ON THE STRUCTURE TENSOR OF $\mathfrak{sl}_n$

A Thesis

by

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ABSTRACT

The structure tensor of $\mathfrak{sl}_n$, denoted $T_{\mathfrak{sl}_n}$, is the tensor arising from the Lie bracket bilinear operation on the set of traceless $n \times n$ matrices over $\mathbb{C}$. This tensor is intimately related to the well studied matrix multiplication tensor. Studying the structure tensor of $\mathfrak{sl}_n$ may provide further insight into the complexity of matrix multiplication and the "hay in a haystack" problem of finding explicit sequences tensors with high rank or border rank. We aim to find new bounds on the rank and border rank of this structure tensor in the case of $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$. The lower bounds on the border rank of $T_{\mathfrak{sl}_4}$ were obtained via Koszul flattenings and border substitution. The best lower bound on the border rank of $T_{\mathfrak{sl}_3}$ were obtained via a new technique called border apolarity, developed by Conner, Harper, and Landsberg. Upper bounds on the rank of $T_{\mathfrak{sl}_3}$ are obtained via numerical methods that allowed us to find an explicit rank decomposition.

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## Contributors

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The code used for the results in Chapter 4, Sections 3 and 4 used excerpts of code from [1].

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## 1. INTRODUCTION

In 1969, Strassen presented a novel algorithm for matrix multiplication of $n \times n$ matrices. Strassen's algorithm used fewer than the $O(n^3)$ arithmetic operations needed for the standard algorithm. This led to the question: what is the minimal number of arithmetic operations required to multiply $n \times n$ matrices, or in other words, what is the complexity of matrix multiplication [2] [3]. An asymptotic version of the problem is to determine the exponent of matrix multiplication, $\omega$, which is the minimum value such that for all $\epsilon > 0$, multiplying $n \times n$ matrices can be performed in $O(n^{\omega+\epsilon})$ arithmetic operations. Any bilinear operation, including matrix multiplication, may be thought of as tensor in the following way: Let $A, B,$ and $C$ denote vector spaces over $\mathbb{C}$. Given a bilinear map $A^* \times B^* \to C$, the universal property of tensor products induces a linear map $A^* \otimes B^* \to C$. Since $\mathrm{Hom}_{\mathbb{C}}(A^* \otimes B^*, C) \simeq A \otimes B \otimes C$, then we can take our bilinear map to be a tensor in $A \otimes B \otimes C$. Let $M_{\langle n \rangle}$ denote the matrix multiplication tensor arising from the bilinear operation of multiplying $n \times n$ matrices.

An important invariant of a tensor is its rank. For a tensor $T \in A \otimes B \otimes C$ the *rank*, denoted $\mathbf{R}(T)$, is the minimal $r$ such that $T = \sum_{i=1}^r a_i \otimes b_i \otimes c_i$ with $a_i \in A, b_i \in B, c_i \in C$ for $1 \leq i \leq r$. Given precise $T_i = a_i \otimes b_i \otimes c_i$, then we call $T = \sum_{i=1}^r T_i$ a *rank decomposition* of $T$. Strassen also showed that the rank of the matrix multiplication tensor is a valid measure of its complexity; in particular, he proved $\omega = \inf\{\tau \in \mathbb{R} \mid \mathbf{R}(M_{\langle n \rangle}) = O(n^\tau)\}$ [2].

For a tensor $T \in A \otimes B \otimes C$, the *border rank* of $T$, denoted $\underline{\mathbf{R}}(T)$, is another invariant of interest and defined to be the minimal $r$ such that $T = \lim_{\epsilon \to 0} T_\epsilon$ where for all $\epsilon > 0$, $T_\epsilon$ has rank $r$. Given rank decompositions of $T_\epsilon = \displaystyle\sum_{i=1}^{r} T_i(\epsilon)$, we then call $\lim_{\epsilon \to 0} \sum_{i=1}^r T_i(\epsilon)$ a *border rank decomposition* of $T$. Later, in 1980, it was shown by Bini that the border rank of matrix multiplication is also a valid measure of its complexity by proving that $\omega = \inf\{\tau \in \mathbb{R} \mid \underline{\mathbf{R}}(M_{\langle n \rangle}) = O(n^\tau)\}$ [4].

Intimately related to the matrix multiplication tensor is the structure tensor of the Lie algebra $\mathfrak{sl}_n$, the set of traceless $n \times n$ matrices over $\mathbb{C}$ equipped with the Lie bracket $[x, y] = xy - yx$.

The *structure tensor* of $\mathfrak{sl}_n$ is defined as the tensor arising from the Lie bracket bilinear operation, and we denote it by $T_{\mathfrak{sl}_n}$. One example of how matrix multiplication is related to $T_{\mathfrak{sl}_n}$ is by closer examination of a skew-symmetric version of the matrix multiplication tensor; consider the tensor arising from the Lie bracket bilinear operation on $\mathfrak{gl}_n$, (which is just $M_n$, but considered as a Lie algebra) [5]. Since $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus z$, where $z$ indicates the scalar matrices which are central in $\mathfrak{gl}_n$, $T_{\mathfrak{sl}_n}$ determines the commutator action on all of $\mathfrak{gl}_n$. While the matrix multiplication tensor has been well studied [5] [1], the structure tensor of $\mathfrak{sl}_n$ has not been studied to the same extent. Currently, the only known non-trivial results are lower bounds on the rank of the structure tensor of $\mathfrak{sl}_n$ [6]. Studying the structure tensor of $\mathfrak{sl}_n$ may provide some further insight into two central problems in complexity theory.

In complexity theory, it is of interest to find *explicit* objects that behave generically. This type of problem is known as a "hay in a haystack" problem. Algebraic geometry tells us that a "random" tensor $T$ in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ will have rank/border rank $\lceil \frac{m^3}{3m-2} \rceil$. By an *explicit* sequence of tensors, we will mean a collection of tensors $T_m \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ such that the coefficients of $T_m$ are computable in polynomial time in m. The "hay in a haystack" problem for tensors is to find an example of an explicit sequence of tensors of high rank or border rank, asymptotically in $m$. Currently, there exists an explicit sequence of tensors, $S_m$, such that $\mathbf{R}(S_m) \geq 3m - o(\log(m))$[7] and a different explicit sequence of tensors, $T_m$, such that $\underline{\mathbf{R}}(T_m) \geq 2.02m - o(m)$ [8]. One should note that the sequence $T_m$ of [8] has border rank equal to $2m$ when $m = 13$ and has been shown to exceed $2m$ for $m > 364175$. It would be of interest to find sequences of tensors for which the border rank exceeds $2m$ for smaller values of $m$. Finding lower bounds on border ranks of tensors over $\mathbb{C}$ is equivalent to a problem called "complexity theory's Waterloo"[9]. It would be groundbreaking to find any lower bounds that are superlinear.

The second problem is Strassen's problem of computing of the complexity of matrix multiplication. The exponent of $\mathfrak{sl}_n$ is defined as $\omega(\mathfrak{sl}_n) := \liminf_{n\to\infty} \log_n(\mathbf{R}(T_{\mathfrak{sl}_n}))$. By Theorem 4.1 from [10], the exponent of matrix multiplication is equal to the exponent of $\mathfrak{sl}_n$. Consequently, upper bounds on the rank and even the border rank of $T_{\mathfrak{sl}_n}$ provide upper bounds on $\omega$.

These two problems motivate our study of the border rank of $T_{\mathfrak{sl}_n}$.

The above definition of $T_{\mathfrak{sl}_n}$ is independent of choice of basis, but we may also write the tensor in terms of bases. Let $\{a_i\}_{i=1}^{n^2-1}$ be a basis of $\mathfrak{sl}_n$ and $\{\alpha^i\}_{i=1}^{n^2-1}$ a dual basis. Recall that $\mathfrak{sl}_n$ has a bilinear operation called the Lie bracket, given by $[a_i, a_j] = a_i a_j - a_j a_i = \sum_{k=1}^{n^2-1} A_{ij}^k a_k$. The structure tensor of $\mathfrak{sl}_n$ in this basis is $T_{\mathfrak{sl}_n} = \sum_{i,j,k} A_{ij}^k \alpha^i \otimes \alpha^j \otimes a_k \in \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n$

We establish some basic definitions of algebraic geometry and representation theory that will be used in our study of the structure tensor of $\mathfrak{sl}_n$.

## 2.1 Algebraic Geometry

The language of algebraic geometry will prove useful in studying our problems. Let $V$ be a vector space over $\mathbb{C}$. Let $\pi : V \to \mathbb{P}V$ be the projection map from $V$ to its projectivization. Denote $[v] := \pi(v) \in \mathbb{P}V$ for nonzero $v \in V$. For $v_1, \cdots, v_k \in V$, let $\langle v_1, \cdots, v_k \rangle$ denote the projectivization of the linear span of $v_1, \cdots, v_k$.

See [11] for the definitions of the Zariski topology, projective variety, and the dimension of a projective variety. For a projective variety, $X \subset \mathbb{P}V$, let S[X] denote its coordinate ring and let $I(X) \subset Sym(A^*)$ denote the ideal of $X$. A projective variety is nondegenerate if it is not contained in any hyperplane. Given $Y \subset \mathbb{P}V$ a nondegenerate projective variety, we define the $r$th secant variety of $Y$, denoted $\sigma_r(Y) \subset \mathbb{P}V$.

**Definition 1.** The $r$th secant variety of $Y$ is $\sigma_r(Y) = \overline{\bigcup_{y_i \in Y} \langle y_1, \cdots, y_r \rangle}$

The closure operation indicated in Definition 1 is the Zariski closure. As polynomials are continuous, this closure also contains the Euclidean closure of the set. In this case, we have that the Euclidean closure is in fact equal to the Zariski closure.

**Theorem 2.** *Let $Z \subset \mathbb{P}V$. Suppose that the Zariski closure of $Z$ is irreducible. If $Z$ contains a Zariski open subset of its Zariski closure, then the Euclidean closure will coincide with the Zariski closure.*

For a proof, see Theorem 3.1.6.1 in [12] or Theorem 2.3.3 in [13].

**Definition 3.** Let $A, B, C$ be vector spaces over $\mathbb{C}$. The Segre embedding is defined as

$$Seg : \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C \to \mathbb{P}(A \otimes B \otimes C)$$
$$([a], [b], [c]) \mapsto [a \otimes b \otimes c]$$

Let $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ denote the image of the Segre embedding, which is a projective variety of dimension $\dim A + \dim B + \dim C - 3$ (See [11] for proof). Note that $X$ is the space of rank one tensors in $\mathbb{P}(A \otimes B \otimes C)$. Consequently, one can redefine the border rank, $\underline{\mathbf{R}}(T)$, for $T \in A \otimes B \otimes C$ as the $r$ such that $T \in \sigma_r(X)$ and $T \notin \sigma_{r-1}(X)$. We make a remark here that $\mathbf{R}(T) \geq \underline{\mathbf{R}}(T)$, trivially, as a tensor rank decomposition can be considered as a border rank decomposition by taking the tensor rank decomposition as a constant sequence.

The dimension of these secant varieties in general can be computed using the classical Terracini lemma on the dimension of tangent spaces of the join of two varieties, which we recall below. Given a projective variety $Y \subset \mathbb{P}V$, let $\hat{Y} = \pi^{-1}(Y) \cup \{0\} \subset V$ be the affine cone over $Y$. The affine tangent space $\hat{T}_{[y]}Y$ at a point $[y] \in Y$ ($y \neq 0$) is the span of tangent vectors at $y$ to analytic curves on $\hat{Y}$. Note that this is independent of choice of $y \in \pi^{-1}([y])$. We also note that we primarily use $T$ to denote a tensor, but will be careful to make clear which object we are using. A point $y \in Y$ is smooth if the dimension of the affine tangent space is constant in a neighborhood of $y$. A general point on variety $Y$ is a point not lying on an explicit Zariski closed subset of $Y$. Denote the set general points on $Y$ by $Y_{gen}$. In our case, we take the explicit closed subvariety to be the singular locus of $Y$, i.e. the set of points that are not smooth. Define the join of two projective varieties $Y, Z \subset \mathbb{P}V$ to be $J(Y, Z) = \overline{\bigcup_{y \in Y, z \in Z, y \neq z} \langle y, z \rangle}$.

**Lemma 4** (Terracini's Lemma)**.** *Given* $(y, z) \in (\hat{Y} \times \hat{Z})_{gen}$ *and* $[x] = [y + z] \in J(Y, Z)$*, then* $\hat{T}_{[x]}J(Y, Z) = \hat{T}_{[y]}Y + \hat{T}_{[z]}Z$

For a proof, see Lemma 5.3.1.1 in [14].

**Corollary 5.** *Let* $(y_1, \cdots, y_r) \in (Y^{\times r})_{gen}$*, then* $\dim \sigma_r(Y) = \dim(\hat{T}_{y_1}Y + \cdots + \hat{T}_{y_r}Y) - 1$.

*Proof.* For a smooth point $y \in Y$, $\dim Y = \dim \hat{T}_{[y]}Y - 1$. Applying this fact with Terracini's Lemma yields the result. □

For a secant variety $\sigma_r(Y) \subset \mathbb{P}V$ in general, the expected dimension will be $\min\{r \dim Y + (r-1), \dim \mathbb{P}V\}$, since we expect to choose $r$ points from $Y$ and have an additional $r-1$ parameters to span the $r$-plane generated by those points. It is not always the case that the dimension of the secant variety will have its expected dimension for an arbitrary projective variety $Y$. However, the following theorem from [15] shows that for the secant varieties of the Segre variety that we will be working with, the expected dimension given below will in fact be the actual dimension.

**Theorem 6** (Lickteig)**.** *For* $Seg(\mathbb{P}V \times \mathbb{P}V \times \mathbb{P}V) \subset \mathbb{P}(V \otimes V \otimes V)$*, the dimension of* $\sigma_r(Seg(\mathbb{P}V \times \mathbb{P}V \times \mathbb{P}V))$ *is the expected dimension* $\min\{r \dim Seg(\mathbb{P}V \times \mathbb{P}V \times \mathbb{P}V) + r - 1, \dim \mathbb{P}(V \otimes V \otimes V)\}$ *except when* $r = 4$ *and* $\dim V = 3$.

See [15] for the proof of this theorem. For $\dim \sigma_r(Seg(\mathbb{P}V \times \mathbb{P}V \times \mathbb{P}V))$, the expected dimension will be $r(3 \dim V - 3) + r - 1 = 3r \dim V - 2r - 1$. This theorem allows us to compute that a "random" tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ will have border rank $\lceil \frac{m^3}{3m-2} \rceil$. Let $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{C}^m)$ and $\mathbb{P}^{m^3-1} = \mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m)$. We note that $\sigma_r(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1})) = \mathbb{P}^{m^3-1}$ when $r(3m-3) + r - 1 \geq m^3 - 1$, or equivalently, $r \geq \frac{m^3}{3m-2}$. Solving for the minimal such integer $r$, we get the desired value. One may say more precisely that the set of tensors not having this property is a proper algebraic subvariety and thus will be a set of measure 0.

Another variety we will make use of is the Grassmannian, which we define below. Given $T \in A \otimes B \otimes C$, analogous to the correspondence of a tensor with a bilinear form, we may regard it as a trilinear form $T \in \mathrm{Hom}_{\mathbb{C}}(A^* \otimes B^* \otimes C^*, \mathbb{C})$, which we will denote by $T(x_1, x_2, x_3)$. Let $v_1, \cdots, v_k, \in V$ and $V^{\otimes k}$ be the tensor product of $V$ with itself $k$ times. Also, let $\mathfrak{S}_k$ be the symmetric group on $k$ elements. Define $v_1 \wedge \cdots \wedge v_k := \sum_{\sigma \in \mathfrak{S}_k} sgn(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in V^{\otimes k}$. Let $\Lambda^k V := \{T \in V^{\otimes k} \mid T(x_1, \cdots, x_k) = sgn(\sigma) T(x_{\sigma(1)}, \cdots, x_{\sigma(k)}) \ \forall \sigma \in \mathfrak{S}_k\}$, which we call the set of skew symmetric tensors. Note that by the above definitions, $v_1 \wedge \cdots \wedge v_k \in \Lambda^k V$ for all $v_i \in V$.

**Definition 7.** The Grassmannian variety is $G(k, V) := \mathbb{P}\{T \in \Lambda^k V \mid \exists v_1, \cdots, v_k \in V$ such that $T = v_1 \wedge \cdots \wedge v_k\}$

See [11] for proof that this is in fact a projective variety. This variety parametrizes the set of $k$-planes in $V$, i.e. $v_1 \wedge \cdots \wedge v_k$ corresponds to the $k$-plane spanned by the $k$ vectors $v_1, \cdots, v_k$.

### 2.2 Representation Theory

Recall that $A, B, C$, and $V$ are complex vector spaces. A guiding principle in geometric complexity theory is to use symmetry to reduce the problem of testing a space of tensors to testing particular representatives of families of tensors. To describe the symmetry of our tensor, we use the language of the representation theory of linear algebraic groups and Lie algebras. See [16] for the definitions of linear algebraic groups, semisimple Lie algebras, representations, orbits of group actions, $G$-modules, irreducibility of a representation/module, Borel subgroups, a maximal torus, and the correspondence between Lie groups and Lie algebras.

The group $GL(A) \times GL(B) \times GL(C)$ acts on $A \otimes B \otimes C$ by the product of the natural actions of $GL(A)$ on $A$, etc. Identify $(\mathbb{C}^*)^{\times 2}$ with the subgroup $\{(aId_A, bId_B, cId_C) \in GL(A) \times GL(B) \times GL(C) \mid abc = 1\}$ of $GL(A) \times GL(B) \times GL(C)$. Note that $(\mathbb{C}^*)^{\times 2}$ acts trivially on $A \otimes B \otimes C$.

**Definition 8.** For a tensor $T \in A \otimes B \otimes C$, define the *symmetry group of* $T$ to be the group $G_T := \{g \in GL(A) \times GL(B) \times GL(C)/(\mathbb{C}^*)^{\times 2} \mid gT = T\}$.

In the case where $A, B, C$ all have the same dimension, we addtionally have $\mathfrak{S}_3$ symmetry corresponding to permuting the factors $A, B, C$ (after having explicit isomorphisms between them), so we define $G_T := \{g \in (GL(A) \times GL(B) \times GL(C)/(\mathbb{C}^*)^{\times 2}) \rtimes \mathfrak{S}_3 \mid gT = T\}$.

In the case of $T_{\mathfrak{sl}_n}$, our symmetry group, $G_{T_{\mathfrak{sl}_n}}$, is in fact isomorphic to $SL_n$. For any element $g \in SL_n$, we have the element $g^* \otimes g^* \otimes g$ acts on $\mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n$ and leave $T_{\mathfrak{sl}_n}$ invariant. It is always the case that for any automorphism of $\mathfrak{sl}_n$, we will have an automorphism of $\mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n$. See [**?**] for a proof that these are all elements of the symmetry group for $T_{\mathfrak{sl}_n}$.

Let $B_T \subset G_T$ denote a *Borel subgroup*. In the case of $T_{\mathfrak{sl}_n}$, where our symmetry group is isomorphic to $SL_n$, we take $B_T$ to be the Borel subgroup of upper triangular matrices of determinant 1. We note that Borel subgroups in general are not unique, but are all conjugate. For this Borel subgroup, let $N \subset B_T$ denote the group of upper triangular matrices with diagonal entries equal to 1, called the *maximal unipotent group*, and let $\mathbb{T}$ denote the subgroup of diagonal matrices, also called the *maximal torus*.

**Definition 9.** A vector $v \in V^{\otimes k}$ is a *weight vector* if $\mathbb{T}[v] = [v]$. In particular, for $t \in \mathbb{T}$, where $t = \text{diag}\{t_1, \cdots, t_n\}$, if $tv = t_1^{p_1} \cdots t_n^{p_n} v$, then $v$ is said to have weight $(p_1, \cdots, p_n) \in \mathbb{Z}^n$.

Furthermore, call $v$ a highest weight vector if $B_T[v] = [v]$, i.e. if $Nv = v$.

In our case where $G_T \simeq SL_n$, every irreducible $G_T$-module will have a highest weight line and addtionally will be be uniquely determined by this highest weight line.

Given a symmetry group, $G_T$, we also have a symmetry Lie algebra, denoted $\mathfrak{g}_T$, which will be more convenient to work with. Let $\mathfrak{b}_T$ denote the Borel subalgebra and $\mathfrak{h}$ denote the Cartan subalgebra, which will be the Lie algebras of the Borel subgroup, $B_T$, and maximal torus, $\mathbb{T}$, respectively. In the case of our symmetry Lie algebra, $\mathfrak{sl}_n$, we take $\mathfrak{h}$ to be the subalgebra of traceless diagonal matrices. Additionally, for $N \subset B_T$, we have the corresponding Lie subalgebra $\mathfrak{n} \subset \mathfrak{b}_T$, which will consist of the strictly upper triangular elements of $\mathfrak{sl}_n$. We will refer to elements of $\mathfrak{n}$ as raising operators.

For a Lie algebra representation, a vector is a *weight vector*, if $\mathfrak{h}[v] = [v]$. One may regard the weight of a vector $v$ as a linear functional $\lambda \in \mathfrak{h}^*$, such that for all $H \in \mathfrak{h}$, $Hv = \lambda(H)v$. Analogous to the above definition, $[v]$ is a highest weight vector if and only if $\mathfrak{n}[v] = 0$. Let $e_i^j$ denote the $n \times n$ matrix with $1$ in the $(i, j)$th entry. Consider the weights $\omega_i \in \mathfrak{h}^*$ satisfying $\omega_i(h_j) = \delta_{ij}$ for all $1 \leq i, j \leq n-1$, where $h_j = e_j^j - e_{j+1}^{j+1} \in \mathfrak{h}$. Call these weights $\omega_1, \cdots, \omega_{n-1} \in \mathfrak{h}^*$ the fundamental weights of $\mathfrak{sl}_n$. It is well known that the highest weight $\lambda$ of an irreducible representation can be represented as an integral linear combination of fundamental weights. For notational convenience, we denote the irreducible representation with highest weight $\lambda = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$ by $[a_1 \ \cdots \ a_{n-1}]$ and denote its highest weight vector by $v_{[a_1 \ \cdots \ a_{n-1}]}$. We note that this notation is different from the widely used diagram notation of Weyl [16]

**Definition 10.** A variety $X \subset \mathbb{P}V$ is *G-homogeneous* if it is a closed orbit of some point $x \in \mathbb{P}V$ under the action of some group $G \subset GL(V)$. If $P \subset G$ is the subgroup fixing $x$, write $X = G/P$.

**Example 2.2.1.** Let $a \in A, b \in B, c \in C$ and $v_1, \cdots, v_k \in V$. Note that both the Segre and Grassmannian varieties are both homogeneous varieties as $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) = GL(A) \times GL(B) \times GL(C) \cdot [a \otimes b \otimes c] \subset \mathbb{P}(A \otimes B \otimes C)$ and $G(k, V) = GL(V) \cdot v_1 \wedge \cdots \wedge v_k \subset \mathbb{P}(\Lambda^k V)$.

The purpose for introducing the language of homogeneous varieties is to introduce the following Normal Form Lemma. Recall that $V$ be a complex vector space.

**Lemma 11** (Normal Form Lemma)**.** *Let $X = G/P \subset \mathbb{P}V$ be a homogeneous variety and $v \in V$ such that $G_v = \{g \in G \mid g[v] = [v]\}$ has a single closed orbit $\mathcal{O}_{min}$ in $X$. Then any border rank $r$ decomposition of $v$ may be modified using $G_v$ to a border rank $r$ decomposition* $\lim_{\epsilon \to 0} x_1(\epsilon) + \cdots + x_r(\epsilon)$ *such that there is a stationary point $x_1(t) \equiv x_1$ (i.e. $x_1$ is independent of $t$) lying in $\mathcal{O}_{min}$.*

*If, moreover, every orbit of $G_v \cap G_{x_1}$ contains $x_1$ in its closure, we may further assume that for all $j \neq 1$, $\lim_{\epsilon \to 0} x_j(\epsilon) = x_1$.*

See Lemma 3.1 in [17] for the proof. This Lemma allows one describe the interactions between the different $G$-orbits. It can be thought of as a consequence of the following theorem.

**Theorem 12** (Lie's Theorem)**.** *Let $H$ be a solvable group and $W$ be an $H$-module. For $[w] \in \mathbb{P}W$, $\overline{H[w]}$ contains an $H$-fixed point.*

See Theorem 9.11 of [16] for proof. We show a simple example to foreshadow how this idea will be used.

**Example 2.2.2.** Let $\{e_i\}_{i=1}^4$, be a basis of a vector space, $V$. Consider the element $[v] = [e_1 \wedge e_2 + e_3 \wedge e_4] \in \mathbb{P}(\Lambda^2(V))$. Let $g_t \in GL(V)$ be the element that maps $e_1$ to $\frac{1}{t}e_1$ and fixes the remaining basis elements. Then acting on $[v]$, we get $[\frac{1}{t}e_1 \wedge e_2 + e_3 \wedge e_4] = [e_1 \wedge e_2 + te_3 \wedge e_4]$. As $t \to 0$, then we limit towards $[e_1 \wedge e_2] \in G(2, V)$.

This Lemma and Theorem are used to assert that a limiting point is in fact $B_T$-fixed.

## 3. METHODOLOGY

The most fruitful current techniques for finding lower bounds on the border rank of a tensor are Koszul flattenings, the border substitution method, and border apolarity. In this chapter, we review each of these techniques.

### 3.1 Koszul flattenings

For $T \in A \otimes B \otimes C$, we may consider it as a linear map $T_B : B^* \to A \otimes C$. We have analogous maps $T_A, T_B, T_C$, which are called the *coordinate flattenings* of $T$. Consider the linear map obtained by composing the map $T_B \otimes Id_{\Lambda^p A} : B^* \otimes \Lambda^p A \to \Lambda^p A \otimes A \otimes C$ with the map $\pi \otimes Id_C : \Lambda^p A \otimes A \otimes C \to \Lambda^{p+1} A \otimes C$. Note that $\pi \otimes Id_C$ is the tensor product of the exterior multiplication map with the identity on $C$. Denote this composition by $T_A^{\wedge p}$. Let $rank$ denote the rank of a linear map.

**Proposition 13** (Landsberg-Ottaviani)**.** *Let $T \in A \otimes B \otimes C$ and $t = a \otimes b \otimes c \in A \otimes B \otimes C$, then $\underline{\boldsymbol{R}}(T) \geq \frac{rank(T_A^{\wedge p})}{rank(t_A^{\wedge p})} = \frac{rank(T_A^{\wedge p})}{\binom{\dim A-1}{p}}$*

*Proof.* Note that in terms of bases $T = \sum_{i,j,k} t^{ijk} a_i \otimes b_j \otimes c_k$ and for $\beta \otimes f_1 \wedge \cdots \wedge f_k \in B^* \otimes \Lambda^p A$, the Koszul flattening map is $T_A^{\wedge p}(\beta \otimes f_1 \wedge \cdots \wedge f_k) = \sum_{ijk} t^{ijk} \beta(b_j) f_1 \wedge \cdots \wedge f_k \wedge a_i \otimes c_k$, Therefore, for a rank one tensor $t = a \otimes b \otimes c$, then the image of $t_A^{\wedge p}$ is $\{f_1 \wedge \cdots \wedge f_k \wedge a \otimes c \in \Lambda^{p+1} A \otimes C \mid a \notin \operatorname{span}\{f_1, \cdots, f_k\}\}$, so then $\operatorname{rank}(t_A^{\wedge p}) = \binom{\dim A-1}{p}$.

Suppose $\underline{\mathbf{R}}(T) = r$ and let $T = \lim T_\epsilon$ with rank $r$ decompositions $T_\epsilon = \sum_{i=1}^r T_i(\epsilon)$. The map $T \mapsto T_A^{\wedge p}$ is linear, and so we have

$$\operatorname{rank}(T_A^{\wedge p}) \leq \operatorname{rank}((T_\epsilon)_A^{\wedge p}) \leq \sum_{i=1}^{r} \operatorname{rank}(T_i(\epsilon)_A^{\wedge p}) = r\binom{\dim A - 1}{p}$$

$\square$

One should note that we achieve the best bounds when $\dim A = 2p+1$. Thus, if $\dim A >$

$2p+1$, we may restrict $T$ to subspaces $A' \subset A$ of dimension $2p+1$, since border rank is upper semi-continuous with respect to restriction i.e. for a restriction $T'$ of $T$, $\underline{\mathbf{R}}(T') \leq \underline{\mathbf{R}}(T)$. Koszul flattenings alone are insufficient to prove $\underline{\mathbf{R}}(T_{\mathfrak{sl}_n}) \geq 2(n^2-1)$, as the limit of the method for $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is below $2m-3$(m even) and $2m-5$(m odd). See [18] for more on this method.

As a remark, we have that Koszul flattening do provide a lower bound on the rank of a tensor, as $\mathbf{R}(T) \geq \underline{\mathbf{R}}(T)$. Koszul flattenings are not able to produce lower bounds on the rank alone. The conditions on the ranks of the Koszul flattenings furnish equations (in particular, the minors of the linear maps) for the secant variety. So we are obtaining equations for a Zariski closed set containing the set of tensors of rank $r$, which is not closed. We note here that trivially $\mathbf{R}(T) \geq \max\{\text{rank}(T_A), \text{rank}(T_B), \text{rank}(T_C)\}$, and in some cases the substitution method, presented in the next section, improves upon this trivial lower bound.

### 3.2 Border Substitution

The only known technique for computing lower bounds on the rank of a tensor is the *substitution method*. A tensor $T$ is *A-concise* if the coordinate flattening map $T_A$ is injective, and define similarly for $B$-concise and $C$-concise. If a tensor is $A$-concise, $B$-concise, and $C$-concise, then we simply call it *concise*. We remark that $T_{\mathfrak{sl}_n}$ is in fact a concise tensor, since $\mathfrak{sl}_n$ is a simple Lie algebra and the coordinate flattening maps do not send everything to 0.

**Proposition 14** (Alexeev-Forbes-Tsimerman)**.** *Let $T \in A \otimes B \otimes C$ be $A$-concise. Let $\dim A = m$ and fix $\tilde{A} \subset A$ of dimension $k$. Then*

$$\boldsymbol{R}(T) \geq \min_{\{A' \in G(k,A^*) \mid A' \cap \tilde{A}^{\perp} = 0\}} \boldsymbol{R}(T\big|_{A' \otimes B^* \otimes C^*}) + (m-k)$$

See Proposition 5.3.1.1 in [12] for proof. In practice, the substitution method applies this proposition iteratively, while also allowing $B$ and $C$ to play the role of $A$.

This proposition can be extended to border rank.

**Proposition 15** (Landsberg-Michałek)**.** *Let $T \in A\otimes B\otimes C$ be $A$-concise. Let $\dim A = m$ and let $k < m$. Then*

$$\underline{\boldsymbol{R}}(T) \geq \min_{A' \in G(k,A^*)} \underline{\boldsymbol{R}}(T\big|_{A'\otimes B^*\otimes C^*}) + (m-k)$$

*Proof.* Suppose $T$ has border rank $r$ with border rank decomposition $T = \lim_{\epsilon\to 0} T_\epsilon$, with $T_\epsilon = \sum_{k=1}^{r} a_k(\epsilon)\otimes b_k(\epsilon)\otimes c_k(\epsilon)$. Without loss of generality, let $a_i(\epsilon)$ for $i = 1,\cdots,m$ be a basis of $A$. Let $A'_\epsilon = \langle a_{k+1}(\epsilon),\cdots,a_m(\epsilon)\rangle^\perp \subset A^*$. Applying the substitution method, we obtain $r = \mathbf{R}(T_\epsilon) \geq (m-k) + \mathbf{R}(T_\epsilon\big|_{A'_\epsilon\otimes B^*\otimes C^*})$. Let $A' = \lim_{\epsilon\to 0} A'_\epsilon$. Taking limits as $\epsilon \to 0$, we may no longer have the restriction that the limiting plane in the grassmannian trivially intersects $\langle a_1(0),\cdots,a_k(0)\rangle^\perp$. Therefore, we must minimize over all elements of the Grassmannian. $\square$

Note that the notation $T\big|_{A'\otimes B^*\otimes C^*}$ is a restriction of $T$ when considering $T$ as a trilinear form $T : A^*\otimes B^*\otimes C^* \to \mathbb{C}$. If we let $\tilde{A} = A/(A')^\perp$ then our restricted tensor will be an element of $\tilde{A}\otimes B\otimes C$.

Also note that in the border substitution proposition, we are minimizing over all elements in the Grassmannian. In practice, applying border substitution uses tensors with large symmetry groups $G_T$. The utility is that one may restrict to looking at representatives of closed $G_T$-orbits in the Grassmannian, rather than by examining all elements of the Grassmannian. One often achieves the best results on the rank of a tensor by using border substitution in conjunction with Koszul flattenings. Naively, the largest lower bound obtainable by the method, i.e. the limit of the method, is at most $\dim A + \dim B + \dim C - 3$, however, the limit is in fact slightly less. For tensors $T \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$, the limit of the method is $3m - 3\sqrt{3m+\frac{9}{4}}+\frac{9}{2}$. See [17] for a proof of this and more on this method. The best lower bound achieved on the border rank that is mentioned in the introduction is achieved using this method [8].

### 3.3 Border Apolarity

In order to establish larger lower bounds on $\underline{\mathbf{R}}(T_{\mathfrak{sl}_3})$ than can be achieved by Koszul flattenings and border substitution for $T_{\mathfrak{sl}_3}$, we will use the idea of border apolarity, as developed in [19] and [1].

Suppose $T$ has a border rank $r$ decomposition, $T = \lim_{\epsilon\to 0} T_\epsilon$, where $T_\epsilon = \sum_{i=1}^r T_i(\epsilon)$. If the rank summands $T_i(\epsilon)$ are in general position in $A \otimes B \otimes C$, then we may identify the border rank decomposition with a curve $E_\epsilon$ in the Grassmannian variety $G(r, A\otimes B\otimes C)$, by taking the exterior product of the $T_i(\epsilon)$, i.e. $E_\epsilon = [T_1(\epsilon) \wedge \cdots \wedge T_r(\epsilon)]$.

Now we define a $\mathbb{Z}^3$-grading on ideals of subsets of $\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C$ (i.e. the Segre variety) from the natural $\mathbb{Z}^3$-grading of $Sym(A \oplus B \oplus C)^*$. Let $Irrel := \{0 \oplus B \oplus C\} \cup \{A \oplus 0 \oplus C\} \cup \{A \oplus B \oplus 0\} \subset A \oplus B \oplus C$. Since $\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C \simeq (A \oplus B \oplus C \backslash Irrel)/(\mathbb{C}^*)^{\times 3}$, then we may consider the quotient map $q : (A \oplus B \oplus C)\backslash Irrel \to \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C$, which will be invariant under the action of $(\mathbb{C}^*)^{\times 3}$. Therefore, for a set $Z \subset \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, the ideal of this set $I(Z) = I(q^{-1}(Z)) \subset Sym(A \oplus B \oplus C)^*$ will have a $\mathbb{Z}^3$ grading. In particular, for a single point $([a],[b],[c]) \in \mathbb{P}A \times \mathbb{P}A \times \mathbb{P}C$, corresponding to a rank one tensor ($[a \otimes b \otimes c] \in \mathbb{P}(A \otimes B \otimes C)$), we are considering the ideal in $Sym(A \oplus B \oplus C)^*$ of polynomials vanishing along the lines $a, b$, and $c$.

Let $I_\epsilon$ denote the $\mathbb{Z}^3$-graded ideal of the set of the r points $[T_i(\epsilon)]$, i.e. $I_{ijk,\epsilon} \subset S^iA^* \otimes S^jB^* \otimes S^kC^*$. Since the $r$ points are in general position, then $\operatorname{codim} I_{ijk,\epsilon} = r$. Define $I_{ijk} := \lim\limits_{\epsilon\to 0} I_{ijk,\epsilon}$ as the limit of points in the Grassmannian $G(\dim(S^iA^* \otimes S^jB^* \otimes S^kC^*) - r, \dim(S^iA^* \otimes S^jB^* \otimes S^kC^*))$. $I_{ijk}$ will exist, since the Grassmannian is compact, however, the resulting ideal $I$ may not be saturated. See [19] for further discussion on this.

Recall that a tensor $T$ is concise if all the coordinate flattening maps $T_A, T_B, T_C$ are injective. For a subspace $U \subset V$, define $U^\perp := \{\alpha \in V^* \mid \alpha(u) = 0 \, \forall u \in U\}$. In a nutshell, border apolarity gives us some necessary conditions on the possible limiting ideals, $I$, that can arise from a border rank decomposition.

**Theorem 16.** *(Weak Border Apolarity) Let $X = \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C$ and $S[X]$ be its coordinate ring. Suppose a tensor $T$ has $\underline{\mathbf{R}}(T) \leq r$. Then there exists a (multi)homogeneous ideal $I \subset S[X]$ such that*

- $I \subset Ann(T)$
- *For each multidegree $ijk$, the $ijk$th graded piece, $I_{ijk}$, of $I$ has* $\operatorname{codim} I_{ijk} = \min\{r, \dim S[X]_{ijk}\}$

*In addition, if $G_T$ is a group acting on $X$ and preserving $T$, then there exists an $I$ as above which in addition is invariant under a Borel subgroup of $G_T$.*

In [19], see Theorems 3.15 (Border Apolarity) for proof of the first part and see Theorem 4.3 (Fixed Ideal Theorem) for a proof of the second part. Theorem 3.15 and 4.3 of [19] are not stated here as they are stated in greater generality than we require using the language of schemes. We remark that the Weak Border Apolarity Theorem provides sufficiency that if a border rank $r$ decomposition exists, then there will exist another border rank $r$ decomposition satisfying given conditions.

We note that the second condition says that we may in fact take the $r$ points of the border rank decomposition to be in general position, and so our initial supposition that the $r$ points are in general position is justified. Lie's Theorem and the Normal Form Lemma allow us to take $I_{111}$ to be $B_T$-fixed. The Fixed Ideal Theorem of [19] uses the same reasoning to generalize this to prove $B_T$-invariance for all multigraded components $I_{ijk}$, not just a finite number of multigraded components.

In [1], using Weak Border Apolarity Theorem, they assert that for $T$ a concise tensor with a border rank $r$ decomposition, there will exist an ideal $I$ satisfying the following:

1. $I_{ijk}$ is $B_T$-stable ($I_{ijk}$ is a Borel fixed weight space)
2. $I \subset Ann(T)$ i.e. $I_{110} \subset T(C^*)^\perp \subset A^* \otimes B^*$, etc. and $I_{111} \subset T^\perp \subset A^* \otimes B^* \otimes C$
3. For all $i, j, k$ such that $i + j + k > 1$, then $\operatorname{codim} I_{ijk} = r$, i.e. the condition that we may take the $r$ points of the border rank decomposition to be in general position.
4. Since $I$ is an ideal, the image of the multiplication map $I_{i-1,j,k} \otimes A^* \oplus I_{i,j-1,k} \otimes B^* \oplus I_{i,j,k-1} \otimes C^* \to S^i A^* \otimes S^j B^* \otimes S^k C^*$ is contained in $I_{ijk}$

We note that the last condition is simply the condition that $I$ is an ideal where the multiplication respects the grading. The border apolarity algorithm presented in [1] makes use of these conditions and attempts to iteratively construct all possible ideals, $I$, in each multidegree. In particular, if at any multidegree $ijk$, there does not exist an $I_{ijk}$ satisfying the above, then we may conclude that $\underline{\mathbf{R}}(T) > r$. We remark that the candidate ideals, $I$, do not necessarily correspond to an actual border rank decomposition of the tensor. We describe the algorithm of [1] precisely below:

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**Algorithm 1** Border Apolarity Algorithm

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Input: $T \in A \otimes B \otimes C$, $r$

Output: Candidate ideals or $\underline{\mathbf{R}}(T) > r$

1. For each $B_T$-fixed space $F_{110}$ of codim $r - \dim C$ in $T(C^*)^\perp$ (i.e. codim $r$ in $A^* \otimes B^*$) compute ranks of maps $F_{110} \otimes A^* \to S^2A^* \otimes B^*$ and $F_{110} \otimes B^* \to A^* \otimes S^2B^*$

   If both have images of codim at least $r$, then $F_{110}$ is possible $I_{110}$. These are called the (210) and (120) tests.

2. Perform analogously for possible $I_{101} \subset T(B^*)^\perp$ and $I_{011} \subset T(A^*)^\perp$ for candidate $F_{101}$ and $F_{011}$

3. For each triple $F_{110}, F_{101}, F_{011}$, compute rank of map $(F_{110} \otimes C^*) \oplus (F_{101} \otimes B^*) \oplus (F_{011} \otimes A^*) \to A^* \otimes B^* \otimes C^*$

   If codim of image is at least $r$, then have a candidate triple. $F_{111}$ is candidate for $I_{111}$ if it is codim $r$, it is contained in $T^\perp$ and contains image of above map.

4. Analogous higher degree tests

5. If at any point there are no such candidates $\underline{\mathbf{R}}(T) > r$, otherwise stabilization of candidate ideals will occur at worst multi-degree $(r, r, r)$

---

The condition that $I_{ijk}$ is $B_T$-fixed allows us to greatly reduce the search for possible candidate ideals. The $B_T$-fixed spaces are easier to list than trying to list all possible $I_{ijk}$. This condition allows the algorithm of [1] to be feasible for tensors with large symmetry groups. Then, using the assumption that our points are in general position, one has rank conditions on the multiplication maps, as the images must have codim at least $r$.

#### 3.3.1 Implementation of Border Apolarity for $T_{\mathfrak{sl}_n}$

We show how to implement the algorithm of [1] for $T_{\mathfrak{sl}_n}$ by describing how to compute all possible $B_T$-fixed $I_{110}$. Additionally, we can leverage the skew-symmetry of $T_{\mathfrak{sl}_n}$ to reduce the amount of computation involved for determining potential $I_{110}, I_{101}, I_{011}$.

Let $T = T_{\mathfrak{sl}_3} \in \mathfrak{sl}_3^* \otimes \mathfrak{sl}_3^* \otimes \mathfrak{sl}_3 = A \otimes B \otimes C$. The first and third condition from border apolarity tells us to compute all $B_T$-fixed weight subspaces $F_{110} \subset A^* \otimes B^*$ of codimension $r$; however, since $T$ is concise with $\dim T(C^*) = \dim \mathfrak{sl}_3 = 8$ and $F_{110} \subset T(C^*)^\perp$ by the second condition of border apolarity, we compute all $B_T$-fixed weight spaces $F_{110} \subset T(C^*)^\perp$ of codimension $r - 8$.

Standard computational methods, see [16], yield that the irreducible decomposition of $A^* \otimes B^*$ as $\mathfrak{sl}_n$-modules is as follows.

$$A^* \otimes B^* = \mathfrak{sl}_n \otimes \mathfrak{sl}_n \tag{3.1}$$

$$\simeq [2\ 0\ \cdots\ 0\ 2] \oplus [2\ 0 \cdots 0\ 1\ 0] \oplus [0\ 1\ 0 \cdots\ 0\ 2] \oplus [0\ 1\ 0 \cdots\ 0\ 1\ 0] \oplus 2[1\ 0\ \cdots 0\ 1] \oplus [0\ \cdots\ 0] \tag{3.2}$$

In particular, for $\mathfrak{sl}_3$, we have the following decomposition into $\mathfrak{sl}_3$-modules.

$$\mathfrak{sl}_3 \otimes \mathfrak{sl}_3 \simeq [2\ 2] \oplus [3\ 0] \oplus [0\ 3] \oplus 2[1\ 1] \oplus [0\ 0] \tag{3.3}$$

Using this decomposition and the conciseness of $T$, we have the $\mathfrak{sl}_3$-module decomposition $T(C^*)^\perp \simeq [2\ 2] \oplus [3\ 0] \oplus [0\ 3] \oplus [1\ 1] \oplus [0\ 0]$ (Note that $\dim T(C^*)^\perp = 56$). Using this $\mathfrak{sl}_3$-module decomposition, we can obtain a decomposition of $T(C^*)^\perp$ into weight spaces (decomposition into $\mathfrak{h}$-modules) by combining the weight space decompositions of each $\mathfrak{sl}_3$-module into one poset. The result is the Figure 3.1. Each node represents a weight space of weight $\lambda$ labeled by $(\lambda,$ dimension of weight space in $T(C^*)^\perp)$. Also note that arrows go from lower weights to higher weights, so the highest weight occuring in $T(C^*)^\perp$ is [2 2]. The weight space [3 0] is of dimension 2, where one of the basis elements for this weight space is the highest weight vector of the $\mathfrak{sl}_3$-module $[3\ 0]$ and the other basis element for this weight space is the vector arising from lowering the highest weight vector of the $\mathfrak{sl}_3$-module $[2\ 2]$.

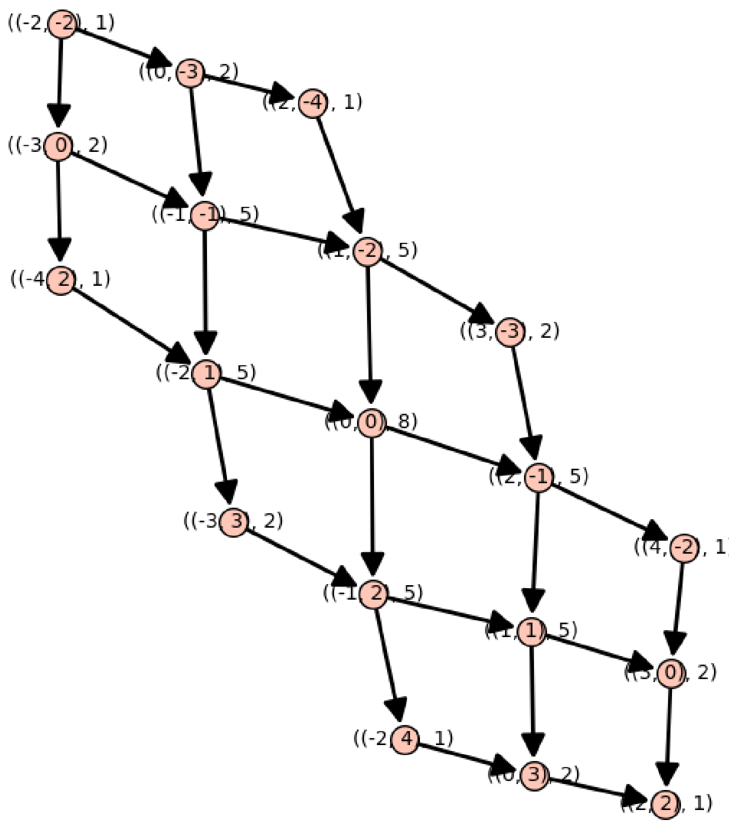


Figure 3.1: Weight Decomposition of $T(C^*)^\perp$

The utility of this decomposition is that we can generate all $B_T$-fixed weight subspaces $F_{110}$ by taking a collection of weight vectors $v_i$ from the poset such that $v_1 \wedge v_2 \wedge \cdots \wedge v_{56-(r-8)}$ is a highest weight vector in $G(56-(r-8), T(C^*)^{\perp})$ and consequently is closed under the raising operators. One should note that if $v_i$ comes from a weight space of dimension greater than 1 then one needs to include linear combinations of basis vectors of that weight space.

**Example 3.3.1.** We show a few small examples of possible $F_{110}$ $B_T$-fixed subspaces to provide intuition of how these spaces are computed.

For $r = 63$, we have that $F_{110}$ will be of the form $v_1$. Since it must be closed under raising operators, it will necessarily be a highest weight vector. Therefore, our choices will be $v_1 = v_\lambda$ where of $v_\lambda$ is a highest weight vector of weight $\lambda = [2\ 2]$, $[3\ 0]$, $[0\ 3]$, $[1\ 1]$, or $[0\ 0]$.

For $r = 62$, $F_{110}$ will be of the form $v_1 \wedge v_2$. Necessarily, we must have that $v_1$ must be a highest weight vector. The second vector $v_2$ may either be another highest weight vector, a weight vector that can be raised to $v_1$, or a linear combination of the two previous cases if they are vectors of the same weight. For example, if we take $v_1 = v_{[2\ 2]}$ to be the highest weight vector of $[2\ 2]$. Let $v_{[3\ 0]}$ and $u_{[3\ 0]}$ be a weight basis for $[3\ 0]$ with $v$ being a highest weight vector. Assume similarly for $[0\ 3]$ weight space. The possible choices for $v_2$ are weight vectors of the following types: $v_{[3\ 0]}, sv_{[3\ 0]} + tu_{[3\ 0]}, v_{[0\ 3]}, sv_{[0\ 3]} + tu_{[0\ 3]}, v_{[1\ 1]}, v_{[0\ 0]}$ with $s, t$ parameters. Applying all possible raising operators to $v_1 \wedge v_2$ where $v_2$ has parameters will provide equations for what values of $s, t$ give us a highest weight vector.

For smaller values of $r$, the number of possible Borel fixed spaces is much larger and more difficult to list by hand without the aide of a computer. The computationally difficult step in this algorithm lies in computing the ranks of the multiplication maps such as $F_{110} \otimes A^* \to S^2A^* \otimes B^*$. In some cases there are many parameters which arise from choosing weight vectors from high dimensional weight spaces, such as the $[0\ 0]$ weight space in Figure 3.1. Recall that a linear map has rank at most $k$ is the $k+1$ minors all vanish. In order to determine whether the multiplication map has image codimension $r$, we look at the appropriate minors of this linear map. When there are no parameters involved, this is a simple linear algebra calculation. However, in some cases, the entries of the multiplication map are linear polynomials in the parameters coming from choosing a linear combination of weight vectors. In order to determine whether the multiplication map has image of codimension $r$, one needs to look at the ideal of these minors, as well as some polynomial equations in the parameters that are needed for the space to be Borel fixed. One must do a Groebner Basis computation on this ideal to determine whether all the minors vanish or not. This can become an unfeasible computation if there are too many parameters and/or equations.

### 3.3.2 Flag Condition

In addition to the necessary conditions on $I$ that come from border apolarity, we have some more necessary conditions called the Flag Conditions in [20]. These additional conditions should help to mitigate the computational issue of having many parameters. We recall that the candidate ideals $I$ generated from implementing border apolarity on a tensor, $T$, may not necessarily arise from an actual border rank decomposition of $T$. For a given $E_{stu} := I_{stu}^{\perp}$ for the ideal defined above, we call it *viable* if it arises from an actual border rank decomposition.

**Proposition 17** (Flag Conditions)**.** *If $E_{110}$ is viable then there exists a $B_T$-fixed filtrand of $E_{110}$, namely $F_1 \subset \cdots \subset F_r = E_{110}$ such that $F_j \subset \sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B))$. Let $T_j \in \mathbb{C}^j \otimes C^j \otimes \mathbb{C}^j$ be a tensor equivalent to the tensor restricted to subspace $F_j$. Then $\underline{\mathbf{R}}(T_j) = j$.*

*Generally, if $E_{stu}$ is viable, there are complete flags in $A, B, C$ such that $\mathbb{P}(E_{stu} \cap S^sA_j \otimes S^tB_j \otimes S^uC_j) \subset \sigma_j(Seg(\mathbb{P}S^sA_j \times \mathbb{P}S^tB_j \times \mathbb{P}S^uC_j))$.*

See [20] for the proof of this proposition. The restricted tensors having minimal border rank for $j \leq m$ are the new necessary conditions. The classification theorem, Theorem 1.2 from [21], provides choices for the forms of the first three filtrands $F_1, F_2$, and $F_3$. For example, the first filtrand must be of the form $F_1 = \langle a \otimes b \rangle$. The second filtrand will have one of the two forms, $F_2 = \langle a \otimes b, a' \otimes b' \rangle$ or $F_2 = \langle a \otimes b, a \otimes b' + a' \otimes b \rangle$, where $a'$ and $b'$ denote tangent vectors of $a$ and $b$, respectively. Note that since the tangent space $\hat{T}_x\mathbb{P}A = A$, we may take $a'$ and $b'$ to be arbitrary vectors. There are five choices for $F_3$; see [21] or [20] for all five explicitly listed. These choices allow us to put conditions on the rank of some of the highest weight vectors occuring.

In particular for $T_{\mathfrak{sl}_3}$, we may eliminate candidate $E_{110}$ which contain $v_{[0\ 0]}$, the highest weight vector in $[0\ 0]$ weight space, since the rank of this weight vector is too high. The $[0\ 0]$ weight space has the highest dimension in $T(C^*)^\perp$ and so limiting the number of choices of weight vectors from that space decreases the number of parameters needed. While this condition helps to eliminate certain cases which may contain too many parameters, it does not help in the cases where the Groebner basis computation has too many equations.

## 4. CURRENT RESULTS

It is known that $\underline{\mathbf{R}}(T_{\mathfrak{sl}_2}) = 5$ [22], so we aim to find bounds on $\underline{\mathbf{R}}(T_{\mathfrak{sl}_n})$ for $n = 3$ and $4$ using the above techniques.

### 4.1 Koszul flattenings

In the case of $T_{\mathfrak{sl}_3}$, we achieve the best results when $p = 3$ and we restrict to a generic 7 dimensional subspace of $\mathfrak{sl}_3$, since $\dim \mathfrak{sl}_3 = 8$. The best bound achieved is $\underline{\mathbf{R}}(T_{\mathfrak{sl}_3}) \geq 14$ (See Table 4.2).

Table 4.1: $T_{\mathfrak{sl}_3}$ Results

| p | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|
| 1 | (64,224) | 0 | 10 |
| 2 | (224,448) | 1 | 11 |
| 3 | (448,560) | 8 | 13 |

Table 4.2: $T_{\mathfrak{sl}_3}$ Restriction to a generic subspace of dim $k$ Results

| p | k | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|---|
| 1 | 3 | (24,24) | 0 | 12 |
| 2 | 5 | (80,80) | 4 | 13 |
| 3 | 7 | (280,280) | 7 | 14 |

In the case of $T_{\mathfrak{sl}_4}$, we achieve the best lower bound of $27$ when $p = 4$ or $5$ while restricting to a subspace (See Table 4.4).

Table 4.3: $T_{\mathfrak{sl}_4}$ Results

| p | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|
| 1 | (225,1575) | 0 | 17 |
| 2 | (1575,6825) | 1 | 18 |
| 3 | (6825,20475) | 15 | 19 |
| 4 | (20475,45045) | 106 | 21 |
| 5 | (45045,75075) | 470 | 23 |
| 6 | (75075,96525) | 2680 | 25 |
| 7 | (96525,96525) | 11039 | 25 |

Table 4.4: $T_{\mathfrak{sl}_4}$ Restriction to a generic subspace of dim $k$ Results

| p | k | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|---|
| 1 | 3 | (45,45) | 0 | 23 |
| 2 | 5 | (150,150) | 2 | 25 |
| 3 | 7 | (525,525) | 7 | 26 |
| 4 | 9 | (1890,1890) | 38 | 27 |
| 5 | 11 | (6930,6930) | 176 | 27 |
| 6 | 13 | (25740,25740) | 2254 | 26 |

As stated above, Koszul flattenings alone are insufficient to obtain border rank lower bounds exceeding $2m$, i.e. Koszul flattenings will not prove $\underline{\mathbf{R}}(T_{\mathfrak{sl}_3}) \geq 16$ and $\underline{\mathbf{R}}(T_{\mathfrak{sl}_4}) \geq 30$.

## 4.2 Border Substitution

For $T_{\mathfrak{sl}_n} \in \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n$, we may identify the space $\mathfrak{sl}_n^*$ with $\mathfrak{sl}_n$ (by sending an element to its negative transpose). Therefore, we may identify $T_{\mathfrak{sl}_n}$ as an element of $\mathfrak{sl}_n \otimes \mathfrak{sl}_n \otimes \mathfrak{sl}_n$. As a first step in applying border substitution, we restrict $T_{\mathfrak{sl}_n} \in A \otimes B \otimes C$ in the $A$ tensor factor. Since we may restrict to looking at representatives of closed $G_{T_{\mathfrak{sl}_n}}$-orbits, then the only planes we need to check are the highest weight planes in $G(k, \mathfrak{sl}_n)$. In order to compute the border rank of the restricted tensor, we use Koszul flattenings on the restricted tensor. Once again, let $v_\lambda$ denote the unique weight vector in weight space $\lambda$.

For $T_{\mathfrak{sl}_3}$, border substitution did not generate a better lower bound than the Koszul flattenings. However, we were able to obtain a better lower bound for $T_{\mathfrak{sl}_4}$. Let $A'$, as in Proposition 15, be $\tilde{A}^\perp$ where we take $\tilde{A}$ to be a space of dimension $m - k$. If we restrict our tensor by a one dimensional subspace, then the only choice for $\tilde{A}$ will be the space spanned by $v_{[1\ 0\ 1]}$, which is the highest weight vector of $\mathfrak{sl}_n$.

Table 4.5: $T_{\mathfrak{sl}_4}$ with Restriction $\tilde{A} = v_{[1\ 0\ 1]}$

| p | k | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|---|
| 1 | 3 | (45,45) | 0 | 23 |
| 2 | 5 | (150,150) | 2 | 25 |
| 3 | 7 | (525,525) | 7 | 26 |
| 4 | 9 | (1890,1890) | 38 | 27 |
| 5 | 11 | (6930,6930) | 248 | 27 |
| 6 | 13 | (25740,25740) | 2254 | 26 |

Restricting by a two dimensional subspace, we have one choice for $\tilde{A}$ up to symmetry in the weight space decomposition for $\mathfrak{sl}_4$, namely $v_{[1\ 0\ 1]} \wedge v_{[-1\ 1\ 1]}$.

Table 4.6: $T_{\mathfrak{sl}_4}$ with Restriction $\tilde{A} = v_{[1\ 0\ 1]} \wedge v_{[-1\ 1\ 1]}$

| p | k | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|---|
| 1 | 3 | (45,45) | 0 | 23 |
| 2 | 5 | (150,150) | 2 | 25 |
| 3 | 7 | (525,525) | 7 | 26 |
| 4 | 9 | (1890,1890) | 78 | 26 |
| 5 | 11 | (6930,6930) | 498 | 26 |

Restricting by a three dimensional subspace, we have three choices for $\tilde{A}$ up to symmetry. $\tilde{A}$ may be $v_{[1\ 0\ 1]} \wedge v_{[1\ 1\ -1]} \wedge v_{[-1\ 1\ 1]}$, $v_{[1\ 0\ 1]} \wedge v_{[1\ 1\ -1]} \wedge v_{[2\ -1\ 0]}$, or $v_{[1\ 0\ 1]} \wedge v_{[1\ 1\ -1]} \wedge v_{[-1\ 2\ -1]}$.

Table 4.7: $T_{\mathfrak{sl}_4}$ with Restriction $\tilde{A} = v_{[1\ 0\ 1]} \wedge v_{[1\ 1\ -1]} \wedge v_{[-1\ 1\ 1]}$

| p | k | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|---|
| 1 | 3 | (45,45) | 0 | 23 |
| 2 | 5 | (150,150) | 2 | 25 |
| 3 | 7 | (525,525) | 31 | 25 |
| 4 | 9 | (1890,1890) | 168 | 25 |
| 5 | 11 | (6930,6930) | 755 | 25 |

Table 4.8: $T_{\mathfrak{sl}_4}$ with Restriction $\tilde{A} = v_{[1\ 0\ 1]} \wedge v_{[1\ 1\ -1]} \wedge v_{[2\ -1\ 0]}$

| p | k | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|---|
| 1 | 3 | (45,45) | 0 | 23 |
| 2 | 5 | (150,150) | 2 | 25 |
| 3 | 7 | (525,525) | 7 | 25 |
| 4 | 9 | (1890,1890) | 72 | 25 |
| 5 | 11 | (6930,6930) | 498 | 25 |

Table 4.9: $T_{\mathfrak{sl}_4}$ with Restriction $\tilde{A} = v_{[1\ 0\ 1]} \wedge v_{[1\ 1\ -1]} \wedge v_{[-1\ 2\ -1]}$

| p | k | Dimensions of Linear Map | Dimension of Kernel | Koszul Bound |
|---|---|---|---|---|
| 1 | 3 | (45,45) | 3 | 21 |
| 2 | 5 | (150,150) | 14 | 23 |
| 3 | 7 | (525,525) | 42 | 25 |
| 4 | 9 | (1890,1890) | 254 | 24 |
| 5 | 11 | (6930,6930) | 1072 | 24 |

The best bound we obtain is $\underline{\mathbf{R}}(T_{\mathfrak{sl}_4}\big|_{(v_{[1\ 0\ 1]})^{\perp}\otimes\mathfrak{sl}_n\otimes\mathfrak{sl}_n}) \geq 27$ (See Table 4.5). By Proposition 15,

**Theorem 18.** $\underline{\boldsymbol{R}}(T_{\mathfrak{sl}_4}) \geq 28$

After restricting in the $A$ tensor factor, we cannot restrict by the highest weight vector of $\mathfrak{sl}_n$ in the $B$ or $C$ factor as the symmetry group of the restricted tensor will have a different symmetry group.

### 4.3 Border Apolarity

We use border apolarity to disprove that $T_{\mathfrak{sl}_3}$ has rank $r = 15$. We first compute candidate $F_{110}$ spaces which passed the $(210)$-test. There were a total of 5 candidate $F_{110}$ subspaces out of a total of more than 1245 possible $F_{110}$ spaces. The candidate $F_{110}$ spaces came in three types of weight space decompositions:

Table 4.10: Two candidate $F_{110}$ planes have the following weight decomposition

| Weight | Dimension of weight space in $A^* \otimes B^*$ | Dimension of weight space in $F_{110}$ |
|---|---|---|
| $[2, 2]$ | 1 | 1 |
| $[3, 0]$ | 2 | 2 |
| $[4, -2]$ | 1 | 1 |
| $[0, 3]$ | 2 | 2 |
| $[1, 1]$ | 5 | 5 |
| $[2, -1]$ | 5 | 5 |
| $[3, -3]$ | 2 | 2 |
| $[-2, 4]$ | 1 | 1 |
| $[-1, 2]$ | 5 | 5 |
| $[0, 0]$ | 8 | 8 |
| $[1, -2]$ | 5 | 5 |
| $[2, -4]$ | 1 | 1 |
| $[-3, 3]$ | 2 | 2 |
| $[-2, 1]$ | 5 | 4 |
| $[-1, -1]$ | 5 | 4 |
| $[0, -3]$ | 2 | 1 |

Table 4.11: One candidate $F_{110}$ plane has the following weight decomposition

| Weight | Dimension of weight space in $A^* \otimes B^*$ | Dimension of weight space in $F_{110}$ |
|---|---|---|
| $[2, 2]$ | 1 | 1 |
| $[3, 0]$ | 2 | 2 |
| $[4, -2]$ | 1 | 1 |
| $[0, 3]$ | 2 | 2 |
| $[1, 1]$ | 5 | 5 |
| $[2, -1]$ | 5 | 5 |
| $[3, -3]$ | 2 | 2 |
| $[-2, 4]$ | 1 | 1 |
| $[-1, 2]$ | 5 | 5 |
| $[0, 0]$ | 8 | 8 |
| $[1, -2]$ | 5 | 5 |
| $[2, -4]$ | 1 | 1 |
| $[-3, 3]$ | 2 | 2 |
| $[-2, 1]$ | 5 | 5 |
| $[-1, -1]$ | 5 | 3 |
| $[-4, 2]$ | 1 | 1 |

Table 4.12: Two candidate $F_{110}$ planes have the following weight decomposition

| Weight | Dimension of weight space in $A^* \otimes B^*$ | Dimension of weight space in $F_{110}$ |
|---|---|---|
| $[2, 2]$ | 1 | 1 |
| $[3, 0]$ | 2 | 2 |
| $[4, -2]$ | 1 | 1 |
| $[0, 3]$ | 2 | 2 |
| $[1, 1]$ | 5 | 5 |
| $[2, -1]$ | 5 | 5 |
| $[3, -3]$ | 2 | 2 |
| $[-2, 4]$ | 1 | 1 |
| $[-1, 2]$ | 5 | 5 |
| $[0, 0]$ | 8 | 8 |
| $[1, -2]$ | 5 | 5 |
| $[2, -4]$ | 1 | 1 |
| $[-3, 3]$ | 2 | 2 |
| $[-2, 1]$ | 5 | 5 |
| $[-1, -1]$ | 5 | 4 |
| $[-4, 2]$ | 1 | 1 |
| $[-3, 0]$ | 2 | 1 |

The computation to produce these 5 candidate $F_{110}$ planes took extensive time in some cases, due to the parameters creating a difficult groebner basis computation when determining whether an $F_{110}$ plane passes $(210)$-test. The large number of candidates was not as much of a computational issue as all the $(210)$-tests can be parallelized. Some of these computations were done on Texas A&M's High Performance Research Cluster as well as Texas A&M's Math Department Cluster.

Using the skew-symmetry of $T_{\mathfrak{sl}_n}$, we are able to produce $F_{011}$ and $F_{101}$ candidate weight spaces from the candidate $F_{110}$ spaces. A computer calculation verified that for each candidate triple $F_{110}, F_{011}, F_{101}$, the rank condition is not met for the $(111)$-test and consequently, there are no candidate $F_{111}$ spaces. Therefore, the rank of $T_{\mathfrak{sl}_3}$ is greater than 15.

**Theorem 19.** $\underline{\boldsymbol{R}}(T_{\mathfrak{sl}_3}) \geq 16$

This result is significant as it is the first example of an explicit tensor such that the border rank is at least $2m$ when $m < 13$.

### 4.3.1 Computational improvements to Border Apolarity

The flag condition helps to eliminate cases where there are many parameters. In the case of testing $r = 15$ for $T_{\mathfrak{sl}_3}$, the flag condition was able to eliminate two cases for which the $(210)$-test had taken months to compute. These two cases had taken the most time to compute and all other cases took a significantly less time (on the order of a week at most). The flag condition is currently being used to reduce the number of $F_{110}$ planes that need to be tested for $r = 16$.

In addition to the flag condition, reducing the Groebner basis computation to be performed over a finite field of characteristic $p$ has been implemented in order to reduce the computational cost in some of the more extreme cases. This reduction mod $p$ does not benefit cases where there the computational difficulty is due to a large number of equations.

## 4.4 Upper Bounds

A numerical computer search has given a rank 20 decomposition of $T_{\mathfrak{sl}_3}$. The technique used was a combination of Newton's Method and Lenstra–Lenstra–Lovász Algorithm to find rational approximations [23]. This technique formulated the problem as a nonlinear optimization problem that was solved to machine precision and then subsequently modified using the Lenstra-Lenstra-Lovász Algorithm to generate a precise solution with algebraic numbers given the numerical solution. As $T_{\mathfrak{sl}_3} \in \mathbb{C}^8 \otimes \mathbb{C}^8 \otimes \mathbb{C}^8$, a rank 20 decomposition consists of finding $a_i, b_i, c_i \in \mathbb{C}^8$ such that $T = \sum_{i=1}^{20} a_i \otimes b_i \otimes c_i$. We take each vector $a_i, b_i, c_i$ to be a vector in 8 variables, and using properties of elements of tensor products, we can multiply out the right hand side and have a system of equations for each entry of the tensor. This amounts to solving a system of 512 polynomial equations of degree 3 in 480 variables. We then use Newton's method to find roots to this system of equations. If it appears to converge to a solution, then we compute it to machine precision and use Lenstra-Lenstra-Lovász to find an algebraic solution that satisfies the initial polynomial conditions. Therefore,

**Theorem 20.** $\boldsymbol{R}(T_{\mathfrak{sl}_3}) \leq 20$

Let $\zeta_6$ denote a primitive 6th root of unity. The following is the rank 20 decomposition of $T_{\mathfrak{sl}_3}$. One may verify that this is in fact a rank decomposition by showing that it satisfies the polynomial equations described above. Note that since $\zeta_6$ is a primitive root of unity, then $\zeta_6^2 = \zeta_6 - 1$.

$$T_{\mathfrak{sl}_3} = \tag{4.1}$$

$$\left(\frac{1}{3^4 2}\right)\begin{bmatrix} 0 & \zeta_6 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} -6 & -4\zeta_6^2 & -6\zeta_6 \\ 9\zeta_6 & 18 & 0 \\ 6\zeta_6^2 & 0 & -12 \end{bmatrix} \otimes \begin{bmatrix} 6\zeta_6 & -4 & 0 \\ 9\zeta_6^2 & 0 & -9 \\ 0 & 4\zeta_6^2 & -6\zeta_6 \end{bmatrix} + \tag{4.2}$$

$$\left(\frac{1}{3^4 2^3}\right)\begin{bmatrix} -6 & -4\zeta_6^2 & -6\zeta_6 \\ 9\zeta_6 & 18 & 0 \\ 6\zeta_6^2 & 0 & -12 \end{bmatrix} \otimes \begin{bmatrix} 0 & \zeta_6 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} -6\zeta_6 & 4 & -6\zeta_6^2 \\ 0 & 6\zeta_6 & 9 \\ -6 & 0 & 0 \end{bmatrix} + \tag{4.3}$$

$$\left(\frac{1}{3^4 2^1}\right)\begin{bmatrix} 6 & 4\zeta_6^2 & 0 \\ 9\zeta_6 & 0 & 9\zeta_6^2 \\ 0 & 4\zeta_6 & -6 \end{bmatrix} \otimes \begin{bmatrix} 12\zeta_6 & -4 & 6\zeta_6^2 \\ 0 & -18\zeta_6 & -9 \\ 6 & 0 & 6\zeta_6 \end{bmatrix} \otimes \begin{bmatrix} 0 & \zeta_6 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \tag{4.4}$$

$$\left(\frac{1}{3^5 2}\right)\begin{bmatrix} 0 & 0 & 6\zeta_6 \\ -18\zeta_6 & -18\zeta_6^2 & -9 \\ -6 & -4\zeta_6 & 18\zeta_6^2 \end{bmatrix} \otimes \begin{bmatrix} 18\zeta_6^2 & 0 & 6\zeta_6 \\ 9\zeta_6 & -18\zeta_6^2 & 18 \\ -6 & -4\zeta_6 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -\zeta_6 & 0 \end{bmatrix} + \tag{4.5}$$

$$(\frac{1}{3^4 2^2})\begin{bmatrix} -6\zeta_6 & 4\zeta_6^2 & -6 \\ 9 & 18\zeta_6 & 0 \\ -6\zeta_6^2 & 0 & -12\zeta_6 \end{bmatrix} \otimes \begin{bmatrix} 0 & \zeta_6 - 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} -12\zeta_6 & 0 & -6 \\ -9 & 6\zeta_6 & 18\zeta_6^2 \\ -6\zeta_6^2 & 4 & 6\zeta_6 \end{bmatrix} + \tag{4.6}$$

$$(\frac{1}{3^4})\begin{bmatrix} 0 & \zeta_6 & 0 \\ 0 & 0 & 0 \\ 0 & \zeta_6^2 & 0 \end{bmatrix} \otimes \begin{bmatrix} -3\zeta_6^2 & -2 & -3\zeta_6 \\ 0 & 9\zeta_6^2 & 0 \\ 1 & 0 & -6\zeta_6^2 \end{bmatrix} \otimes \begin{bmatrix} 6\zeta_6 & 0 & -6 \\ 1 & 6\zeta_6 & -9\zeta_6^2 \\ -6\zeta_6^2 & 4 & -12\zeta_6 \end{bmatrix} + \tag{4.7}$$

$$(\frac{1}{3^4 2^3})\begin{bmatrix} -12 & -4\zeta_6^2 & -6\zeta_6 \\ 0 & 18 & -9\zeta_6^2 \\ 6\zeta_6^2 & 0 & -6 \end{bmatrix} \otimes \begin{bmatrix} -6\zeta_6^2 & 4\zeta_6 & 0 \\ 9 & 0 & 9\zeta_6 \\ 0 & 4 & 6\zeta_6^2 \end{bmatrix} \otimes \begin{bmatrix} 0 & -4 & 6\zeta_6^2 \\ -9\zeta_6^2 & -6\zeta_6 & 0 \\ 6 & 0 & 6\zeta_6 \end{bmatrix} + \tag{4.8}$$

$$(\frac{1}{3^5 2^3})\begin{bmatrix} -18 & 0 & 6\zeta_6^2 \\ 9\zeta_6^2 & 18 & 18\zeta_6 \\ -6\zeta_6 & -4\zeta_6^2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & -6\zeta_6^2 \\ 18\zeta_6^2 & -18 & 9\zeta_6 \\ 6\zeta_6 & 4\zeta_6^2 & 18 \end{bmatrix} \otimes \begin{bmatrix} 0 & 4\zeta_6 & 6\zeta_6^2 \\ -9\zeta_6^2 & 6 & 0 \\ -6\zeta_6 & 0 & -6 \end{bmatrix} + \tag{4.9}$$

$$(\frac{1}{3^3 2^3})\begin{bmatrix} -12\zeta_6 & 8 & -6\zeta_6^2 \\ 0 & 18\zeta_6 & 9 \\ -6 & -4\zeta_6^2 & -6\zeta_6 \end{bmatrix} \otimes \begin{bmatrix} -2\zeta_6^2 & 0 & 0 \\ 3 & 0 & 3\zeta_6 \\ 0 & 0 & 2\zeta_6^2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & -6\zeta_6 \\ 9\zeta_6 & 6 & 0 \\ 6\zeta_6^2 & 4\zeta_6 & -6 \end{bmatrix} + \tag{4.10}$$

$$(\frac{1}{3^5 2^3})\begin{bmatrix} 18\zeta_6^2 & 12 & 6\zeta_6 \\ 9\zeta_6 & -18\zeta_6^2 & 18 \\ -6 & 8\zeta_6 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -6 & -6\zeta_6 \\ 18\zeta_6 & 18\zeta_6^2 & 9 \\ 6 & -2\zeta_6 & -18\zeta_6^2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 6\zeta_6 \\ -9\zeta_6 & -6\zeta_6^2 & 0 \\ -6 & -4\zeta_6 & 6\zeta_6^2 \end{bmatrix} + \tag{4.11}$$

$$
(\frac{1}{3^4 2^3})\begin{bmatrix} 12 & 4 & -6 \\ 0 & -18 & 9 \\ -6 & 0 & 6 \end{bmatrix} \otimes \begin{bmatrix} -6 & 4 & 0 \\ 9 & 0 & -9 \\ 0 & -4 & 6 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 6 \\ -9 & 6 & 0 \\ 6 & -4 & -6 \end{bmatrix} + \tag{4.12}
$$

$$
(\frac{1}{3^3 2^3})\begin{bmatrix} -12 & 0 & 6 \\ 0 & 18 & -9 \\ 6 & -4 & -6 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 4 & -6 \\ 9 & -6 & 0 \\ -6 & 0 & 6 \end{bmatrix} + \tag{4.13}
$$

$$
(\frac{1}{3^3 2^1})\begin{bmatrix} -2 & 0 & 0 \\ 3 & 0 & -3 \\ 0 & 0 & 2 \end{bmatrix} \otimes \begin{bmatrix} 12 & 0 & -6 \\ 0 & -18 & 9 \\ -6 & 4 & 6 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \tag{4.14}
$$

$$
(\frac{1}{3^4 2^1})\begin{bmatrix} 6 & 4 & -6 \\ 9 & -18 & 0 \\ -6 & 0 & 12 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 6 & 0 & -6 \\ 0 & -6 & 9 \\ -6 & 4 & 0 \end{bmatrix} + \tag{4.15}
$$

$$
(\frac{1}{3^3 2^1})\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} -6 & -4 & 6 \\ -9 & 18 & 0 \\ 6 & 0 & -12 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix} + \tag{4.16}
$$

$$
(\frac{1}{3^3 2^5})\begin{bmatrix} -4 & -6 & -6 \\ 0 & 0 & 0 \\ 9 & 4 & 4 \end{bmatrix} \otimes \begin{bmatrix} -4 & 6 & -6 \\ 0 & 0 & 0 \\ -9 & 4 & 4 \end{bmatrix} \otimes \begin{bmatrix} -4 & -6 & 6 \\ 0 & 0 & 0 \\ 9 & -4 & 4 \end{bmatrix} + \tag{4.17}
$$

$$(\frac{1}{3^3 2^5})\begin{bmatrix}4 & -6 & -6\\ 0 & 0 & 0\\ 9 & 4 & -4\end{bmatrix}\otimes\begin{bmatrix}-4 & -6 & 6\\ 0 & 0 & 0\\ 9 & -4 & 4\end{bmatrix}\otimes\begin{bmatrix}4 & -6 & 6\\ 0 & 0 & 0\\ 9 & -4 & -4\end{bmatrix}+ \tag{4.18}$$

$$(\frac{1}{3^3 2^5})\begin{bmatrix}-4 & -6 & 6\\ 0 & 0 & 0\\ 9 & -4 & 4\end{bmatrix}\otimes\begin{bmatrix}-4 & 6 & 6\\ 0 & 0 & 0\\ -9 & -4 & 4\end{bmatrix}\otimes\begin{bmatrix}4 & 6 & 6\\ 0 & 0 & 0\\ -9 & -4 & -4\end{bmatrix}+ \tag{4.19}$$

$$(\frac{1}{3^3 2^5})\begin{bmatrix}-4 & 6 & -6\\ 0 & 0 & 0\\ -9 & 4 & 4\end{bmatrix}\otimes\begin{bmatrix}-4 & -6 & -6\\ 0 & 0 & 0\\ 9 & 4 & 4\end{bmatrix}\otimes\begin{bmatrix}4 & -6 & -6\\ 0 & 0 & 0\\ 9 & 4 & -4\end{bmatrix}+ \tag{4.20}$$

$$(\frac{2}{3^2})\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\otimes\begin{bmatrix}0 & 0 & 3\\ 0 & 0 & 0\\ 0 & -2 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & -2 & 0\\ 0 & 0 & 0\\ 3 & 0 & 0\end{bmatrix} \tag{4.21}$$

In an attempt to find a smaller rank decomposition, we found numerical evidence suggesting that $\underline{\mathbf{R}}(T_{\mathfrak{sl}_3}) \leq 18$. The above method was unable to determine exact algebraic numbers for it to be an honest border rank decomposition. We include the approximate border rank decomposition, which was obtained as a numerical solution to machine precision using Newton's method, in Appendix A. This decomposition is satisfies the equation $T_{\mathfrak{sl}_3} = \sum_{k=1}^{18} a_k(t) \otimes b_k(t) \otimes c_k(t) + O(t)$ to a maximum error in each entry of $3.88578058618805\, 10^{-16}$ ($\ell_0$ error). It also is satisfied with a sum of squares error of $1.85900227125328\, 10^{-15}$ ($\ell_2$ error), which is the square root of the sum of the squares of all errors in each entry.

# 5. CONCLUSION

We have found new bounds on the rank and border rank of $T_{\mathfrak{sl}_3}$ as well as a lower bound on $T_{\mathfrak{sl}_4}$. The lower bound on the border rank of $T_{\mathfrak{sl}_3}$ is the first case of a tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ with $m < 13$ and border rank at least $2m$. We have used all available techniques to obtain these bounds. The limitations going forward are computational as this tensor lives in a high dimensional space. Future work to determine better bounds on border rank for $\mathfrak{sl}_3$ is geared towards trying to improve upon the implementation of border apolarity as that has given the best lower bound thus far. Currently, we are applying border apolarity to test if $\underline{\mathbf{R}}(T_{\mathfrak{sl}_3}) > 16$.

It is also ongoing work to find a lower bound on the border rank for $T_{\mathfrak{sl}_n}$ for $n$ in general. Since koszul flattenings are $\mathfrak{sl}_n$-module maps, it suffices to find all highest weight vectors of $\mathfrak{sl}_n \otimes \Lambda^k \mathfrak{sl}_n^*$ and see which highest weight vectors are in the kernel of the koszul flattening in order to determine the rank of the koszul flattening. Currently, we are computing highest weight vectors of $\mathfrak{sl}_n \otimes \Lambda^3 \mathfrak{sl}_n^*$ for all $n \geq 6$, which should help give us a lower bound on $\underline{\mathbf{R}}(T_{\mathfrak{sl}_n})$ for all $n \geq 6$. One should note that this is also a computationally difficult task as this is a high dimensional space even in the case when $n = 6$.

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# APPENDIX A

## APPROXIMATE BORDER RANK 18 DECOMPOSITION

```
## Approximate Border Rank 18 Decomposition for sl3. B is list of
   ↪  length 18,
## Each element of B is 3 lists, one for each tensor factor

t = var('t')
decomp_sl3_br18 = [
[[0.8582131228193816*t^4, 1.0*t^3, -0.656183917735616, 0,
   ↪ -0.6991867952664118*t^4, 0.48158077142326267*t, 0, 0],
[0.4276000554886944, 0, -0.18145759099190728*t^-4,
   ↪ -0.615151864463497*t, 0.3663065358105463,
   ↪ 0.41017982352375143*t^-3, 0.32919233217347255*t^4,
   ↪ -0.37760897344619454*t^3],
[-0.1349207590097993*t^-4, -0.6002859548136603*t^-3, 1.0,
   ↪ 0.5115159306456755*t^-5, -0.1595527810915205*t^-4,
   ↪ -0.6048099252034903*t^-1, -0.7277947872836206*t^-8,
   ↪ -0.40421722543545996*t^-7]],
[[-1.0*t^4, 0.35415463745033937*t^3, 0.0064263290832379735, 0,
   ↪ -0.36545526641405424*t^4, -0.07853056373003943*t,
   ↪ -0.032960816562400096*t^8, 0],
[0, 0, 1.0*t^-4, -0.2566348901375672*t, -0.22472886423613508, 0,
   ↪ -1.5211311526049585*t^4, 0],
[0.8999601061346961*t^-4, 0, -0.9891997951816988,
   ↪ 0.7757989654047442*t^-5, 0.6720573676204581*t^-4, 0,
```

```
↪ -0.7680275258871294*t^-8, 0.23098457888352464*t^-7]],
[[1.4652283519838352*t^4, 0, -0.49981037975335263,
  ↪ -0.1095789691297215*t^5, 0, -0.39151017186719006*t,
  ↪ -0.0035109361876165053*t^8, 0],
[-0.0370198277154002, -0.4868421872551675*t^-1,
  ↪ -0.6112939642144736*t^-4, 1.0*t, 0, -0.2400052666531245*t
  ↪ ^-3, 0.3994050510000636*t^4, 0.38578320751551654*t^3],
[-0.976481792017985*t^-4, -0.6067214925499242*t^-3, 0,
  ↪ -0.723941360626237*t^-5, -1.0*t^-4, 0.5548674236521388*t
  ↪ ^-1, 0.43144009347718587*t^-8, 0]],
[[0, 0, 0, 0.7384182957633163*t^5, 0, 0.644445145257651*t, 1.0*t
  ↪ ^8, 0.14298165951600136*t^7],
[0.48917042278155703, -1.1676021066814524*t^-1,
  ↪ 0.0742778280974771*t^-4, 0, 0, 0.43490564244204044*t^-3, 0,
  ↪  -0.35107756870823037*t^3],
[0.35864016315209*t^-4, 0.08753789324397686*t^-3,
  ↪ 0.09745671407220928, 0.8747382868849779*t^-5,
  ↪ -0.08844676171127677*t^-4, 0, 1.0*t^-8,
  ↪ 0.26767660269283405*t^-7]],
[[0, 0.7154062494455652*t^3, 0.753065093225573,
  ↪ 0.25916815694138623*t^5, 1.0*t^4, -0.13328651389721624*t,
  ↪ -0.43754268152319187*t^8, 0.33712746822644685*t^7],
[0.31175249346291917, 0.4893435772426664*t^-1,
  ↪ -0.675003023029101 5*t^-4, 0, 0, 0, -1.0*t^4,
  ↪ 0.2730200083141182*t^3],
[-0.48768995751368266*t^-4, 0.3666405285647919*t^-3,
  ↪ -0.509119354246841, -0.4700930934822484*t^-5,
```

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  ↪ -0.22797281366964972*t^-4, -0.35924834770877284*t^-1,
  ↪ 0.21574925698409564*t^-8, -0.4457498667959218*t^-7]],
[[-0.3726117655275953*t^4, -0.6850149555795361*t^3,
  ↪ -0.47485190359116686, -0.3845493471094187*t^5,
  ↪ 1.3294054010695648*t^4, 0, 0, -0.360876333169775*t^7],
[-0.20585599800494148, 0.14163424899135532*t^-1,
  ↪ -0.07154541239160381*t^-4, 0, 0, 0.6161959625923827*t^-3,
  ↪ -1.0*t^4, 1.1951737645411018*t^3],
[0.32862886361063015*t^-4, -1.0*t^-3, 0.02421044619636145,
  ↪ -0.10037637090787854*t^-5, 0.8394716140987267*t^-4,
  ↪ 0.6969300342438073*t^-1, 0.33915176005821984*t^-8,
  ↪ -0.8710663402014699*t^-7]],
[[-0.26018060869681375*t^4, -0.5986510147484245*t^3, 0, 0,
  ↪ 0.34975430620458303*t^4, 0.6295753676920407*t,
  ↪ -0.008909285185907506*t^8, -1.0*t^7],
[0.1869896012120452, 0.21074450964709926*t^-1,
  ↪ 0.15931993306826575*t^-4, -0.5916688936748106*t, -1.0,
  ↪ 0.4704225477242227*t^-3, 0.4764298640490715*t^4, 0],
[-0.11381180152788646*t^-4, 0, -0.6321964884450085,
  ↪ 1.1046891281785145*t^-5, 0, 1.0*t^-1, -0.39603240282608254*
  ↪ t^-8, 0.9263232781144852*t^-7]],
[[0.5249458233325552*t^4, 0.48647993414509433*t^3,
  ↪ 0.288317913855889, -0.6006289011129895*t^5,
  ↪ -0.43339360309318065*t^4, -0.9021189701254521*t,
  ↪ -0.3580507322918174*t^8, 0],
[0, -0.6140778368802516*t^-1, 0, 0.3157042904634412*t, 1.0,
  ↪ 0.3354809468306432*t^-3, -0.4344836650252679*t^4,
```

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↪ 0.3640820338419208*t^3],
[0, 0.021642017184031522*t^-3, 0, 1.0*t^-5, -0.16272749957295446*
↪ t^-4, 0, 0.330227018568934*t^-8, -0.5277803864906128*t
↪ ^-7]],
[[0, 0.5078943260216121*t^3, -0.8437068467257393,
↪ -0.16779079767696245*t^5, 0, -0.22773097773605452*t, 1.0*t
↪ ^8, 0],
[-0.6816689949383756, 0, -0.5858467479048473*t^-4,
↪ 0.4936500799168931*t, -1.0, -0.06688550512654293*t^-3, 0,
↪ 0],
[0, 0.6059010244748405*t^-3, 0, 1.3353428165296193*t^-5, 0, 0,
↪ -0.6434299512786738*t^-8, -0.4609346348556645*t^-7]],
[[0.5342376006675218*t^4, -0.6086422770257539*t^3, 0,
↪ 0.4884368142723877*t^5, -0.718333221563019*t^4, 0, 0, -1.0*
↪ t^7],
[0.4628574417338272, 0.8309327397168714*t^-1,
↪ 0.21142383552160535*t^-4, -1.0560448029801677*t, 0, -1.0*t
↪ ^-3, 0, 0.5907349063234486*t^3],
[-0.19068598535726494*t^-4, 1.251113151532601*t^-3, 0,
↪ 0.801440153916977*t^-5, -0.4621883155369931*t^-4,
↪ 0.649922008892636*t^-1, 0.451762403193013 5*t^-8,
↪ 0.6177441885560955*t^-7]],
[[-0.46119950535589427*t^4, 0.7406376078020875*t^3,
↪ 0.29057187128303724, 0, 0.6143929403505343*t^4,
↪ 0.4568692437943906*t, 1.0*t^8, 0.36703376745263416*t^7],
[0, 1.080760392781858*t^-1, -0.14934226596262576*t^-4, 0,
↪ 0.463851046688496, -0.5286724400096557*t^-3, 0,
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  ↪ -0.5122850384466106*t^3],
[0, -0.17675493191050737*t^-3, -0.7935243857498888,
  ↪ 1.0525678466492765*t^-5, 0, 0, 0.8232503249794846*t^-8,
  ↪ 1.0*t^-7]],
[[-0.1128613231857281*t^4, -1.004601481243395*t^3,
  ↪ -0.18790258087897918, 0.40149916700646043*t^5,
  ↪ -0.14415337265479422*t^4, 0, 0, 1.0*t^7],
[-0.6862576195287183, 0.5664746389899703*t^-1,
  ↪ 0.2908675685084361*t^-4, 1.3714715312276369*t, 1.0,
  ↪ -0.5713223321771269*t^-3, 0.5522456890804313*t^4, 0],
[-0.3006833454545763*t^-4, -1.5673937490149197*t^-3, 0, 0,
  ↪ 0.38713909105007405*t^-4, 0.9955939462610987*t^-1,
  ↪ 0.5677563588587456*t^-8, 0.5353436694572539*t^-7]],
[[-0.6245100623794343*t^4, 0, -1.0, 0.5065225347182659*t^5,
  ↪ 0.5317253204886139*t^4, 0, 0, -0.3719669835035259*t^7],
[0.5699237870531472, -0.2500784148082814*t^-1, 0.442945128286494*
  ↪ t^-4, 0.45638933734578824*t, 1.3973251849132609, 0, 0, 0],
[0, 0.5427787886626944*t^-3, 0.6987561651998788, 1.0*t^-5, 0,
  ↪ 0.20324504499293194*t^-1, -0.21496324065341224*t^-8,
  ↪ -0.35477523806474964*t^-7]],
[[0, -0.8732161724035096*t^3, -0.2040186481697669,
  ↪ -1.2010646422303124*t^5, -0.23508531984800027*t^4,
  ↪ 0.5654807700289866*t, 0, 1.0788666817176849*t^7],
[-1.0, 0.23352368487408065*t^-1, 0.2848223349469312*t^-4,
  ↪ 0.1768073101741446*t, 1.1645543351076657, 0, 0, 0],
[0.4548846200775119*t^-4, 0, -1.0, 0.7165423092004359*t^-5,
  ↪ -0.2616313451578387*t^-4, -1.2932734528263643*t^-1,
```

```
  ↪ -0.30361754210884495*t^-8, 0.12055753515645255*t^-7]],
[[-0.36174656010597567*t^4, 0, 0, 1.0*t^5, 0,
  ↪ -0.9529930944532633*t, 0, 0],
[-0.249684891380413, -0.48049270284391143*t^-1,
  ↪ 0.27222663948587644*t^-4, 0.06402554598955673*t,
  ↪ 0.32897469345165686, -0.29230508678584777*t^-3,
  ↪ 0.3854984554913255*t^4, 1.084386049943073*t^3],
[0.5943543096084136*t^-4, 0, -0.4586866892423329, -1.0*t^-5,
  ↪ -0.2869864460438895*t^-4, 0.7692509696955846*t^-1,
  ↪ -0.5902387210012353*t^-8, -0.10600670730550378*t^-7]],
[[0.2878925845174677*t^4, 0.5281046432775963*t^3,
  ↪ -0.2234313203594912, 0, -0.14791845243016916*t^4,
  ↪ -0.33262458187432487*t, 1.108538091915835*t^8,
  ↪ 0.2489733522919065*t^7],
[-0.19980163401634468, -0.4519313364099573*t^-1,
  ↪ -0.9400278789365702*t^-4, 1.7023099438662026*t,
  ↪ 0.47743291480661737, 0.40637104558436565*t^-3,
  ↪ -0.9678003132293527*t^4, -1.0*t^3],
[1.0*t^-4, 0.391031885184312*t^-3, 0, -1.0*t^-5,
  ↪ 0.8654063566747124*t^-4, 0, 0, 0]],
[[1.0110775513705723*t^4, -0.3307689454870473*t^3,
  ↪ -0.45870287159609613, 0, -0.2918463909613691*t^4,
  ↪ 0.30167345561087666*t, 0.582399773106054*t^8, 0],
[0, -1.0*t^-1, 0.7191039431350641*t^-4, 0.539253643257692З*t,
  ↪ -0.39197591630294476, -0.7143544436146056*t^-3, 0,
  ↪ 0.8086746481540814*t^3],
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[0.4561800893644289*t^-4, -0.7356641704491395*t^-3,
   ↪ 0.4710082951189216, 0, 0, 0.7109669616748934*t^-1, -1.0*t
   ↪ ^-8, 0]],
[[-0.41687446856120663*t^4, 0.3197337020122604*t^3,
   ↪ 0.3753986532686001, 0, -0.40495125990774045*t^4,
   ↪ 0.2979428965247865*t, -0.6344817450528121*t^8,
   ↪ -0.212014762523078*t^7],
[-0.2396929297153146, 0.7894128907681753*t^-1, 0, 0,
   ↪ 0.2995112979592937, 1.0*t^-3, 0.09278089610904368*t^4,
   ↪ 0.5160805291028968*t^3],
[0.4011063039077983*t^-4, 0, 0, 0.2503035377295029*t^-5, 1.0*t
   ↪ ^-4, -0.9730216219558445*t^-1, 0.6637049194937615*t^-8,
   ↪ -0.5073657590422873*t^-7]]
]
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