix} = \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \varrho \\ \varsigma \\ \tau - \check{y}\eta^0 \\ \gamma^1 - \check{y}\eta^1 \\ \gamma^2 - \check{y}\eta^2 \end{bmatrix}. \tag{4.18}$$

It remains to determine $R_{\check{g}}^* \psi$, for which we enlist the help of the structure equations (3.71) of $B_4^{(1)}$. We differentiate the equation

$$R_{\check{g}}^*(\tau) = \tau - \check{y}\eta^0$$

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and use (4.18) to conclude

$$-R^*_{\check{g}}(\psi) \wedge \eta^0 = -(\psi - 2\check{y}\tau) \wedge \eta^0,$$

whence we see that

$$R^*_{\check{g}}(\psi) \equiv \psi - 2\check{y}\tau \mod \{\eta^0\}.$$

Let us therefore write

$$R^*_{\check{g}}(\psi) = \psi - 2\check{y}\tau + a\eta^0$$

for some $a \in \mathbb{R}$ and differentiate again, this time reducing by $\eta^0, \eta^2, \eta^3, \eta^{\overline{2}}, \eta^{\overline{3}}$ to get

$$0 \equiv \tfrac{1}{2}(R^*_{\check{g}}(\overline{F}^3_2) - \overline{F}^3_2)\gamma^1 \wedge \eta^1 + \tfrac{1}{2}(R^*_{\check{g}}(\overline{F}^3_2) - \overline{F}^3_2)\gamma^{\overline{1}} \wedge \eta^{\overline{1}} - \mathbf{i}(a - \check{y}^2)\eta^1 \wedge \eta^{\overline{1}}$$
$$\mod \{\eta^0, \eta^2, \eta^3, \eta^{\overline{2}}, \eta^{\overline{3}}\}.$$

Thus we conclude

$$R^*_{\check{g}}(\psi) = \psi - 2\check{y}\tau + \check{y}^2\eta^0,$$

which along with (4.18) shows

$$R^*_{\check{g}}\omega = \begin{bmatrix} -(\tau - \check{y}\eta^0) - \mathbf{i}\tfrac{1}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma & -\mathbf{i}(\gamma^2 - \check{y}\eta^2) & -\mathbf{i}(\gamma^{\overline{1}} - \check{y}\eta^{\overline{1}}) & -\mathbf{i}(\psi - 2\check{y}\tau + \check{y}^2\eta^0) \\[1.5em] -\epsilon\eta^{\overline{2}} & -\mathbf{i}\tfrac{1}{4}\varrho - \mathbf{i}\tfrac{3}{4}\varsigma & \epsilon\eta^{\overline{3}} & -\epsilon\mathbf{i}(\gamma^{\overline{2}} - \check{y}\eta^{\overline{2}}) \\[1.5em] \eta^1 & \eta^3 & \mathbf{i}\tfrac{3}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma & \mathbf{i}(\gamma^1 - \check{y}\eta^1) \\[1.5em] -\mathbf{i}\eta^0 & \eta^2 & \eta^{\overline{1}} & (\tau - \check{y}\eta^0) - \mathbf{i}\tfrac{1}{4}\varrho + \mathbf{i}\tfrac{1}{4}\varsigma \end{bmatrix}.$$

$$(4.19)$$

It is clear that $\mathfrak{g}^{(1)}_4$ is isomorphic to $P^2_\star$ as they are both one-dimensional, abelian Lie groups.

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We formally define an isomorphism $\varphi : \mathfrak{g}_4^{(1)} \to P_\star^2$ by mapping the element represented by the inverse of the matrix (4.17) to the $P_\star^2$ matrix in (4.4). In particular,

$$
\varphi(\breve{g}^{-1}) = \begin{bmatrix} 1 & 0 & 0 & \mathbf{i}\breve{y} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

so it is straightforward to check that $\mathrm{Ad}_{\varphi(\breve{g}^{-1})} \circ \omega$ agrees with the matrix (4.19). Thus we have shown that $\hat{\pi} : B_4^{(1)} \to B_4$ is a principal $P_\star^2$-bundle for which $\omega \in \Omega^1(B_4^{(1)}, \mathfrak{su}_\star)$ is a Cartan connection.

Recall that the bundle $\underline{\pi} : B_4 \to M$ from §3.4 is locally trivialized as $B_4 \cong G_4 \times M$ by fixing a 4-adapted coframing $\theta_{\mathbb{1}}$ of M. This trivialization parameterizes local 4-adapted coframings by $g^{-1}\theta_{\mathbb{1}}$ where g^{-1} is the matrix (3.45). Furthermore, the tautological forms on B_4 have the local expression

$$
\begin{bmatrix} \underline{\eta}^0 \\ \underline{\eta}^1 \\ \underline{\eta}^2 \\ \underline{\eta}^3 \end{bmatrix} = \begin{bmatrix} t^2 & 0 & 0 & 0 \\ c^1 & t\mathbf{e}^{\mathbf{i}r} & 0 & 0 \\ c^2 & 0 & t\mathbf{e}^{\mathbf{i}s} & 0 \\ \frac{\mathbf{i}}{t^2}c^1c^2 & \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}r}c^2 & \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}s}c^1 & \mathbf{e}^{\mathbf{i}(r+s)} \end{bmatrix} \begin{bmatrix} \underline{\pi}^*\theta_{\mathbb{1}}^0 \\ \underline{\pi}^*\theta_{\mathbb{1}}^1 \\ \underline{\pi}^*\theta_{\mathbb{1}}^2 \\ \underline{\pi}^*\theta_{\mathbb{1}}^3 \end{bmatrix} ; \qquad \begin{array}{l} r, s, 0 \neq t \in C^\infty(B_4); \\ c^1, c^2 \in C^\infty(B_4, \mathbb{C}), \end{array}
$$

As such, the coframing η_y of B_4 in (4.17) above may be expanded

$$
\begin{bmatrix} t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c^1 & t\mathbf{e}^{\mathbf{i}r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c^2 & 0 & t\mathbf{e}^{\mathbf{i}s} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mathbf{i}}{t^2}c^1c^2 & \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}r}c^2 & \frac{\mathbf{i}}{t}\mathbf{e}^{\mathbf{i}s}c^1 & \mathbf{e}^{\mathbf{i}(r+s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ yt^2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ yc^1 & yt\mathbf{e}^{\mathbf{i}r} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ yc^2 & 0 & yt\mathbf{e}^{\mathbf{i}s} & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{\pi}^*\theta_{\mathbb{1}}^0 \\ \underline{\pi}^*\theta_{\mathbb{1}}^1 \\ \underline{\pi}^*\theta_{\mathbb{1}}^2 \\ \underline{\pi}^*\theta_{\mathbb{1}}^3 \\ \underline{\varrho} \\ \underline{\varsigma} \\ \underline{\tau} \\ \underline{\gamma}^1 \\ \underline{\gamma}^2 \end{bmatrix}, \qquad (4.20)
$$

and this defines a local trivialization of the bundle $\pi : B_4^{(1)} \to M$ as $B_4^{(1)} \cong G_4^{(1)} \times M$ where the

structure group $G_4^{(1)} \cong \mathfrak{g}_4^{(1)} \times G_4$ is parametrized as shown. We extend the isomorphism φ above to an isomorphism $G_4^{(1)} \to P_\star$ by mapping the inverse of the matrix (4.20) to the matrix (4.3). In this way we realize $\pi : B_4^{(1)} \to M$ as a principal $P_\star$-bundle over M.

We need not attempt to verify the equivariancy condition on this bundle; ω cannot be a Cartan connection for $\pi : B_4^{(1)} \to M$ since the curvature tensor C given by (4.9) is not π-semibasic; see [ČS09, Lem 1.5.1].

4.4   A Non-Flat Example

Recall from §4.2 that a necessary and sufficient condition for a 2-non-degenerate CR manifold M to be locally CR equivalent to the homogeneous model $M_\star$ is that the coefficients F^1, F^2 of the fundamental invariants (4.11) vanish. We saw that this implies the curvature tensor C as in (4.9) is trivial, and such M is therefore called flat. To demonstrate the existence of non-flat M, we consider $\mathbb{C}^4$ with complex coordinates $\{z^i, z^{\bar{i}}\}_{i=1}^4$, and let M be the hypersurface given by the level set $\rho^{-1}(0)$ of a smooth function $\rho : \mathbb{C}^4 \to \mathbb{R}$. In this setting, we can take the contact form $\theta^0 \in \Omega^1(M)$ to be

$$\theta^0 := -\mathbf{i}\partial\rho = -\mathbf{i}\frac{\partial\rho}{\partial z^i}\mathrm{d}z^i. \tag{4.21}$$

After a change of coordinates if necessary, the equation $\rho = 0$ may be written

$$F(z^1, z^2, z^3, z^{\bar{1}}, z^{\bar{2}}, z^{\bar{3}}) = z^4 + z^{\bar{4}},$$

for $F : \mathbb{C}^3 \to \mathbb{R}$, and the forms $\mathrm{d}z^j, \mathrm{d}z^{\bar{j}}$ ($1 \leq j \leq 3$) complete θ^0 to a local coframing of M. In the simplified case that F is given by

$$F(z^1, z^2, z^3, z^{\bar{1}}, z^{\bar{2}}, z^{\bar{3}}) = f(z^1 + z^{\bar{1}}, z^2 + z^{\bar{2}}, z^3 + z^{\bar{3}})$$

for some $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$F_j := \frac{\partial F}{\partial z^j} = \frac{\partial F}{\partial z^{\bar{j}}} =: F_{\bar{j}},$$

and we denote their common expression by f_j. Thus, (4.21) may be written

$$\theta^0 = -\mathbf{i}f_j \mathrm{d}z^j + \mathbf{i}\mathrm{d}z^4. \tag{4.22}$$

Second order partial derivatives are indicated by two subscripts, so that differentiating (4.22) gives the following matrix representation of the Levi form of M with respect to the coframing $\{\mathrm{d}z^j, \mathrm{d}z^{\bar{j}}\}_{j=1}^3$:

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}.$$

If we impose the condition that $f_{12} = 0$ while all other f_{jk} are nonvanishing, then Levi-degeneracy is equivalent to the partial differential equation

$$0 = \det(f_{jk}) = f_{11}f_{22}f_{33} - f_{11}(f_{23})^2 - f_{22}(f_{13})^2, \tag{4.23}$$

which is satisfied, for example, when

$$(f_{23})^2 = \tfrac{1}{2}f_{22}f_{33}, \qquad\qquad (f_{13})^2 = \tfrac{1}{2}f_{11}f_{33}. \tag{4.24}$$

We further assume that $f_{jj} > 0$ for $j = 1, 2, 3$, so that when (4.24) holds, $f_{k3} = \pm\sqrt{\tfrac{1}{2}f_{kk}f_{33}}$ for $k = 1, 2$, and the coframing given by

$$\begin{bmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{f_{11}} & 0 & \pm\sqrt{\tfrac{1}{2}f_{33}} \\ 0 & 0 & \sqrt{f_{22}} & \pm\sqrt{\tfrac{1}{2}f_{33}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta^0 \\ \mathrm{d}z^1 \\ \mathrm{d}z^2 \\ \mathrm{d}z^3 \end{bmatrix} \tag{4.25}$$

diagonalizes the Levi form,

$$\mathrm{d}\theta^0 = \mathbf{i}\theta^1 \wedge \theta^{\bar{1}} + \mathbf{i}\theta^2 \wedge \theta^{\bar{2}}.$$

We will compute the structure equations for a concrete example: let x_1, x_2, x_3 be coordinates

for $\mathbb{R}^3$ and take $\mathbb{R}^3_+$ to be the subspace where all coordinates are strictly positive. Define

$$f(x_1, x_2, x_3) = -x_3 \ln\left(\frac{x_1 x_2}{(x_3)^2}\right). \tag{4.26}$$

In the sequel, we will continue to denote $x_j = z^j + z^{\overline{j}}$ in order to compactify notation. Thus, (4.22) is given by

$$\theta^0 = \mathbf{i}\frac{x_3}{x_1}\mathrm{d}z^1 + \mathbf{i}\frac{x_3}{x_2}\mathrm{d}z^2 + \mathbf{i}\left(\ln\left(\frac{x_1 x_2}{(x_3)^2}\right) - 2\right)\mathrm{d}z^3 + \mathbf{i}\mathrm{d}z^4,$$

and our first approximation (4.25) at an adapted coframing is

$$\begin{bmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{x_3}}{x_1} & 0 & -\frac{1}{\sqrt{x_3}} \\ 0 & 0 & \frac{\sqrt{x_3}}{x_2} & -\frac{1}{\sqrt{x_3}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta^0 \\ \mathrm{d}z^1 \\ \mathrm{d}z^2 \\ \mathrm{d}z^3 \end{bmatrix}. \tag{4.27}$$

We differentiate to determine the structure equations so far,

$$\mathrm{d}\theta^0 = \mathbf{i}\theta^1 \wedge \theta^{\overline{1}} + \mathbf{i}\theta^2 \wedge \theta^{\overline{2}},$$

$$\mathrm{d}\theta^1 = \frac{1}{x_3}\theta^3 \wedge \theta^{\overline{1}} + \frac{1}{\sqrt{x_3}}\theta^1 \wedge \theta^{\overline{1}} - \frac{1}{2x_3}\theta^1 \wedge \theta^3 + \frac{1}{2x_3}\theta^1 \wedge \theta^{\overline{3}},$$

$$\mathrm{d}\theta^2 = \frac{1}{x_3}\theta^3 \wedge \theta^{\overline{2}} + \frac{1}{\sqrt{x_3}}\theta^2 \wedge \theta^{\overline{2}} - \frac{1}{2x_3}\theta^2 \wedge \theta^3 + \frac{1}{2x_3}\theta^2 \wedge \theta^{\overline{3}},$$

$$\mathrm{d}\theta^3 = 0.$$

$$\tag{4.28}$$

Recall that the structure group G_0 of all 0-adapted coframings is parametrized by (3.3), and that the subgroup G_1 which preserves 1-adaptation is given by the additional conditions (3.13). The structure equations (4.28) show that our coframing is 1-adapted as in (3.12), and we maintain this property when we submit it to a G_1-transformation to get a new coframing

$$\begin{bmatrix} \eta^0 \\ \theta^{1'} \\ \theta^{2'} \\ \theta^{3'} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{i} & 0 \\ 0 & 1 & -\mathbf{i} & 0 \\ 0 & 0 & 0 & \frac{1}{x_3} \end{bmatrix} \begin{bmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix}. \tag{4.29}$$

The new structure equations are

$$\mathrm{d}\eta^0 = \mathbf{i}\theta^{1'} \wedge \theta^{\overline{1}'} + \mathbf{i}\theta^{2'} \wedge \theta^{\overline{2}'},$$

$$\mathrm{d}\theta^{1'} = \theta^{3'} \wedge \theta^{\overline{2}'} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\theta^{1'} \wedge \theta^{\overline{1}'} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\theta^{1'} \wedge \theta^{\overline{2}'} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\theta^{2'} \wedge \theta^{\overline{1}'} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\theta^{2'} \wedge \theta^{\overline{2}'}$$
$$\quad + \tfrac{1}{2}\theta^{1'} \wedge (\theta^{\overline{3}'} - \theta^{3'}),$$

$$\mathrm{d}\theta^{2'} = \theta^{3'} \wedge \theta^{\overline{1}'} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\theta^{1'} \wedge \theta^{\overline{1}'} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\theta^{1'} \wedge \theta^{\overline{2}'} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\theta^{2'} \wedge \theta^{\overline{1}'} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\theta^{2'} \wedge \theta^{\overline{2}'}$$
$$\quad + \tfrac{1}{2}\theta^{2'} \wedge (\theta^{\overline{3}'} - \theta^{3'}),$$

$$\mathrm{d}\theta^{3'} = \theta^{3'} \wedge \theta^{\overline{3}'},$$

(4.30)

so our coframing (4.29) is now 2-adapted according to (3.19). The structure group G_2 of the bundle of 2-adapted coframes is parametrized by (3.20), so our 2-adaptation is preserved when we apply a G_2 transformation to get a new coframing

$$\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \theta^{3''} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ c^1 & 1 & 0 & 0 \\ c^2 & 0 & 1 & 0 \\ 0 & b_1 & b_2 & 1 \end{bmatrix} \begin{bmatrix} \eta^0 \\ \theta^{1'} \\ \theta^{2'} \\ \theta^{3'} \end{bmatrix},$$

(4.31)

for some $c^1, c^2, b_1, b_2 \in C^\infty(M, \mathbb{C})$. The effect of this transformation on the first three structure equations may be written

$$\mathrm{d}\eta^0 = \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \mathbf{i}\eta^2 \wedge \eta^{\overline{2}} + \mathbf{i}\eta^0 \wedge (\bar{c}^1\eta^1 - c^1\eta^{\overline{1}} + \bar{c}^2\eta^2 - c^2\eta^{\overline{2}}),$$

$$\mathrm{d}\eta^1 \equiv \eta^3 \wedge \eta^{\overline{2}} + \tfrac{b_2}{2}\eta^1 \wedge \eta^2 - \tfrac{2\sqrt{x_3}(\bar{b}_1 - 2\mathbf{i}c^1) - 1 - \mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{1}} - \tfrac{2\sqrt{x_3}(2b_1 + \bar{b}_2) - 1 + \mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}}$$
$$\quad + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \tfrac{4\sqrt{x_3}(\mathbf{i}c^1 - b_2) + 1 + \mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{2}} + \tfrac{1}{2}\eta^1 \wedge (\theta^{\overline{3}''} - \theta^{3''}) \qquad \bmod\{\eta^0\},$$

$$\mathrm{d}\eta^2 \equiv \eta^3 \wedge \eta^{\overline{1}} - \tfrac{b_1}{2}\eta^1 \wedge \eta^2 + \tfrac{4\sqrt{x_3}(\mathbf{i}c^2 - b_1) + 1 - \mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{1}} - \tfrac{2\sqrt{x_3}(\bar{b}_1 + 2b_2) - 1 - \mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}}$$
$$\quad + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} - \tfrac{2\sqrt{x_3}(\bar{b}_2 - 2\mathbf{i}c^2) - 1 + \mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{2}} + \tfrac{1}{2}\eta^2 \wedge (\theta^{\overline{3}''} - \theta^{3''}) \qquad \bmod\{\eta^0\}.$$

(4.32)

We choose functions b, c that eliminate the coefficients of $\eta^1 \wedge \eta^{\overline{1}}$ and $\eta^2 \wedge \eta^{\overline{2}}$ in the identities for $\mathrm{d}\eta^1, \mathrm{d}\eta^2$ in (4.32). Therefore, set

$$c^1 := \frac{-1+\mathbf{i}}{4\sqrt{x_3}}, \qquad\qquad c^2 := \frac{1+\mathbf{i}}{4\sqrt{x_3}}, \qquad\qquad b_1 = b_2 = 0.$$

Now we have

$$\mathrm{d}\eta^0 = \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \mathbf{i}\eta^2 \wedge \eta^{\overline{2}} + \tfrac{1}{4\sqrt{x_3}}\eta^0 \wedge ((1-\mathbf{i})\eta^1 + (1+\mathbf{i})\eta^{\overline{1}} + (1+\mathbf{i})\eta^2 + (1-\mathbf{i})\eta^{\overline{2}}),$$

$$\mathrm{d}\eta^1 = \eta^3 \wedge \eta^{\overline{2}} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \tfrac{1}{2}\eta^1 \wedge (\theta^{\overline{3}''} - \theta^{3''}) - \tfrac{1}{8x_3}\eta^0 \wedge (\mathbf{i}\eta^1 + \eta^{\overline{1}} + \eta^2 + \mathbf{i}\eta^{\overline{2}}),$$

$$\mathrm{d}\eta^2 = \eta^3 \wedge \eta^{\overline{1}} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} + \tfrac{1}{2}\eta^2 \wedge (\theta^{\overline{3}''} - \theta^{3''}) + \tfrac{1}{8x_3}\eta^0 \wedge (\eta^1 - \mathbf{i}\eta^{\overline{1}} - \mathbf{i}\eta^2 + \eta^{\overline{2}}),$$

$$\mathrm{d}\theta^{3''} = \theta^{3''} \wedge \theta^{\overline{3}''}.$$

$$(4.33)$$

Finally, we apply a G_3-transformation – see (3.34) – to get

$$
\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c^3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \theta^{3''} \end{bmatrix},
$$

$$(4.34)$$

which effects the following alteration of the latter three structure equations (4.33)

$$\mathrm{d}\eta^1 = \eta^3 \wedge \eta^{\overline{2}} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \tfrac{1}{2}\eta^1 \wedge (\theta^{\overline{3}''} - \theta^{3''})$$
$$+ \tfrac{1}{8x_3}\eta^0 \wedge ((4x_3(\overline{c}^3 - c^3) - \mathbf{i})\eta^1 - \eta^{\overline{1}} - \eta^2 - (8x_3 c^3 + \mathbf{i})\eta^{\overline{2}}),$$

$$\mathrm{d}\eta^2 = \eta^3 \wedge \eta^{\overline{1}} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} + \tfrac{1}{2}\eta^2 \wedge (\theta^{\overline{3}''} - \theta^{3''}) \qquad (4.35)$$
$$+ \tfrac{1}{8x_3}\eta^0 \wedge (\eta^1 - (8x_3 c^3 + \mathbf{i})\eta^{\overline{1}} + (4x_3(\overline{c}^3 - c^3) - \mathbf{i})\eta^2 + \eta^{\overline{2}}),$$

$$\mathrm{d}\eta^3 \equiv \eta^3 \wedge \eta^{\overline{3}} + \mathbf{i}c^3\eta^1 \wedge \eta^{\overline{1}} + \mathbf{i}c^3\eta^2 \wedge \eta^{\overline{2}} \qquad\qquad \mathrm{mod}\ \{\eta^0\}.$$

If we take

$$\gamma^1 \equiv \tfrac{1}{8x_3}((4x_3(\overline{c}^3 - c^3) - \mathbf{i})\eta^1 - \eta^{\overline{1}} - \eta^2 - (8x_3 c^3 + \mathbf{i})\eta^{\overline{2}}) \quad \mathrm{mod}\ \{\eta^0\},$$

$$\gamma^2 \equiv \tfrac{1}{8x_3}(\eta^1 - (8x_3 c^3 + \mathbf{i})\eta^{\overline{1}} + (4x_3(\overline{c}^3 - c^3) - \mathbf{i})\eta^2 + \eta^{\overline{2}}) \quad \mathrm{mod}\ \{\eta^0\},$$

then we can equivalently express (4.35) as

$$\mathrm{d}\eta^1 = -\gamma^1 \wedge \eta^0 + \eta^3 \wedge \eta^{\overline{2}} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} + \tfrac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \tfrac{1}{2}\eta^1 \wedge (\theta^{\overline{3}''} - \theta^{3''}),$$

$$\mathrm{d}\eta^2 = -\gamma^2 \wedge \eta^0 + \eta^3 \wedge \eta^{\overline{1}} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \tfrac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} + \tfrac{1}{2}\eta^2 \wedge (\theta^{\overline{3}''} - \theta^{3''}),$$

$$\mathrm{d}\eta^3 \equiv -\mathbf{i}\gamma^2 \wedge \eta^1 - \mathbf{i}\gamma^1 \wedge \eta^2 + \eta^3 \wedge \eta^{\overline{3}} + \tfrac{\mathbf{i}16x_3 c^3 - 1}{8x_3}(\eta^1 \wedge \eta^{\overline{1}} + \eta^2 \wedge \eta^{\overline{2}})$$
$$- \tfrac{\mathbf{i}}{8x_3}\eta^1 \wedge \eta^{\overline{2}} + \tfrac{\mathbf{i}}{8x_3}\eta^2 \wedge \eta^{\overline{1}} \qquad\qquad \mathrm{mod}\ \{\eta^0\}.$$

$$(4.36)$$

We select c^3 to eliminate the $\eta^1 \wedge \eta^{\overline{1}}$ and $\eta^2 \wedge \eta^{\overline{2}}$ terms in the identity (4.36) for $d\eta^3$, viz,

$$c^3 := -\frac{\mathbf{i}}{16x_3}.$$

Now the forms $\eta^0, \eta^1, \eta^2, \eta^3$ on M are completely determined. We summarize in terms of our $\mathbb{C}^4$ coordinates z^1, z^2, z^3, z^4, whose real parts we assume to be strictly positive (except for z^4),

$$\eta^0 = 2\mathbf{i}\frac{z^3+z^{\overline{3}}}{z^1+z^{\overline{1}}}dz^1 + 2\mathbf{i}\frac{z^3+z^{\overline{3}}}{z^2+z^{\overline{2}}}dz^2 + 2\mathbf{i}\left(\ln\left(\frac{(z^1+z^{\overline{1}})(z^2+z^{\overline{2}})}{(z^3+z^{\overline{3}})^2}\right) - 2\right)dz^3 + 2\mathbf{i}dz^4,$$

$$\eta^1 = \frac{(1-\mathbf{i})\sqrt{z^3+z^{\overline{3}}}}{2(z^1+z^{\overline{1}})}dz^1 - \frac{(1-\mathbf{i})\sqrt{z^3+z^{\overline{3}}}}{2(z^2+z^{\overline{2}})}dz^2 - \frac{(1+\mathbf{i})}{2\sqrt{z^3+z^{\overline{3}}}}\ln\left(\frac{(z^1+z^{\overline{1}})(z^2+z^{\overline{2}})}{(z^3+z^{\overline{3}})^2}\right)dz^3 - \frac{1+\mathbf{i}}{2\sqrt{z^3+z^{\overline{3}}}}dz^4,$$

$$\eta^2 = \frac{(1+\mathbf{i})\sqrt{z^3+z^{\overline{3}}}}{2(z^1+z^{\overline{1}})}dz^1 - \frac{(1+\mathbf{i})\sqrt{z^3+z^{\overline{3}}}}{2(z^2+z^{\overline{2}})}dz^2 - \frac{(1-\mathbf{i})}{2\sqrt{z^3+z^{\overline{3}}}}\ln\left(\frac{(z^1+z^{\overline{1}})(z^2+z^{\overline{2}})}{(z^3+z^{\overline{3}})^2}\right)dz^3 - \frac{1-\mathbf{i}}{2\sqrt{z^3+z^{\overline{3}}}}dz^4,$$

$$\eta^3 = \frac{1}{8(z^1+z^{\overline{1}})}dz^1 + \frac{1}{8(z^2+z^{\overline{2}})}dz^2 + \frac{6 + \ln\left(\frac{(z^1+z^{\overline{1}})(z^2+z^{\overline{2}})}{(z^3+z^{\overline{3}})^2}\right)}{8(z^3+z^{\overline{3}})}dz^3 + \frac{1}{8(z^3+z^{\overline{3}})}dz^4.$$

The structure equations for these forms are

$$d\eta^0 = \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \mathbf{i}\eta^2 \wedge \eta^{\overline{2}} + \frac{1}{4\sqrt{x_3}}\eta^0 \wedge ((1-\mathbf{i})\eta^1 + (1+\mathbf{i})\eta^{\overline{1}} + (1+\mathbf{i})\eta^2 + (1-\mathbf{i})\eta^{\overline{2}}),$$

$$d\eta^1 = \eta^3 \wedge \eta^{\overline{2}} + \frac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} + \frac{1-\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \frac{1}{2}\eta^1 \wedge (\eta^{\overline{3}} - \eta^3) - \frac{1}{16x_3}\eta^0 \wedge (\mathbf{i}\eta^1 + 2\eta^{\overline{1}} + 2\eta^2 + \mathbf{i}\eta^{\overline{2}}),$$

$$d\eta^2 = \eta^3 \wedge \eta^{\overline{1}} + \frac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^2 \wedge \eta^{\overline{1}} + \frac{1+\mathbf{i}}{4\sqrt{x_3}}\eta^1 \wedge \eta^{\overline{2}} + \frac{1}{2}\eta^2 \wedge (\eta^{\overline{3}} - \eta^3) + \frac{1}{16x_3}\eta^0 \wedge (2\eta^1 - \mathbf{i}\eta^{\overline{1}} - \mathbf{i}\eta^2 + 2\eta^{\overline{2}}),$$

$$d\eta^3 = \frac{1}{64(x_3)^{3/2}}((1+\mathbf{i})\eta^1 - (1-\mathbf{i})\eta^{\overline{1}} - (1-\mathbf{i})\eta^2 + (1+\mathbf{i})\eta^{\overline{2}}) \wedge \eta^0$$

$$+ \frac{1}{16x_3}\eta^1 \wedge \eta^{\overline{1}} + \frac{1}{16x_3}\eta^2 \wedge \eta^{\overline{2}} + \eta^3 \wedge \eta^{\overline{3}},$$

$$(4.37)$$

which shows that the coframing $\eta^0, \eta^1, \eta^2, \eta^3$ of M defines a section of the bundle $B_4 \to M$ of 4-adapted coframes. If we denote the pullbacks along this section of the pseudoconnection forms

on B_4 by their same names, then we write

$$\tau = \tfrac{1}{8\sqrt{x_3}}((1-\mathbf{i})\eta^1 + (1+\mathbf{i})\eta^{\overline{1}} + (1+\mathbf{i})\eta^2 + (1-\mathbf{i})\eta^{\overline{2}}),$$

$$\mathbf{i}\varrho = \tfrac{1}{2}(\eta^{\overline{3}} - \eta^3) + \tfrac{1}{8\sqrt{x_3}}((1-\mathbf{i})\eta^1 - (1+\mathbf{i})\eta^{\overline{1}} - (1+\mathbf{i})\eta^2 + (1-\mathbf{i})\eta^{\overline{2}}),$$

$$\mathbf{i}\varsigma = \tfrac{1}{2}(\eta^{\overline{3}} - \eta^3) - \tfrac{1}{8\sqrt{x_3}}((1-\mathbf{i})\eta^1 - (1+\mathbf{i})\eta^{\overline{1}} - (1+\mathbf{i})\eta^2 + (1-\mathbf{i})\eta^{\overline{2}}), \qquad (4.38)$$

$$\gamma^1 = \tfrac{1+\mathbf{i}}{64(x_3)^{3/2}}\eta^0 - \tfrac{1}{16x_3}(\mathbf{i}\eta^1 + 2\eta^{\overline{1}} + 2\eta^2 + \mathbf{i}\eta^{\overline{2}}),$$

$$\gamma^2 = \tfrac{1-\mathbf{i}}{64(x_3)^{3/2}}\eta^0 + \tfrac{1}{16x_3}(2\eta^1 - \mathbf{i}\eta^{\overline{1}} - \mathbf{i}\eta^2 + 2\eta^{\overline{2}}),$$

and the structure equations (4.37) may be written according to (3.47)

$$\mathrm{d}\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = -\begin{bmatrix} 2\tau & 0 & 0 & 0 \\ \gamma^1 & \tau + \mathbf{i}\varrho & 0 & 0 \\ \gamma^2 & 0 & \tau + \mathbf{i}\varsigma & 0 \\ 0 & \mathbf{i}\gamma^2 & \mathbf{i}\gamma^1 & \mathbf{i}\varrho + \mathbf{i}\varsigma \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{i}\eta^1 \wedge \eta^{\overline{1}} + \mathbf{i}\eta^2 \wedge \eta^{\overline{2}} \\ \eta^3 \wedge \eta^{\overline{2}} + F^1\eta^{\overline{1}} \wedge \eta^2 \\ \eta^3 \wedge \eta^{\overline{1}} + F^2\eta^{\overline{2}} \wedge \eta^1 \\ T^3_{\overline{1}}\eta^{\overline{1}} \wedge \eta^0 + T^3_{\overline{2}}\eta^{\overline{2}} \wedge \eta^0 + F^3_1\eta^{\overline{2}} \wedge \eta^1 + F^3_2\eta^{\overline{1}} \wedge \eta^2 \end{bmatrix},$$

for

$$F^1 = -\frac{1-\mathbf{i}}{4\sqrt{z^3 + z^{\overline{3}}}}, \qquad\qquad F^2 = -\frac{1+\mathbf{i}}{4\sqrt{z^3 + z^{\overline{3}}}}, \qquad (4.39)$$

$$T^3_{\overline{1}} = -\frac{1-\mathbf{i}}{64(z^3 + z^{\overline{3}})^{3/2}}, \quad T^3_{\overline{2}} = \frac{1+\mathbf{i}}{64(z^3 + z^{\overline{3}})^{3/2}}, \quad F^3_1 = \frac{\mathbf{i}}{8(z^3 + z^{\overline{3}})}, \quad F^3_2 = -\frac{\mathbf{i}}{8(z^3 + z^{\overline{3}})}.$$

In particular, the coefficients (4.39) of the fundamental invariants (4.11) are nonvanishing, so M is not locally CR equivalent to the homogeneous model $M_\star$.

At this point, the forms $\eta, \varrho, \varsigma, \tau, \gamma$ on M are adapted to the B_4 structure equations, so they define a section of the bundle $B_4^{(1)} \to M$, and they are exactly the pullbacks along this section of the tautological forms with the same names (3.49) on $B_4^{(1)}$. Thus, to find the pullback of the full parallelism $\omega \in \Omega^1(B_4^{(1)}, \mathfrak{su}_\star)$ as in §4.2, it remains to find an expression for the pullback of ψ,

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which we will also call ψ. To accomplish this, we differentiate τ and γ^1 according to the structure equations (3.71). We begin with τ,

$$
\begin{aligned}
\mathrm{d}\tau = {}& \tfrac{\mathbf{i}}{2}\gamma^1 \wedge \eta^{\overline{1}} - \tfrac{\mathbf{i}}{2}\gamma^{\overline{1}} \wedge \eta^1 + \tfrac{\mathbf{i}}{2}\gamma^2 \wedge \eta^{\overline{2}} - \tfrac{\mathbf{i}}{2}\gamma^{\overline{2}} \wedge \eta^2 \\
& + \frac{1}{128(x_3)^{3/2}}\eta^0 \wedge ((1+\mathbf{i})\eta^1 + (1-\mathbf{i})\eta^{\overline{1}} - (1-\mathbf{i})\eta^2 - (1+\mathbf{i})\eta^{\overline{2}}),
\end{aligned}
$$

so we see

$$
\psi \equiv \frac{1}{128(x_3)^{3/2}}((1+\mathbf{i})\eta^1 + (1-\mathbf{i})\eta^{\overline{1}} - (1-\mathbf{i})\eta^2 - (1+\mathbf{i})\eta^{\overline{2}}) \mod \{\eta^0\}.
$$

To find the coefficient of η^0 in the full expansion of ψ, one takes the real part of the coefficient of $\eta^0 \wedge \eta^1$ in the expression

$$
\mathrm{d}\gamma^1 - (\tau - \mathbf{i}\varrho) \wedge \gamma^1 + \gamma^{\overline{2}} \wedge \eta^3 - \mathbf{i}F_2^3\gamma^{\overline{1}} \wedge \eta^0 - F^1\gamma^{\overline{1}} \wedge \eta^2 + F^1\gamma^2 \wedge \eta^{\overline{1}}.
$$

We simply state that the result of this calculation is

$$
\psi = \frac{1}{128(z^3 + z^{\overline{3}})^2}\eta^0 + \frac{1}{128(z^3 + z^{\overline{3}})^{3/2}}((1+\mathbf{i})\eta^1 + (1-\mathbf{i})\eta^{\overline{1}} - (1-\mathbf{i})\eta^2 - (1+\mathbf{i})\eta^{\overline{2}}).
$$

With this one-form in hand, the pullback of the parallelism ω to M is completely determined.

5. CONCLUSION

The computational intensity of §3 reveals how formidable the challenge of classification in Levi-degenerate CR geometry can be. In dimension seven alone, the equivalence problem remains open with regard to those M for which $\mathcal{C}$ is not of conformal unitary type or $\mathrm{rank}_{\mathbb{C}} K = 2$, though homogeneous models may have been discovered in these cases ([San15]). Moreover, the question of whether a Cartan geometry can be associated to the case we studied is also unresolved. Recall that the Isaev-Zaitsev and Medori-Spiro solutions in dimension five differed in this respect, which is not without precedent.

In Cartan's celebrated "five variables" paper [Car10], the parallelism he constructs over a manifold equipped with a generic distribution of growth vector (2,3,5) does not satisfy the equivariance condition of Definition 4.1. However, Tanaka's solution to the equivalence problem ([Tan70, Tan79]) for a much more general class of differential systems proved the existence of a canonical Cartan connection for such a geometry. This discrepancy is attributable to the choices of torsion normalization implemented in each construction (c.f. [Ste64, Ch VII Prop 2.1]).

The ingenuity of Tanaka's procedure lies partially in its utilization of Lie algebra cohomology to ensure that equivariancy is maintained in each stage of torsion normalization, though this comes at the expense of hypotheses on the geometric structure which limit the procedure's applicability. In particular, Levi-degenerate geometries do not fall under Tanaka's purview due to the fact that the Levi kernel is integrable. However, in ongoing work with Igor Zelenko, we adapt Tanaka's construction to generalize the known results in dimensions five and seven, and we anticipate some degree of resolution to the question of when Cartan geometries can be constructed over 2-nondegenerate CR manifolds.

Beyond 2-nondegeneracy, we hinted in §2.3 that Freeman's work also characterizes higher nondegeneracy conditions which have yet to be classified. Similarly, despite limited progress in low dimensions ([SS00],[ČS02],[SS06]), equivalence problems abound for higher-codimensional CR structures. It seems likely that CR geometry and the method of equivalence will continue to motivate each other's evolution as they have so far.

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