EQUATIONS FOR CHOW VARIETIES, THEIR SECANT VARIETIES AND OTHER VARIETIES ARISING IN COMPLEXITY THEORY

A Dissertation

by

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ABSTRACT

The Chow variety of polynomials that decompose as a product of linear forms has been studied for more than 100 years. Brill, Gordon, and others obtained set-theoretic equations for the Chow variety. I compute Brill's equations as a $GL(V)$-module. I find new equations for Chow varieties, their secant varieties, and an additional variety by *flattenings* and *Koszul Young flattenings*. This enables a new lower bound for the symmetric border rank of $x_1x_2\cdots x_d$ when d is odd and a new complexity lower bound for the permanent. I use the method of prolongation to obtain equations for secant varieties of Chow varieties as $GL(V)$-modules. The goal of studying these varieties arising in complexity theory is to separate **VP** from **VNP**, which is an algebraic analog of the famous **P** versus **NP** problem.

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1. INTRODUCTION

1.1 Motivation

1.1.1 Motivation from algebraic geometry

There has been substantial recent interest in the equations of certain algebraic varieties that encode natural properties of polynomials (see e.g. [9, 29, 33, 35, 36]). Such varieties are usually preserved by algebraic groups and it is a natural question to understand the module structures of the spaces of equations. One variety of interest is the *Chow variety* of polynomials that decompose as a product of linear forms, which is defined by $Ch_d(V) = \mathbb{P}\{z \in S^dV | z = w_1 \cdots w_d \text{ for some } w_i \in V\} \subset \mathbb{P}S^dV$, where V is a finite-dimensional complex vector space and $\mathbb{P}S^dV$ is the projective space of homogeneous polynomials of degree d on the dual space V^*.

The ideal of the Chow variety of polynomials that decompose as a product of linear forms has been studied for over 100 years, dating back at least to Gordon [18] and Hadamard [25]. Let $S^\delta(S^dV)$ denote the space of homogeneous polynomials of degree δ on S^dV^*. The *Foulkes-Howe* map $h_{\delta,d} : S^\delta(S^dV) \to S^d(S^\delta V)$ (see §2.5 for the definition) was defined by Hermite [27] when dim $V = 2$, and Hermite proved the map is an isomorphism in his celebrated "Hermite reciprocity". Hadamard [24] defined the map in general and observed that its kernel is $I_\delta(Ch_d(V^*))$, the degree δ component of the ideal of the Chow variety. The conjecture that $h_{\delta,d}$ is always of maximal rank, which dates back to Hadamard [25], has become known as the "Foulkes-Howe conjecture" [15, 28]. Müller and Neunhöffer [41] proved the conjecture is false by showing the map $h_{5,5}$ is not injective. Brion [4, 5] proved the Foulkes-Howe conjecture is true asymptotically, giving an explicit, but very large bound for δ in terms of d and dim V. This map is not understood when $d > 4$ (see

[4, 5, 15, 25, 28, 38]).

Brill and Gordon (see [17, 18, 31]) wrote down set-theoretic equations for the Chow variety of degree $d+1$, called "Brill's equations". Brill's equations give a geometric derivation of set-theoretic equations for the Chow variety, and it is a natural question to understand these equations as a $GL(V)$-module, where $GL(V)$ denotes the *general linear group* of invertible linear maps from V to V.

1.1.2 Motivation from complexity theory

Informally speaking, the **P** versus **NP** problem (see e.g.[44]) asks whether every problem whose solution can be quickly verified by a computer can also be quickly solved by a computer. An early mention of it was a 1956 letter written by Kurt Gödel to John von Neumann. Gödel asked whether a certain NP-complete problem could be solved in quadratic or linear time [26]. The precise statement of the P versus NP problem was introduced in 1971 by Stephen Cook in [11] and is considered to be the most important open problem in theoretical computer science [14].

In computational complexity theory, a *decision problem* is a question in some formal system with a yes-or-no answer, depending on the values of input parameters. The class **P** consists of all those decision problems that can be solved in an amount of time that is polynomial in the size of the input; the class **NP** consists of all those decision problems whose positive solutions can be verified in polynomial time given the right information. For example, given a set A of n integers and a subset B of A, the statement that "B adds up to zero"can be quickly verified with at most $(n-1)$ additions. However, there is no known algorithm to find a subset of A adding up to zero in polynomial time.

Leslie Valiant [48] defined in 1979 an algebraic analogue of the **P** versus **NP** problem The class **VP** is an algebraic analogue of the class **P**, and the class **VNP** is an algebraic analog of the class **VP**. Valiant Conjectured $\mathbf{VP} \neq \mathbf{VNP}$. If Valiant Conjecture failed i.e. $\mathbf{VP} = \mathbf{VNP}$, then $\mathbf{P} = \mathbf{NP}$ [48, 49]. Valiant's Conjecture $\mathbf{VP} \neq \mathbf{VNP}$ [48] may be rephrased as "there does not exist polynomial size circuit that computes the permanent", defined by $\mathrm{perm}_n = \sum_{\sigma \in \mathfrak{S}_\mathfrak{n}} x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{n\sigma(n)} \in S^n\mathbb{C}^{n^2}$, where $\mathfrak{S}_\mathfrak{n}$ is the symmetric group and $\mathbb{C}^{n^2}$ has a basis $\{x_{ij}\}_{1 \leq i,j \leq n}$. The readers can refer to Appendix A to learn more about circuits, complexity classes and Valiant's Conjecture.

1.1.3 A geometric approach to Valiant's conjecture

A geometric method to approach Valiant's conjecture implicitly proposed by Gupta, Kamath, Kayal and Saptharishicite [23] is to determine equations for certain secant varieties (defined below).

Let W be a complex vector space and $X \subset \mathbb{P}W$ be an algebraic variety, define $\sigma_r^0(X) = \bigcup_{p_1,\ldots,p_r \in X} \langle p_1, \ldots, p_r \rangle \subset \mathbb{P}W$, where $\langle p_1, \ldots, p_r \rangle$ denotes the projective plane spanned by $p_1, \ldots, p_r$. Define the r-th *secant variety* of X to be $\sigma_r(X) = \overline{\sigma_r^0(X)} \subset \mathbb{P}W$, where the over line denotes closure in the Zariski topology.

Let $X \subset \mathbb{P}V$ be an algebraic variety, define the *Veronese embedding* $v_d(X) \subset \mathbb{P}S^dV$ of X by $v_d(X) = \mathbb{P}\{z \in S^dV | z = w^d \text{ for some } [w] \in X\}$, $v_d(\mathbb{P}V)$ is called the *Veronese variety.*

Let h_n and g_n be two positive sequences, define $h_n = \omega(g_n)$ if $\lim_{n\to\infty} \frac{h_n}{g_n} = \infty$, define $h_n = \Omega(g_n)$ if $\lim_{n\to\infty} \frac{h_n}{g_n} \geq C$ for some positive constant C.

The following two theorems appeared in [32], they are geometric rephrasings of results in [23].

Theorem 1.1.1. [23, 32] *If for all but a finite number of m, for all r, n with $rn <$*

$2^{\omega(\sqrt{m}\log(m))}$,

$$[l^{n-m}\mathrm{perm}_m] \notin \sigma_r(Ch_n(\mathbb{C}^{m^2+1})),$$

then Valiant's Conjecture **VP** $\neq$ **VNP** [48] *holds.*

Theorem 1.1.2. [23, 32] *If for all but finite number of n, and for $\delta_1, \delta_2 \sim \sqrt{n}$, for r,ρ with $r\rho < 2^{\omega(\sqrt{n}\log(n))}$,*

$$[\mathrm{perm}_n] \notin \sigma_\rho(v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}^{n^2-1})))),$$

then Valiant's conjecture **VP** $\neq$ **VNP** [48] *holds.*

Theorems 1.1.1 and 1.1.2 motivated me to study the equations for $\sigma_r(Ch_d(V))$ and $\sigma_\rho(v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}V)))$. The results obtained here are not in the ranges needed to separate **VP** from **VNP**. However, the results come from a geometric perspective and are amenable to generalizations. For the first problem I obtain equations for secant varieties of Chow varieties using two different methods. For the second problem, this is the first time that equations for $\sigma_\rho(v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}V)))$ are approached geometrically.

1.2 Results

1.2.1 Brill's equations as a $GL(V)$-module

The group $GL(V)$ has an induced action on S^dV (see §2.2). The Chow variety $Ch_d(V)$ is invariant under the action of $GL(V)$, therefore the ideal of $Ch_d(V)$ is a $GL(V)$-module (see §2.2). For any partition λ, let $S_\lambda V$ be the irreducible $GL(V)$-module determined by the partition λ (see §2.3). For example $S_{(d)}V = S^dV$, while $S_{(1^d)}V = \Lambda^dV$ is the d-th exterior power of V.

In Chapter 3, I prove

Theorem 1.2.1. *Assume* $\dim V \geq 3$ *and* $d \geq 2$*. The degree* $d+1$ *equations for* $Ch_d(V)$ *discovered by Brill, as a* $GL(V)$*-module, are:*

$$\begin{cases} S_{(7,3,2)}V^*, & \text{if } d = 3; \\ \bigoplus_{j=2}^{d} S_{(d^2-j,d,j)}V^*, & \text{if } d \neq 3. \end{cases}$$

Remark 1.2.2. *Compare the codimension of* $Ch_d(\mathbb{C}^3)$ *with the dimension of all the modules in Theorem 3.2.1 that define* $Ch_d(\mathbb{C}^3)$ *set- theoretically: When* $d = 2$*, the codimension of* $Ch_2(\mathbb{C}^3)$ *is* 1 *and the dimension of* $S_{(2,2,2)}\mathbb{C}^{*3}$ *is* 1*. When* $d = 3$*, the codimension of* $Ch_3(\mathbb{C}^3)$ *is* 3 *while the dimension of* $S_{(7,3,2)}\mathbb{C}^{*3}$ *is* 35*. In general the dominant term of the codimension of* $Ch_d(\mathbb{C}^3)$ *is* $\frac{d^2}{2}$ *, but the dominant term of the dimension of the modules from Brill's equations that define* $Ch_d(\mathbb{C}^3)$ *is* $\frac{d^7}{2}$*. Therefore, the Chow variety is far from being a complete intersection.*

1.2.2 Symmetric border rank of monomials

For a given polynomial $P \in S^dV$, the *symmetric rank* $\mathbf{R}_S(P)$ of P is the smallest integer r such that $[P] \in \sigma_r^0(v_d(\mathbb{P}V))$, and the *symmetric border rank* $\underline{\mathbf{R}}_S(P)$ of P is the smallest integer r such that $[P] \in \sigma_r(v_d(\mathbb{P}V))$. Notice that $\mathbf{R}_S(P) \geq \underline{\mathbf{R}}_S(P)$.

It is an open problem to determine the symmetric border rank of $x_1 \cdots x_d$. Classical results show that $\underline{\mathbf{R}}_S(x_1 \cdots x_d) \geq \binom{d}{\lfloor \frac{d}{2} \rfloor} \sim \frac{2^d}{\sqrt{d}}$. Ranestad and Schreyer [42] showed $\mathbf{R}_S(x_1 \cdots x_d) = 2^{d-1}$. Therefore $\binom{d}{\lfloor \frac{d}{2} \rfloor} \leq \underline{\mathbf{R}}_S(x_1 \cdots x_d) \leq \mathbf{R}_S(x_1 \cdots x_d) = 2^{d-1}$.

In §4.1 I prove a new lower bound for $\underline{\mathbf{R}}_S(x_1 \cdots x_{2n+1})$ when $d = 2n+1$, which is $\binom{2n+1}{n}$ plus an additional exponential term in n.

Theorem 1.2.3. $\underline{\mathbf{R}}_S(x_1 \cdots x_{2n+1}) \geq \binom{2n+1}{n}(1 + \frac{n^2}{(n+1)^2(2n-1)})$.

Remark 1.2.4. *When $d = 2n$, I conjecture that with the same method one can show $\underline{\mathbf{R}}_S(x_1 \cdots x_{2n}) \geq \binom{2n}{n}(1 + \frac{C}{2n})$ for some constant C and for n big enough. I did not prove that, but I verified small cases with a computer. When $d = 3$, $\underline{\mathbf{R}}_S(x_1x_2x_3) \geq 4$, so $\underline{\mathbf{R}}_S(x_1x_2x_3) = 4 > \binom{3}{1}$. When $d = 5$, $\binom{5}{2} < 13 \leq \underline{\mathbf{R}}_S(x_1x_2x_3x_4x_5) \leq 16$.*

1.2.3 A new lower complexity bound for the permanent

In §4.2, I prove a new lower complexity bound for the permanent:

Theorem 1.2.5. *Let $\delta_1, \delta_2 \sim \sqrt{n}$ and $\dim V = n^2$, when $\frac{2n - \sqrt{n}\log(r)}{\sqrt{n}\log(n)} = \omega(1)$, i.e. $r = 2^{2\sqrt{n} - \log(n)\omega(1)}$,*

$$\mathrm{perm}_n \notin \sigma_\rho(v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}V))))$$

for $\rho < 2^{\omega(\sqrt{n}\log(n))}$.

1.2.4 Equations for secant varieties of Chow varieties

Let W be a complex vector space and $X \subset \mathbb{P}W$ be an algebraic variety, and let $I_d(X)$ denote the degree d component of the ideal of X.

In §5.2, I prove:

Theorem 1.2.6. *If $\dim V \leq 6$, then $I_7(\sigma_2(Ch_3(V^*))) = 0$.*

Remark 1.2.7. *When $\dim V \leq 4$, $\sigma_2(Ch_3(V^*)$ is the ambient space and the ideal is 0. When $\dim V \geq 7$, any module with 7 rows in $S^7(S^3V)$ is in $I_7(\sigma_2(Ch_3(V^*)))$.*

Also I prove:

Theorem 1.2.8. *If $\dim V \geq 6$, then $S_{(5,5,5,5,3,1)}V \subset I_8(\sigma_2(Ch_3(V^*)))$.*

Remark 1.2.9. *When $\dim V = 5$, I expect $I_8(\sigma_2(Ch_3(V^*))) = 0$, but I did not prove it. When $\dim V = 6$, I expect $I_8(\sigma_2(Ch_3(V^*))) = S_{(5,5,5,5,3,1)}V$. When $\dim V \geq 7$,* *any module with more than 7 rows in $S^8(S^3V)$ is in $I_8(\sigma_2(Ch_3(V^*)))$, in addition to the module $S_{(5,5,5,5,3,1)}V$.*

In §5.3, I prove:

Theorem 1.2.10. *Consider* $\dim V \geq 4r$,

$$S_{(6,6,4^{4r-2})}V \subset I_{4r+1}(\sigma_r(Ch_4(V^*)).$$

Remark 1.2.11. *The lowest degree in the ideal of $Ch_4(V^*)$ is 5, and by Theorem 5.1.3, the lowest possible degree in the ideal of $\sigma_r(Ch_4(V^*))$ is $4r+1$, so $I_{4r}(\sigma_r(Ch_4(V^*)) = 0$. When $\dim V = 4r$, I expect $S_{(6,6,4^{4r-2})}V = I_{4r+1}(\sigma_r(Ch_4(V^*))$. When $\dim V > 4r$, any module with $4r+1$ rows in $S^{4r+1}(S^4V)$ is in $I_{4r+1}(\sigma_r(Ch_4(V^*)))$, in addition to the module $S_{(6,6,4^{4r-2})}V$.*

In §5.4, I prove:

Theorem 1.2.12. *The isotypic component of $S_{((2m+2)^m,(2m)^{2mr-m})}V$ in $S^{2mr+1}(S^{2m}V)$ is in $I_{2mr+1}(\sigma_r(Ch_{2m}(V^*)))$.*

1.3 Overview of methods

1.3.1 Computing the image of Brill's map

Brill's equations [17, 18, 31] are set-theoretic equations for the Chow variety $Ch_d(V)$. The Chow variety $Ch_d(V)$ is the zero set of a polynomial map $\mathfrak{B} : S^dV \to S_{(d,d)}V \otimes S^{d^2-d}V$ of degree $d+1$ (see §2.4). Brill's equations are the span of the coefficients of the polynomial map $\mathfrak{B}$. The polynomial map $\mathfrak{B}$ is complicated and it is hard to write down the coefficients explicitly from Brill's presentation. I determine Brill's equations as a $GL(V)$-module to understand these equations and write down these equations explicitly.

The idea is to construct the polarization (see §2.1) $\bar{\mathfrak{B}}$ of $\mathfrak{B}$, where $\bar{\mathfrak{B}} : S^{d+1}(S^dV) \to S_{(d,d)}V \otimes S^{d^2-d}V$, and then determine the image of $\bar{\mathfrak{B}}$, whose dual is isomorphic to the $GL(V)$-module corresponding to Brill's equations. I call $\bar{\mathfrak{B}}$ Brill's map.

Brill's map $\bar{\mathfrak{B}}$ is a $GL(V)$-module map, the space $S_{(d,d)}V \otimes S^{d^2-d}V$ can be decomposed by Pieri's rule (see e.g. [16] or §2.3),

$$S_{(d,d)}V \otimes S^{d^2-d}V = \bigoplus_{j=0}^{d} S_{(d^2-j,d,j)}V,$$

I determine which irreducible $GL(V)$-modules are in the image of Brill's map.

1.3.2 Flattenings, Koszul Young flattenings and determinantal equations

Equations for the secant varieties of Chow varieties are mostly unknown, and even for the secant varieties of Veronese varieties very little is known. One class of equations is obtained from the so-called *flattenings* or *catalecticants*, which date back to Sylvester: for any $1 \leq k \leq d$, there is an inclusion $F_{k,d-k} : S^dV \hookrightarrow S^kV \otimes S^{d-k}V$, called a polarization map. For any $P \in S^dV$, define the *k-th polarization* $P_{k,d-k}$ of P to be $F_{k,d-k}(P)$. Then $P_{k,d-k} \in S^kV \otimes S^{d-k}V$ can be seen as a linear map $P_{k,d-k} : S^kV^* \to S^{d-k}V$. The image of $P_{k,d-k}$ is the space spanned by all k-th order partial derivatives of P, and is studied in the computer science literature under the name the *method of partial derivatives* (see, e.g. [10] and the references therein). If $\{x_1, \dots, x_n\}$ is a basis of V, then $\{\frac{\partial^k}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}\}_{i_1+\cdots+i_n=k}$ is a basis of S^kV^*, define $P_{k,d-k}(\frac{\partial^k}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}) = \frac{\partial^k P}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$ and extend it linearly.

If $[P] \in v_d(\mathbb{P}V)$, the rank of $P_{k,d-k}$ is one, so the size $(r+1)$-minors of $P_{k,d-k}$ are in the ideal of $I_{r+1}(\sigma_r(v_d(\mathbb{P}V)))$. If $[P] \in Ch_d(V)$ with $\dim V \geq d$, then the rank of $P_{k,d-k}$ is $\binom{d}{k}$, so the size $r\binom{d}{k} + 1$ minors are in the ideal of $\sigma_r(Ch_d(V))$.

Other equations come from *Young flattenings*, see [12, 13, 35] for a discussion of the Young flattenings and the state of the art. For $P \in S^dV$, the Koszul Young flattening is a linear map $P^{\wedge p}_{k,d-k} : S^kV^* \otimes \Lambda^pV \to S^{d-k+1}V \otimes \Lambda^{p+1}V$, it is defined by the composition of the following two maps

$$S^kV^* \otimes \Lambda^pV \to S^{d-k}V \otimes \Lambda^pV \to S^{d-k-1}V \otimes \Lambda^{p+1}V,$$

where the first map is defined by tensoring $P_{k,d-k}$ with the identity map $Id_{\Lambda^pV} : \Lambda^pV \to \Lambda^pV$, and the second map $\wedge_{d-k,p} : S^{d-k}V \otimes \Lambda^pV \to S^{d-k-1}V \otimes \Lambda^{p+1}V$ is defined as follows:

$$l_1 \cdots l_{d-k} \otimes m_1 \wedge m_2 \cdots \wedge m_p \mapsto \sum_{s=1}^{d-k} l_1 l_2 \cdots \hat{l_s} \cdots l_{d-k} \otimes l_s \wedge m_1 \wedge m_2 \cdots \wedge m_p,$$

then extend linearly to the whole space. In the tensor setting, Koszul Young flattenings have led to the current best lower bound for the border rank of matrix multiplication in [34, 37].

Another Young flattening $P_{k,d-k[l]} : S^kV^* \otimes S^lV \to S^{d-k+l}V$ is obtained by tensoring $P_{k,d-k}$ with the identity map $Id_{S^lV} : S^lV \to S^lV$, and projecting (symmetrizing) the image in $S^{d-k}V \otimes S^lV$ to $S^{d-k+l}V$. This map goes under the name "method of shifted partial derivatives" in the computer science literature. The method of shifted partial derivatives is studied in [22], where Gupta, Kamath, Kayal and S-aptharishicite proved if $\delta_1, \delta_2 \sim \sqrt{n}$, dim $V = n^2$ and $[\text{perm}_n] \in \sigma_r(v_{\delta_1}(\mathbb{P}S^{\delta_2}V))$, then $r = 2^{\Omega(\sqrt{n})}$.

By computing the Koszul Young flattenings of Chow varieties and their secant varieties, I obtain equations for these varieties.

Theorem 1.3.1. *Let $V = \mathbb{C}^d$ with a basis $\{x_1, \ldots, x_d\}$ and $P = x_1 \cdots x_d$, and let*

$2 \leq k < \lceil \frac{d}{2} \rceil$, $p < \lceil \frac{d}{2} \rceil$. *The map*

$$P^{\wedge p}_{k,d-k} : S^k V^* \otimes \Lambda^p V \to S^{d-k-1} V \otimes \Lambda^{p+1} V$$

has rank

$$\begin{aligned}
\mathbf{S}(\mathbf{p},\mathbf{d},\mathbf{k}) &= \sum_{s=\max\{0,p-k\}}^{\min\{p,d-k-1\}} \binom{d}{s} \binom{d-s}{d-k+p-2s} \binom{d-k+p-2s-1}{p-s} && (1.1)\\
&= \frac{d!}{p!(d-p-1)!} \sum_{s=\max\{0,p-k\}}^{\min\{p,d-k-1\}} \frac{\binom{p}{s}\binom{d-1-p}{s+k-p}}{d-k+p-2s}. && (1.2)
\end{aligned}$$

Therefore the $(\mathbf{S}(\mathbf{p},\mathbf{d},\mathbf{k})+1) \times (\mathbf{S}(\mathbf{p},\mathbf{d},\mathbf{k})+1)$ *minors of* $P^{\wedge p}_{k,d-k}$ *are in the ideal of* $Ch_d(V)$.

Theorem 1.3.2. *Let* $V = \mathbb{C}^{rd}$ *with a basis* $\{x_1, \ldots, x_{rd}\}$ *and* $P = x_1 \cdots x_d + x_{d+1} \cdots x_{2d} + \cdots + x_{(r-1)d+1} \cdots x_{rd}$. *Assume* $k < \lceil \frac{d}{2} \rceil$, $p < \lceil \frac{d}{2} \rceil$, $r \geq 2$. *Then the map*

$$P^{\wedge p}_{k,d-k} : S^k V^* \otimes \Lambda^p V \to S^{d-k-1} V \otimes \Lambda^{p+1} V$$

has rank

$$\operatorname{rank}(P^{\wedge p}_{k,d-k}) \leq r[\binom{d}{k}(\binom{dr}{p} - \binom{d}{p}) + \mathbf{S}(\mathbf{p},\mathbf{k},\mathbf{d})]. \qquad (1.3)$$

In particular, when $d \geq 2$, *and* $p = k = 1$,

$$\operatorname{rank}(P^{\wedge 1}_{1,d-1}) \leq d^2 r^2 - r.$$

Therefore the $(d^2r^2 - r + 1) \times (d^2r^2 - r + 1)$ *minors of* $P^{\wedge 1}_{1,d-1}$ *are in the ideal of*

$\sigma_r(Ch_d(V))$.

Remark 1.3.3. *Theorems 1.2.3 and 1.3.2 are consequences of Theorem 1.3.1.*

To prove Theorem 1.2.5, I compute the flattening rank of a generic polynomial in $v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}V))) \subset \mathbb{P}S^nV$, where dim $V = n^2$ and $\delta_1, \delta_2 \sim \sqrt{n}$, and then I compare it to that of the permanent.

1.3.3 Prolongation and equations for secant varieties

Let W be a complex vector space and $X \subset \mathbb{P}W$ be an algebraic variety. Suppose we know the ideal of X. Then there is a systematic method called *prolongation* (see §5.1.1 for definition) to compute the ideal of $\sigma_r(X)$, but this method is difficult to implement. This method was studied by J. Sidman and S. Sullivant [43], and J.M. Landsberg and L. Manivel [33].

The group $GL(V)$ has an induced action on $S^k(S^dV)$, so $S^k(S^dV)$ can be decomposed into a direct sum of irreducible $GL(V)$-modules, the multiplicity of $S_\lambda V$ in $S^k(S^dV)$ is the plethysm coefficient $p_\lambda(k, d)$. To obtain equations for secant varieties, on one hand I compute prolongations directly via differential operators and representation theory. On the other hand, I rephrase prolongations and reduce computing prolongations to computing the polarization maps via plethysm coefficients and Littlewood-Richardson coefficients. This gives a path towards obtaining equations for secant varieties of Chow varieties and other varieties.

1.4 Organization

In Chapter 2, I include mathematical preliminaries for this dissertation, which are polarization of a polynomial map, G-variety, Brill's equations, representation theory and Foulkes-Howe map and the ideal of Chow variety.

In Chapter 3, I determine Brill's equations as a $GL(V)$-module.

In Chapter 4, I obtain determinantal equations for the Chow varieties, their secant varieties and secant varieties of Veronese reembedding of secant varieties of Veronese variety by flattenings and Koszul Young flattenings. Consequently, I get a new lower bound for the symmetric border rank of $x_1 x_2 \cdots x_d$ when d is odd, and a new complexity lower bound for the permanent.

In Chapter 5, I use the method of prolongation to obtain equations for secant varieties of Chow varieties as $GL(V)$-modules.

In Chapter 6, I give a summary of the dissertation.

1.5 Notation

1. $Ch_d(V)$: Chow variety of polynomials that decompose as product of linear form.

2. $GL(V)$: the general linear group of invertible linear maps from V to V.

3. $S^d V$: the space of homogeneous polynomials of degree d on the dual space V^*.

4. $\Lambda^d V$: the d-th exterior power of V.

5. $S^\delta(S^d V)$: the space of homogeneous polynomials of degree δ on $S^d V^*$.

6. $\mathfrak{S}_\mathfrak{n}$: the symmetric group of order n.

7. perm_n: the permanent defined by $\mathrm{perm}_n = \sum_{\sigma \in \mathfrak{S}_\mathfrak{n}} x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{n\sigma(n)}$.

8. $l^{n-m}\mathrm{perm}_m$: The padded permanent.

9. $\sigma_r(X)$: the r-th secant variety of X.

10. $v_d(X)$: Veronese embedding of X.

11. $v_d(\mathbb{P}V)$: the Veronese variety.

12. Let h_n and g_n be two positive sequences, define $h_n = \omega(g_n)$ if $\lim_{n\to\infty} \frac{h_n}{g_n} = \infty$, define $h_n = \Omega(g_n)$ if $\lim_{n\to\infty} \frac{h_n}{g_n} \geq C$ for some positive constant C.

13. A partition λ of order d and length m: $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_1 \geq \cdots \geq \lambda_m > 0$, $\lambda_j \in \mathbb{N}$ and $\sum_{i=1}^m \lambda_i = d$.

14. $[\lambda]$: irreducible $\mathfrak{S}_d$ representation corresponding to the partition λ of order d.

15. $S_\lambda V$: irreducible $GL(V)$ representation corresponding to the partition λ.

16. A semi-standard tableau of shape λ and content $k \times d$: a semi-standard tableau associated to λ and filled with $\{1, \dots, k\}$ such that each $i \in \{1, \dots, k\}$ appears d times.

17. Lexicographic order of partitions: $\lambda > \mu$ if the first nonvanishing $\lambda_i - \mu_i$ is positive.

18. Dominance partial order of partitions: $\alpha > \beta$ if $\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i$ for each i.

19. $\mathbf{R}_S(P)$: the symmetric rank of P.

20. $\underline{\mathbf{R}}_S(P)$: the symmetric border rank of P.

21. $I_d(X)$: the degree d component of the ideal of X.

22. $\bar{P}$: the (complete) polarization of a polynomial map P.

23. $P_{k,d-k}$: the k-th polarization of $P \in S^d V$, or the k-th flattening of $P \in S^d V$.

24. $P_{k,d-k}^{\wedge p}$: Koszul Young flattening of $P \in S^d V$.

2.1 Polarization of a polynomial map

Definition 2.1.1. *Let $V_1, \ldots, V_d$ be complex vector spaces, define a map $\varphi : V_1 \times \cdots \times V_d \to V_1 \otimes \cdots \otimes V_d$ by $\varphi(v_1, \ldots, v_d) = v_1 \otimes \cdots \otimes v_d$. The universal property of tensors is the following: given a complex vector space W and a multi-linear map $h : V_1 \times \cdots \times V_d \to W$, there is a unique linear map $\tilde{h} : V_1 \otimes \cdots \otimes V_d \to W$, such that $h = \tilde{h} \circ \varphi$.*

Definition 2.1.2. *Let W be a complex vector space. A map $P : W \to \mathbb{C}^m$ is a polynomial map of degree k if $P = (P_1, \ldots, P_m)$, and each P_i $(i = 1, \ldots, m)$ is called a homogenous polynomial of degree k on W.*
Define the complete polarization $\bar{P} : W \times \cdots \times W \to \mathbb{C}^m$ of P to be

$$\bar{P}(w_1, \cdots, w_k) = \frac{1}{k!} \sum_{I \subset [k], I \neq \emptyset} (-1)^{k-|I|} P(\sum_{i \in I} w_i).$$

Where $[k] = \{1, \ldots, k\}$, $w_i \in W$ and $\bar{P}$ is a symmetric multi-linear map. By the universal property of tensors, $\bar{P}$ is considered as a map $\bar{P} : W^{\otimes k} \to \mathbb{C}^m$. By the symmetry of $\bar{P}$, $\bar{P}$ can be also seen as a map $\bar{P} : S^k W \to \mathbb{C}^m$, such that

$$\bar{P}(w_1 \cdots w_k) = \frac{1}{k!} \sum_{I \subset [k], I \neq \emptyset} (-1)^{k-|I|} P(\sum_{i \in I} w_i), \tag{2.1}$$

and it can be extended linearly to the whole space.

Example 2.1.3. *Let* $\dim V$ *=2, and let $\{e_1, e_2\}$ be a basis of V. Consider the poly-*

nomial map $P : V \to \mathbb{C}^2$ defined by

$$a_1e_1 + a_2e_2 \mapsto (a_1^2, a_1^2 + a_2^2).$$

P is a polynomial map of degree 2, so by (2.1) $\bar{P} : S^2V \to \mathbb{C}^2$ is defined by

$$\begin{aligned}
\bar{P}((a_1e_1 + a_2e_2)(a_3e_1 + a_4e_2)) &= \frac{1}{2}[P(a_1e_1 + a_2e_2 + a_3e_1 + a_4e_2) \\
&\quad -P(a_1e_1 + a_2e_2) - P(a_3e_1 + a_4e_2)] \\
&= \frac{1}{2}[((a_1 + a_3)^2, (a_1 + a_3)^2 + (a_2 + a_4)^2) \\
&\quad -(a_1^2, a_1^2 + a_2^2) - (a_3^2, a_3^2 + a_4^2)] \\
&= (a_1a_3, a_1a_3 + a_2a_4).
\end{aligned}$$

Therefore

$$\begin{aligned}
\bar{P}(ae_1^2 + be_1e_2 + ce_2^2) &= a\bar{P}(e_1^2) + b\bar{P}(e_1e_2) + c\bar{P}(e_2^2) \\
&= (a, a) + (0, 0) + (0, c) \\
&= (a, a + c).
\end{aligned}$$

2.2 G-variety

I follow the notation in [31, §4.7].

Definition 2.2.1. *Let W be a complex vector space. A variety $X \subset \mathbb{P}W$ is called a G-variety if W is a module for the group G and for all $g \in G$ and $x \in X$, $g \cdot x \in X$.*

The group G has an induced action on S^dW^* such that for any $P \in S^dW^*$ and $w \in W$, $g \cdot P(w) = P(g^{-1} \cdot w)$. The degree d component of the ideal of X $I_d(X)$ is a linear subspace of S^dW^* that is invariant under the action of G. Define

$S^\bullet W^* := \oplus_{d=0}^{\infty} S^d W^*$, then:

Proposition 2.2.2. *If $X \subset \mathbb{P}W$ is a G-variety, then the ideal of X is a G-submodule of $S^\bullet W^* = \oplus_{d=0}^{\infty} S^d W^*$.*

Example 2.2.3. *The group $GL(V)$ has an induced action on $S^d V$ and $S^k(S^d V^*)$ similarly. $Ch_d(V)$, $v_d(\mathbb{P}V)$ and their secant varieties are invariant under the action of $GL(V)$, therefore they are $GL(V)$-varieties and their ideals are $GL(V)$-submodules of $S^\bullet(S^d V^*) = \oplus_{k=0}^{\infty} S^k(S^d V^*)$.*

Let $X \subset \mathbb{P}W$ be a G-variety, and M be an irreducible submodule of $S^\bullet W^*$, then either $M \subset I(X)$ or $M \cap I(X) = \emptyset$. Thus to test if M gives equations for X, one only need to test one polynomial in M.

2.3 Representation theory

2.3.1 Young tableaux and semi-standard tableaux

I follow the notation in [16] and [31]. A partition λ of an integer d is $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 \geq \cdots \geq \lambda_m > 0$, $\lambda_j \in \mathbb{N}$ and $\sum_{i=1}^{m} \lambda_i = d$. We say d is the *order* of λ and m is the *length* of λ. We often denote this by $\lambda \vdash d$. To a partition $\lambda \vdash d$, we associate a *Young diagram*, which is a left aligned collection of boxes with λ_i boxes in row i.

A *filling* of a Young diagram using the numbers $\{1, \cdots, l\}$ is an assignment of one number to each box, with repetitions allowed. A filled Young diagram is called a *Young tableau.* A *semi-standard* filling is one in which the entries are strictly increasing in the columns and weakly increasing in the rows. *Semi-standard tableau* is similarly defined.

Let λ be a partition with order kd, a semi-standard tableau of *shape* λ *and content* $k \times d$ is a semi-standard tableau associated to λ and filled with $\{1, \ldots, k\}$ such that each $i \in \{1, \ldots, k\}$ appears d times.

2.3.2 Irreducible representations of the symmetric group $\mathfrak{S}_d$ and the group $GL(V)$

I follow the notation in [16] and [31]. For any partition λ of order d, we can construct the irreducible representations of the symmetric group $\mathfrak{S}_d$ and the group $GL(V)$ as follows:

Definition 2.3.1. *Let G be a finite Group with elements $g_1, \ldots, g_r$, define the group algebra $\mathbb{C}[G]$ of G to be a complex vector space with basis $\{e_{g_1}, \ldots, e_{g_r}\}$ and with the algebra structure $e_{g_i} e_{g_j} = e_{g_i g_j}$.*

Let T_λ be a Young tableau of shape λ and filled with $\{1, 2, \ldots, d\}$ without repetitions, the symmetric group $\mathfrak{S}_d$ acts on T_λ in a natural way. Define $P_{T_\lambda} = \{g \in \mathfrak{S}_d : g \text{ preserves each row}\}$, and $Q_{T_\lambda} = \{g \in \mathfrak{S}_d : g \text{ preserves each colume}\}$. Define elements in $\mathbb{C}[\mathfrak{S}_d]$: $a_\lambda = \sum_{g \in P_{T_\lambda}} e_g$, $b_\lambda = \sum_{g \in Q_{T_\lambda}} \operatorname{sign}(g) e_g$, and $c_{T_\lambda} = a_{T_\lambda} \cdot b_{T_\lambda}$. c_{T_λ} is called a *Young symmetrizer.*

Theorem 2.3.2. *$\mathbb{C}[\mathfrak{S}_d] c_{T_\lambda}$ is an irreducible representation of $\mathfrak{S}_d$. Moreover, if $\tilde{T}_\lambda$ is another Young tableau of shape λ filled with $\{1, 2, \cdots, d\}$ without repetitions, then $\mathbb{C}[\mathfrak{S}_d] c_{T_\lambda}$ and $\mathbb{C}[\mathfrak{S}_d] c_{\tilde{T}_\lambda}$ are isomorphic.*

Definition 2.3.3. *Given a partition λ of order d, the $\mathfrak{S}_d$-module $[\lambda]$ is defined to be the representation corresponding to any of $\mathbb{C}[\mathfrak{S}_d] c_{T_\lambda}$.*

Example 2.3.4. *If $\lambda = (d)$, then $[\lambda] = \mathbb{C} \sum_{g \in \mathfrak{S}_d} e_g$ is the trivial representation of $\mathfrak{S}_d$.*
If $\lambda = (1^d)$, then $[\lambda] = \mathbb{C} \sum_{g \in \mathfrak{S}_d} \operatorname{sign}(g) e_g$ is the alternating representation of $\mathfrak{S}_d$.
If $\lambda = (d-1, 1)$, then $[\lambda]$ is the standard representation of $\mathfrak{S}_d$.

The group $GL(V)$ has a natural action on $V^{\otimes d}$ such that $g \cdot (v_1 \otimes v_2 \cdots \otimes v_d) = g \cdot v_1 \otimes \cdots \otimes g \cdot v_d$. While the group $\mathfrak{S}_d$ has a right action on $V^{\otimes d}$ by $(v_1 \otimes \cdots \otimes$

$v_d) \cdot g = v_{g^{-1}(1)} \otimes \cdots \otimes v_{g^{-1}(d)}$, which induces a right action of $\mathbb{C}[\mathfrak{S}_d]$ on $V^{\otimes d}$. Define $S_{T_\lambda}V := V^{\otimes d} \cdot c_{T_\lambda} \subset V^{\otimes d}$.

Theorem 2.3.5. *$S_{T_\lambda}V$ is an irreducible $GL(V)$-submodule of $V^{\otimes d}$. Moreover, if $\tilde{T}_\lambda$ is another Young tableau of shape λ filled with $\{1, 2, \ldots, d\}$ without repetitions, then $S_{T_\lambda}V$ and $S_{\tilde{T}_\lambda}V$ are isomorphic.*

Definition 2.3.6. *Given a partition λ of order d, the $GL(V)$-module $S_\lambda V$ is defined to be any of $S_{T_\lambda}V$.*

Example 2.3.7. *If $\lambda = (d)$, then $S_\lambda V = V^{\otimes d} \cdot \sum_{g \in \mathfrak{S}_d} e_g = S^d V$.*
If $\lambda = (1^d)$, then $S_\lambda V = V^{\otimes d} \cdot \sum_{g \in \mathfrak{S}_d} \operatorname{sign}(g) e_g = \Lambda^d V$.

2.3.3 The Littlewood-Richardson coefficients and Pieri's rule

Let π and μ be two partitions, the tensor product $S_\lambda V \otimes S_\mu V$ is a $GL(V)$-module. The littlewood-Richardson coefficients $c^\nu_{\pi\mu}$ are defined to be the multiplicity of $S_\nu V$ in $S_\lambda V \otimes S_\mu V$, i.e. $S_\lambda V \otimes S_\mu V = \bigoplus_\nu c^\nu_{\pi\mu} S_\nu V$.

We order partitions *lexicographically*: $\lambda > \mu$ if the first nonvanishing $\lambda_i - \mu_i$ is positive. Necessary conditions for $c^\nu_{\pi\mu}$ to be positive are $|\nu| = |\pi| + |\mu|$ and ν is greater than π and μ.

In particular $S_\lambda V \otimes S^d V = c^\nu_{\lambda,(d)} S_\nu V$.

Theorem 2.3.8. (Pieri's rule)

$$c^\nu_{\lambda,(d)} = \begin{cases} 1 & \text{if } \nu \text{ is obtained from } \lambda \text{ by adding d boxes to} \\ & \text{the rows of } \lambda \text{ with no two in the same column;} \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.3.9. *By Pieri's rule,*

$$S^aV \otimes S^bV = \bigoplus_{0\le t\le s, s+t=a+b} S_{(s,t)}V.$$

$$S_{(d,d)}V \otimes S^{d^2-d}V = \bigoplus_{j=0}^{d} S_{(d^2-j,d,j)}V.$$

2.3.4 Highest weight vectors of an irreducible $GL(V)$-module

I follow the notation in [16]. Let dim $V = n$ and $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Recall that the group $GL(V)$ has a natural action on $V^{\otimes d}$ such that $g \cdot (v_1 \otimes v_2 \cdots \otimes v_d) = g \cdot v_1 \otimes \cdots \otimes g \cdot v_d$. Let $B \subset GL(V)$ be the subgroup of upper-triangular matrices (a *Borel subgroup*). For any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with order d, there is a unique line in $S_\lambda V \subset V^{\otimes d}$ that is preserved by B, called a *highest weight line.* Let $\mathfrak{gl}(V)$ be the Lie algebra of $GL(V)$, there is an induced action of $\mathfrak{gl}(V)$ on $V^{\otimes d}$. For $X \in \mathfrak{gl}(V)$,

$$X.(v_1 \otimes v_2 \cdots \otimes v_d) = X.v_1 \otimes v_2 \cdots \otimes v_d + \cdots + v_1 \otimes v_2 \cdots \otimes v_{d-1} \otimes X.v_d.$$

Let $E^i_j \in \mathfrak{gl}(V)$ such that $E^i_j(e_j) = e_i$ and $E^i_j(e_k) = 0$ when $k \neq j$. If $i < j$, E^i_j is called a *raising operator*; if $i > j$, E^i_j is called a *lowering operator.*

A *highest weight vector* of a $GL(V)$-module is a weight vector that is killed by all raising operators. Each realization of the module $S_\lambda V$ has a unique highest weight line. Let W be a $GL(V)$-module, the multiplicity of $S_\lambda V$ in W is equal to the dimension of the highest weight space with respect to the partition λ.

Define the weight space $W_{(a_1,\ldots,a_n)} \subset S^k(S^dV)$ to be the set of all the weight vectors whose weights are $(a_1, \ldots, a_n)$. Note that S^dV has a natural basis $\{e_1^{\alpha_1} \cdots e_n^{\alpha_n}\}_{\alpha_1+\cdots+\alpha_n=d}$.

Example 2.3.10. *$S_{(4,2)}V \subset S^3(S^2V)$ has multiplicity 1.*

Proof. Let v be a highest weight vector of $S_{(4,2)}V$. The weight space $W_{(4,2)}$ has a basis $\{(e_1^2)^2(e_2^2), (e_1^2)(e_1e_2)^2\}$. Write $v = a(e_1^2)^2(e_2^2) + b(e_1^2)(e_1e_2)^2$, then $E_2^1 v = 0$ implies $(2a+2b)(e_1^2)^2(e_1e_2) = 0$, therefore $a = -b$, so the multiplicity of $S_{(4,2)}V$ in $S^3(S^2V)$ is 1. $\square$

Proposition 2.3.11. *A highest weight vector f of $S_{(2^k)}V \subset S^k(S^2V)$ is the determinant of the $k \times k$ matrix M with $M_{ij} = e_ie_j$ for $1 \leq i, j \leq k$.*

Proof. Since $S_{(2^k)}V \subset S^k(S^2V)$ is of multiplicity one, we only need to prove $\det M$ is killed by all raising operators E_{i+1}^i $(i = 1, 2, ..., k-1)$. By symmetry, we only need to prove $\det M$ is killed by the raising operator E_2^1. It is straightforward to verify $\det M$ is killed by the raising operator E_2^1. $\square$

Proposition 2.3.12. *The highest weight vector f of $S_{(7,3,2)}V \subset S^4(S^3V)$ is*

$$\begin{aligned} f \;=\; & (e_1^3)^2(e_1e_2^2)(e_2e_3^2) - 2(e_1^3)^2(e_1e_2e_3)(e_2^2e_3) + (e_1^3)^2(e_1e_3^2)(e_2^3) - (e_1^3)(e_1^2e_2)^2(e_2e_3^2) \\ & +2(e_1^3)(e_1^2e_2)(e_1^2e_3)(e_2^2e_3) - 4(e_1^3)(e_1^2e_2)(e_1e_2^2)(e_1e_3^2) + 0(e_1^3)(e_1^2e_3)(e_1e_2^2)(e_1e_2e_3) \\ & +3(e_1^2e_2)^3(e_1e_3^2) + 4(e_1e_2e_3)^2(e_1^2e_2)(e_1^3) - (e_1^3)(e_1^2e_3)^2(e_2^3) + 3(e_1^2e_2)(e_1e_2^2)(e_1^2e_3)^2 \\ & -6(e_1^2e_2)^2(e_1^2e_3)(e_1e_2e_3). \end{aligned}$$

Proof. Let $f \in W_{(7,3,2)} \subset S^4(S^3V)$ be a weight vector. The weight space $W_{(7,3,2)} \subset S^4(S^3V)$ has dimension 12. Write f as a linear combination of the basis vectors and apply E_2^1 and E_3^2 to f, we get two systems of linear equations. There is a unique solution up to scale. $\square$

Remark 2.3.13. *The module $S_{(7,3,2)}V$ cuts out $Ch_3(V^*)$ set-theoretically* [19].

Proposition 2.3.14. *The highest weight vector f of $S_{(5,4,2,1)}V \subset S^4(S^3V)$ is*

$$f = e_2^2 e_4 h_1 + e_1 e_3 e_4 h_2 + e_1 e_2 e_4 h_3 + e_1^2 e_4 h_4. \tag{2.2}$$

Here

$$\begin{aligned} h_4 &= (e_1^2 e_2)(e_2^3)(e_1 e_3^2) - (e_1 e_2^2)^2 (e_1 e_3^2) - (e_1^2 e_2)(e_1 e_2 e_3)(e_2^2 e_3) \\ &+ (e_1^2 e_3)(e_1 e_2^2)(e_2^2 e_3) - (e_1 e_2^2)(e_1 e_2 e_3)^2 - (e_1^2 e_3)(e_1 e_2 e_3)(e_2^3), \end{aligned}$$

$h_3 = -E_2^1 h_4$, $h_1 = \frac{1}{2} E_2^1 E_2^1 h_4$ *is a highest weight vector of* $S_{(5,2,2)}V \subset S^3(S^3V)$ *and* $h_2 = E_3^2 E_2^1 h_4$ *is a highest weight vector of* $S_{(4,4,1)}V \subset S^3(S^3V)$.

2.4 Brill's equations

Following the idea in §8.6 in [31], I use the following notation to define Brill's equations. We first define two maps $\pi_{d,d}$ and Q_d, then use them to define Brill's equations.

Define the projection map $\pi_{d,d} : S^dV \otimes S^dV \to S_{(d,d)}V$ by

$$(l_1 \cdots l_d) \otimes (m_1 \cdots m_d) \mapsto \sum_{\sigma \in \mathfrak{S}_d} (l_1 \wedge m_{\sigma(1)}) \cdot (l_2 \wedge m_{\sigma(2)}) \cdots (l_d \wedge m_{\sigma(d)}), \tag{2.3}$$

and then extend linearly to the whole space.

Recall $S^\bullet V = \oplus_{i=0}^\infty S^iV$. Define a multiplication on $S^\bullet V \otimes S^\bullet V$ by, for any $a, b, c, d \in S^\bullet V$,

$$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd, \tag{2.4}$$

and this extends linearly to $S^\bullet V \otimes S^\bullet V$.

Let $f \in S^\delta V$ and let $f_{j,\delta-j} \in S^j V \otimes S^{\delta-j} V$ be the j-th polarization of f. Define maps

$$E_j : S^\delta V \to S^j V \otimes S^{j(\delta-1)} V,$$
$$f \mapsto f_{j,\delta-j} \cdot (1 \otimes f^{j-1}).$$

If $j > \delta$ define $E_j(f) = 0$.

Example 2.4.1. *Let $f = l_1 l_2 l_3 \in S^3 V$, then*

$$\begin{aligned} E_1(f) &= f_{1,2} \cdot (1 \otimes 1) \\ &= l_1 \otimes l_2 l_3 + l_3 \otimes l_1 l_2 + l_2 \otimes l_1 l_3. \end{aligned}$$

$$\begin{aligned} E_2(f) &= f_{2,1} \cdot (1 \otimes l_1 l_2 l_3) \\ &= (l_1 l_2 \otimes l_3 + l_1 l_3 \otimes l_2 + l_2 l_3 \otimes l_1) \cdot (1 \otimes l_1 l_2 l_3). \\ &= l_1 l_2 \otimes l_1 l_2 l_3^2 + l_1 l_3 \otimes l_1 l_2^2 l_3 + l_2 l_3 \otimes l_1^2 l_2 l_3. \end{aligned}$$

$$\begin{aligned} E_3(f) &= f_{3,0} \cdot (1 \otimes f^2) \\ &= f \otimes f^2 \\ &= l_1 l_2 l_3 \otimes l_1^2 l_2^2 l_3^2. \end{aligned}$$

The elementary symmetric and power sum function are:

$$e_j = e_j(x_1, \ldots, x_{\underline{v}}) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le \underline{v}} x_{i_1} \cdots x_{i_j},$$

$$p_j = p_i(x_1, \ldots, x_{\underline{v}}) = \sum_{i=1}^{\underline{v}} x_i^j.$$

The power sum can be written in terms of symmetric function using Girard formula:

$$p_k = \tag{2.5}$$
$$\mathcal{P}_k(e_1, \cdots, e_d) = \sum_{i_1+2i_2+\cdots di_d=k} k(-1)^{k+i_1+i_2+\cdots i_d} \frac{(i_1+i_2+\cdots i_d-1)!}{i_1!\cdots i_d!} e_1^{i_1}\cdots e_d^{i_d}.$$

Example 2.4.2. $p_2 = \mathcal{P}_2(e_1, e_2) = e_1^2 - 2e_2$. $p_3 = \mathcal{P}_3(e_1, e_2, e_3) = e_1^3 - 3e_1e_2 + 3e_3$.

Next, we use Girard formula and E_j to define Q_d . Define polynomial maps

$$Q_{d,\delta} : S^\delta V \longrightarrow S^d V \otimes S^{d(\delta-1)}V$$

by

$$Q_{d,\delta}(f) = \mathcal{P}_d(E_1(f), \ldots, E_d(f)). \tag{2.6}$$

Write $Q_d = Q_{d,d}$. Explicitly

$$Q_d(f) = \tag{2.7}$$
$$\sum_{i_1+2i_2+\cdots+di_d=d} d(-1)^{d+i_1+\cdots+i_d} \frac{(i_1+\cdots+i_d-1)!}{i_1!\cdots i_d!} \left(\prod_{j=1}^{d} f_{j,d-j}^{i_j}\right) \cdot (1 \otimes f^{d-(i_1+\cdots+i_d)}).$$

Example 2.4.3. *Let $d = 2$, and $f \in S^2V$, by (2.7),*

$$Q_2(f) = f_{1,1}^2 - 2f \otimes f.$$

Lemma 2.4.4. (§8.6 [31]) *Let $l_i \in V$ for $i = 1, \cdots, d$, then*

$$Q_d(l_1 \cdots l_d) = \sum_{j=1}^{d} l_j^d \otimes (l_1^d \cdots l_{j-1}^d l_{j+1}^d \cdots l_d^d). \tag{2.8}$$

Now we define Brill's polynomial map $\mathfrak{B} : S^dV \to S_{(d,d)}V \otimes S^{d^2-d}V$ invariantly. It is the composition of the following two maps:

$$S^dV \to S^dV \otimes S^dV \otimes S^{d^2-d}V \to S_{(d,d)}V \otimes S^{d^2-d}V,$$

where the first map sends $f \in S^dV$ to $f \otimes Q_d(f)$, and the second map is $\pi_{d,d} \otimes Id_{S^{d^2-d}V}$. By Lemma 2.4.4,

$$\begin{aligned} \mathfrak{B}(l_1 \cdots l_d) &= \pi_{d,d} \otimes Id_{S^{d^2-d}V}[(l_1 \cdots l_d) \otimes \sum_{j=1}^{d} l_j^d \otimes (l_1^d \cdots l_{j-1}^d l_{j+1}^d \cdots l_d^d)] \\ &= 0. \end{aligned}$$

The converse is also true:

Theorem 2.4.5. (Brill,Gordon [18], Gelfand-Kapranov-Zelevinsky [17], Briand [3]) *Consider the polynomial map*

$$\mathfrak{B} : S^dV \to S_{(d,d)}V \otimes S^{d^2-d}V$$

given by

$$\mathfrak{B}(f) = \pi_{d,d} \otimes Id_{S^{d^2-d}V}[f \otimes Q_d(f)]. \tag{2.9}$$

Then $\mathfrak{B}(f) = 0 \Leftrightarrow [f] \in Ch_d(V)$.

Remark 2.4.6. *There was a gap in Brill's argument, that was repeated in* [17] *and finally fixed by E. Briand in* [3].

2.5 Foulkes-Howe map and the ideal of Chow variety

I follow notation in [31, §8.6]. Define the *Foulkes-Howe map* $FH_{\delta,d} : S^\delta(S^d V) \to S^d(S^\delta V)$ as follows: First include $S^\delta(S^d V) \subset V^{\otimes \delta d}$. Next, regroup and symmetrize the blocks to $(S^\delta V)^{\otimes d}$. Finally, thinking of $S^\delta V$ as a single vector space, symmetrize again to land in $S^\delta(S^d V)$.

Example 2.5.1. $FH_{2,2}(x^2 \cdot y^2) = (xy)^2$*, and* $FH_{2,2}((xy)^2) = \frac{1}{2}[x^2 \cdot y^2 + (xy)^2]$.

$FH_{\delta,d}$ is a $GL(V)$-module map and Hadamard [24] observed and Howe rediscovered the following relationship between Foulkes-Howe map and ideal of Chow variety.

Proposition 2.5.2. (Hadamard [24]) Ker $FH_{\delta,d} = I_\delta(Ch_d(V^*))$.

Corollary 2.5.3. *When* $\delta = d + 1$, Ker $FH_{d+1,d} = I_{d+1}(Ch_d(V^*))$. *Therefore as an abstract* $GL(V)$*-module,* $I_{d+1}(Ch_d(V^*)) \supset S^{d+1}(S^d V) - S^d(S^{d+1} V)$.

Proposition 2.5.4. (Hermite [27], Hadamard [25], J.Müler and M.Neunhöfer)[41]) *When* $d = 2, 3, 4$, $FH_{d,d}$ *are injective and hence surjective.*

Proposition 2.5.5. *(*T. McKay [40]*) If* $FH_{\delta,d}$ *is surjective, then* $FH_{\delta+1,d}$ *is surjective.*

So when $d = 2, 3, 4$, $FH_{d+1,d}$ are surjective, and $I_{d+1}(Ch_d(V^*)) = S^{d+1}(S^d V) - S^d(S^{d+1} V)$ as $GL(V)$- modules.

3. BRILL'S EQUATIONS AS A $GL(V)$-MODULE

This chapter is based on [19], I first construct Brill's map by the polarization of Brill's polynomial map, and then compute the image of Brill's map to determine Brill's equations as a $GL(V)$-module.

3.1 Construction of Brill's map

First consider the polarization $\overline{Q_d}$ of Q_d , where $Q_d : S^dV \to S^dV \otimes S^{d^2-d}V$.

Example 3.1.1. *Let $d = 2$, and $f, g \in S^2V$, by* (2.7)

$$Q_2(f) = f_{1,1}^2 - 2f \otimes f.$$

Therefore by (2.1)*, $\overline{Q_2} : S^2(S^2V) \to S^2V \otimes S^2V$ is defined by:*

$$\begin{aligned} \bar{Q}_2(f \cdot g) &= \frac{1}{2}((f+g)_{1,1}^2 - 2(f+g) \otimes (f+g) - (f_{1,1}^2 - 2f \otimes f) - (g_{1,1}^2 - 2g \otimes g)) \\ &= f_{1,1}g_{1,1} - f \otimes g - g \otimes f. \end{aligned}$$

So by (2.4)

$$\begin{aligned} \bar{Q}_2(e_1e_2 \cdot e_1e_2) &= (e_1e_2)_{1,1}^2 - 2(e_1e_2) \otimes (e_1e_2) \\ &= (e_1 \otimes e_2 + e_2 \otimes e_1)^2 - 2(e_1e_2) \otimes (e_1e_2) \\ &= e_1^2 \otimes e_2^2 + e_2^2 \otimes e_1^2. \end{aligned}$$

$$\begin{aligned}\bar{Q}_2(e_1^2 \cdot e_1e_2) &= (e_1e_2)_{1,1} \cdot (e_1^2)_{1,1} - (e_1^2) \otimes (e_1e_2) - (e_1e_2) \otimes (e_1^2) \\ &= (e_1 \otimes e_2 + e_2 \otimes e_1) \cdot (2e_1 \otimes e_1) - (e_1^2) \otimes (e_1e_2) - (e_1e_2) \otimes (e_1^2) \\ &= e_1^2 \otimes e_1e_2 + e_1e_2 \otimes e_1^2.\end{aligned}$$

$$\begin{aligned}\bar{Q}_2(e_1e_2 \cdot e_1e_3) &= (e_1e_2)_{1,1} \cdot (e_1e_3)_{1,1} - (e_1e_3) \otimes (e_1e_2) - (e_1e_2) \otimes (e_1e_3) \\ &= (e_1 \otimes e_2 + e_2 \otimes e_1) \cdot (e_1 \otimes e_3 + e_3 \otimes e_1) \\ &\quad -(e_1e_3) \otimes (e_1e_2) - (e_1e_2) \otimes (e_1e_3) \\ &= e_1^2 \otimes e_2e_3 + e_2e_3 \otimes e_1^2.\end{aligned}$$

$$\begin{aligned}\bar{Q}_2(e_1e_2 \cdot e_3^2) &= (e_1e_2)_{1,1} \cdot (e_3^2)_{1,1} - (e_3^2) \otimes (e_1e_2) - (e_1e_2) \otimes (e_3^2) \\ &= (e_1 \otimes e_2 + e_2 \otimes e_1) \cdot (2e_3 \otimes e_3) - (e_3^2) \otimes (e_1e_2) - (e_1e_2) \otimes (e_3^2) \\ &= 2e_1e_3 \otimes e_2e_3 + 2e_2e_3 \otimes e_1e_3 - e_3^2 \otimes e_1e_2 - e_1e_2 \otimes e_3^2.\end{aligned}$$

In general, $\overline{Q_d} : S^d(S^dV) \to S^dV \otimes S^{d^2-d}V$ is used to define Brill's map $\bar{\mathfrak{B}}$:

Lemma 3.1.2. *The polarization of Brill's polynomial map* $\mathfrak{B}$

$$\bar{\mathfrak{B}} : S^{d+1}(S^dV) \to S_{(d,d)}V \otimes S^{d^2-d}V$$

is

$$\bar{\mathfrak{B}}(f_1f_2 \dots f_{d+1}) = \frac{1}{d+1} \sum_{i=1}^{d+1} \pi_{d,d} \otimes Id_{S^{d^2-d}V}[f_i \otimes \overline{Q_d}(f_1 \cdots \hat{f}_i \cdots f_{d+1})]. \qquad (3.1)$$

Example 3.1.3. *Consider Brill's map* $\bar{\mathfrak{B}}$:$S^3(S^2V) \to S_{(2,2)}V \otimes S^2V$ *for* $d = 2$*. By*

Lemma 3.1.2,

$$\begin{aligned}
\bar{\mathfrak{B}}(e_1e_2\cdot e_1e_2\cdot e_1^2) &= \frac{1}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1^2\otimes\overline{Q_2}(e_1e_2\cdot e_1e_2)] \\
&\quad +\frac{2}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1e_2\otimes\overline{Q_2}(e_1e_2\cdot e_1^2)] \\
&= \frac{1}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1^2\otimes(e_1^2\otimes e_2^2+e_2^2\otimes e_1^2)] \\
&\quad +\frac{2}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1e_2\otimes(e_1^2\otimes e_1e_2+e_1e_2\otimes e_1^2)] \\
&= \frac{1}{3}[2(e_1\wedge e_2)^2\otimes e_1^2)+\frac{2}{3}(-(e_1\wedge e_2)^2\otimes e_1^2] \\
&= 0. \\
\bar{\mathfrak{B}}(e_1e_2\cdot e_1e_2\cdot e_1e_3) &= \frac{1}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1e_3\otimes\overline{Q_2}(e_1e_2\cdot e_1e_2)] \\
&\quad +\frac{2}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1e_2\otimes\overline{Q_2}(e_1e_2\cdot e_1e_3)] \\
&= \frac{1}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1e_3\otimes(e_1^2\otimes e_2^2+e_2^2\otimes e_1^2)] \\
&\quad +\frac{2}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1e_2\otimes(e_1^2\otimes e_2e_3+e_2e_3\otimes e_1^2)] \\
&= \frac{1}{3}[2(e_1\wedge e_2)(e_1\wedge e_3)\otimes e_1^2)]+\frac{2}{3}[-(e_1\wedge e_2)(e_1\wedge e_3)\otimes e_1^2] \\
&= 0. \\
\bar{\mathfrak{B}}(e_1e_2\cdot e_1e_2\cdot e_3^2) &= \frac{1}{3}\pi_{2,2}\otimes Id_{S^2V}[e_3^2\otimes\overline{Q_2}(e_1e_2\cdot e_1e_2)] \\
&\quad +\frac{2}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1e_2\otimes\overline{Q_2}(e_1e_2\cdot e_3^2)] \\
&= \frac{1}{3}\pi_{2,2}\otimes Id_{S^2V}(e_3^2\otimes(e_1^2\otimes e_2^2+e_2^2\otimes e_1^2)) \\
&\quad +\frac{2}{3}\pi_{2,2}\otimes Id_{S^2V}[e_1e_2\otimes(2e_1e_3\otimes e_2e_3 \\
&\quad +2e_2e_3\otimes e_1e_3-e_3^2\otimes e_1e_2-e_1e_2\otimes e_3^2)] \\
&= \frac{2}{3}[(e_1\wedge e_3)^2\otimes e_2^2+(e_2\wedge e_3)^2\otimes e_1^2+(e_1\wedge e_2)^2\otimes e_3^2 \\
&\quad -2(e_1\wedge e_2)(e_1\wedge e_3)\otimes e_2e_3-2(e_1\wedge e_2)(e_2\wedge e_3)\otimes e_1e_2 \\
&\quad -2(e_1\wedge e_3)(e_2\wedge e_3)\otimes e_1e_2].
\end{aligned}$$

3.2 Brill's map as a $GL(V)$-module map

Consider Brill's map $\bar{\mathfrak{B}} : S^{d+1}(S^dV) \to S_{(d,d)}V \otimes S^{d^2-d}V$. The image of Brill's map is isomorphic to dual of the $GL(V)$-module generated by Brill's equations. Therefore to prove Theorem 1.2.1, we only need to prove the following theorem:

Theorem 3.2.1. *Assume* dim $V \geq 3$. *Consider Brill's map*

$$\bar{\mathfrak{B}} : S^{d+1}(S^dV) \to S_{(d,d)}V \otimes S^{d^2-d}V.$$

Then

$$\mathrm{Im}(\overline{\mathfrak{B}}) = \begin{cases} S_{(7,3,2)}V & d = 3; \\ \bigoplus_{j=2}^{d} S_{(d^2-j,d,j)}V & d \neq 3. \end{cases}$$

Brill's map is a $GL(V)$-module map, therefore by Schur's lemma, the image of Brill's map is a $GL(V)$-submodule of $S_{(d,d)}V \otimes S^{d^2-d}V$. However since we do not know the general decomposition of $S^{d+1}(S^dV)$, it is impossible to compute the image of each isotypic component of $S^{d+1}(S^dV)$ directly. Fortunately, it is easy to decompose the space $S_{(d,d)}V \otimes S^{d^2-d}V$ by Pieri's rule, i.e.

$$S_{(d,d)}V \otimes S^{d^2-d}V = \bigoplus_{j=0}^{d} S_{(d^2-j,d,j)}V \tag{3.2}$$

Each isotypic component $S_{(d,d)}V \otimes S^{d^2-d}V$ is of multiplicity 1, so the image of Brill's map is multiplicity free. Also, we only need to consider the modules with length no more than 3, so we only need to consider V to be 3-dimensional from now on.

3.3 Weight spaces and weight vectors of $S_{(d,d)}V \otimes S^{d^2-d}V$ and $S^{d+1}(S^dV)$

Let $\{e_1, e_2, e_3\}$ be a basis of V.

Lemma 3.3.1. *As a GL_3-module , $S^d(\wedge^2\mathbb{C}^3)$ is $S_{(d,d)}\mathbb{C}^3$.*

Proof. First, since $(e_1 \wedge e_2)^d \in S^d(\wedge^2\mathbb{C}^3)$ is a highest weight vector with weight (d, d), so $S_{(d,d)}\mathbb{C}^3 \subset S^d(\wedge^2\mathbb{C}^3)$. Second, dim $S_{d,d}\mathbb{C}^3 =$ dim $S^d(\wedge^2\mathbb{C}^3) = \binom{d+2}{2}$. The result follows. □

Definition 3.3.2. *Given an integer j such that $j \in \{0, \ldots, d\}$. Define the weight space $W_j \subset S^{d+1}(S^dV)$ to be the set of all the degree $d+1$ homogenous polynomials on S^dV^* such that each monomial has weight $(d^2 - j, d, j)$.*
Define the weight space $\widetilde{W_j} \subset S_{(d,d)}V \otimes S^{d^2-d}V = S^d(\wedge^2V) \otimes S^{d^2-d}V$ to be the set of all the weight vectors in $S^d(\wedge^2V) \otimes S^{d^2-d}V$ whose weights are $(d^2 - j, d, j)$.

Lemma 3.3.3. *The weight space $\widetilde{W_j} \subset S_{(d,d)}V \otimes S^{d^2-d}V = S^d(\wedge^2V) \otimes S^{d^2-d}V$ has indeed basis*

$$\{(e_1 \wedge e_2)^{d+s-j-t}(e_1 \wedge e_3)^t(e_2 \wedge e_3)^{j-s} \otimes e_1^{d^2-d-s}e_2^te_3^{s-t}\}_{0\leq s\leq j, 0\leq t\leq s}.$$

Proof. $S^d(\wedge^2V) \otimes S^{d^2-d}V$ has a indeed basis

$$\{(e_1 \wedge e_2)^{d-a_1-a_2}(e_1 \wedge e_3)^{a_1}(e_2 \wedge e_3)^{a_2} \otimes e_1^{d^2-d-a_3-a_4}e_2^{a_3}e_3^{a_4}\}_{0\leq a_1+a_2\leq d, 0\leq a_3+a_4\leq d^2-d}.$$

Let $v \in W_j$ be a basis vector of $S^d(\wedge^2V) \otimes S^{d^2-d}V$. Then

$$\begin{cases} a_1 + a_2 + a_4 = 0, \\ a_1 - a_3 = 0. \end{cases} \tag{3.3}$$

Let $a_3 = t, a_3 + a_4 = s$, then $0 \leq s \leq j, 0 \leq t \leq s$ and $v = (e_1 \wedge e_2)^{d+s-j-t}(e_1 \wedge e_3)^t(e_2 \wedge e_3)^{j-s} \otimes e_1^{d^2-d-s} e_2^t e_3^{s-t}$. $\square$

Lemma 3.3.4. *The highest weight vector* $\tilde{v_j} \in S_{(d^2-j,d,j)}V \subset S_{(d,d)}V \otimes S^{d^2-d}V = S^d(\wedge^2 V) \otimes S^{d^2-d}V$ *is*

$$\sum_{s=0}^{j}\sum_{t=0}^{s}(-1)^t \binom{j}{s}\binom{s}{t}(e_1 \wedge e_2)^{d+s-j-t}(e_1 \wedge e_3)^t(e_2 \wedge e_3)^{j-s} \otimes e_1^{d^2-d-s} e_2^t e_3^{s-t}. \quad (3.4)$$

Proof. By Lemma 3.3.3, write

$$\tilde{v_j} = \sum_{s=0}^{j}\sum_{t=0}^{s} a_{st}(e_1 \wedge e_2)^{d+s-j-t}(e_1 \wedge e_3)^t(e_1 \wedge e_2)^{j-s} \otimes e_1^{d^2-d-s} e_2^t e_3^{s-t}.$$

Apply raising operators E_2^1 and E_3^2 on $\tilde{v_j}$,

$$\begin{aligned} E_2^1\tilde{v_j} &= \sum_{s=0}^{j}\sum_{t=0}^{s} a_{st}(j-s)(e_1 \wedge e_2)^{d+s-j-t}(e_1 \wedge e_3)^{t+1}(e_1 \wedge e_2)^{j-s-1} \otimes e_1^{d^2-d-s} e_2^t e_3^{s-t}. \\ &\quad + \sum_{s=0}^{j}\sum_{t=0}^{s} t a_{st}(e_1 \wedge e_2)^{d+s-j-t}(e_1 \wedge e_3)^t(e_2 \wedge e_3)^{j-s-1} \otimes e_1^{d^2-d-s+1} e_2^{t-1} e_3^{s-t} \\ &= \sum_{s=0}^{j-1}\sum_{t=1}^{s+1}(t a_{s+1,t} + (j-s)a_{s,t-1})(e_1 \wedge e_2)^{d+s-j-t}(e_1 \wedge e_3)^t(e_2 \wedge e_3)^{j-s-1} \\ &\quad \otimes e_1^{d^2-d-s} e_2^{t-1} e_3^{s-t}. \end{aligned}$$

and

$$
\begin{aligned}
E_3^2\tilde{v}_j &= \sum_{s=0}^{j}\sum_{t=0}^{s} ta_{st}(e_1\wedge e_2)^{d+s-j-t+1}(e_1\wedge e_3)^{t-1}(e_1\wedge e_2)^{j-s}\otimes e_1^{d^2-d-s}e_2^t e_3^{s-t} \\
&\quad +\sum_{s=0}^{j}\sum_{t=0}^{s}(s-t)a_{st}(e_1\wedge e_2)^{d+s-j-t}(e_1\wedge e_3)^{t}(e_2\wedge e_3)^{j-s}\otimes e_1^{d^2-d-s}e_2^{t-+}e_3^{s-t-1} \\
&= \sum_{s=1}^{j}\sum_{t=1}^{s}(ta_{s,t}+(s-t+1)a_{s,t-1})(e_1\wedge e_2)^{d+s-j-t+1}(e_1\wedge e_3)^{t-1}(e_2\wedge e_3)^{j-s} \\
&\quad \otimes e_1^{d^2-d-s}e_2^t e_3^{s-t}.
\end{aligned}
$$

we get two systems of equations for $\{a_{st}\}_{0\le s\le j, 0\le t\le s}$:

$$
\begin{cases} ta_{s+1,t}+(j-s)a_{s,t-1}=0, \\ ta_{s,t}+(s-t+1)a_{s,t-1}=0. \end{cases} \tag{3.5}
$$

And then solve for $\{a_{st}\}_{0\le s\le j, 0\le t\le s}$, we get a unique solution $a_{s,t}=(-1)^t\binom{j}{s}\binom{s}{t}$ up to scale. $\square$

Since Brill's map is a $GL(V)$-module map, we only need to check whether $\tilde{v}_j$ is in the image of Brill's map.

For convenience, write

$$
S^{d+1}(S^dV)=A_d\bigoplus(\bigoplus_{j=0}^{d}S_{(d^2-j,d,j)}V^{\oplus m_j}). \tag{3.6}
$$

Where A_d is the direct sum of the isotypic components of $S^{d+1}(S^dV)$ other than $S_{(d^2-j,d,j)}V$ for $j=0,1,\dots,d$, which is certainly in the kernel of Brill's map.

The idea is to take $v_j=(e_1^{d-1}e_2)^d(e_1^{d-j}e_3{}^j)\in W_j$, compute $\bar{\mathfrak{B}}(v_j)$, and see whether the projection of $\bar{\mathfrak{B}}(v_j)$ to $S_{(d^2-j,d,j)}V\subset S_{(d,d)}V\otimes S^{d^2-d}V$ is 0.

Proposition 3.3.5. *If the projection of* $\bar{\mathfrak{B}}(v_j)$ *to* $S_{(d^2-j,d,j)}V \subset S_{(d,d)}V \otimes S^{d^2-d}V$ *is not 0, then* $\tilde{v_j}$ *is in the image of Brill's map, therefore* $S_{(d^2-j,d,j)}V \subset S_{(d,d)}V \otimes S^{d^2-d}V$ *is in the image of Brill's map.*

Proof. Write $v_j = v_{j1} + v_{j2} + v_{j3}$, where $v_{j1} \in A_d$, $v_{j2} \in \bigoplus_{k=0}^{j-1} S_{(d^2-k,d,k)}V^{\oplus m_k}$, and $v_{j3} \in S_{(d^2-j,d,j)}V^{\oplus m_j}$ is a highest weight vector. By Schur's Lemma, $\bar{\mathfrak{B}}(v_{j1}) = 0$, $\bar{\mathfrak{B}}(v_{j2}) \in \bigoplus_{k=0}^{j-1} S_{(d^2-k,d,k)}V$, and $\bar{\mathfrak{B}}(v_{j3}) \in S_{(d^2-j,d,j)}V$, therefore the projection of $\bar{\mathfrak{B}}(v_j)$ to $S_{(d^2-j,d,j)}V$
$\subset S_{(d,d)}V \otimes S^{d^2-d}V$ is exactly $\bar{\mathfrak{B}}(v_{j3})$, by Schur's Lemma, if it is not 0, it is $\tilde{v_j}$ (see Lemma 3.3.4) up to scale. □

3.4 Computing $\bar{\mathfrak{B}}(v_j)$

Brill's map is very complicated to compute in general. Fortunately, we are able to compute $\bar{\mathfrak{B}}(v_j)$.

$$\begin{aligned}
\bar{\mathfrak{B}}(v_j) &= \bar{\mathfrak{B}}((e_1^{d-1}e_2)^d \cdot (e_1^{d-j}e_3{}^j)) \\
&= \frac{1}{d+1}\pi_{d,d} \otimes Id_{S^{d^2-d}V}((e_1^{d-j}e_3{}^j) \otimes \overline{Q_d}((e_1^{d-1}e_2)^d) \\
&\quad + \frac{d}{d+1}\pi_{d,d} \otimes Id_{S^{d^2-d}V}((e_1^{d-1}e_2) \otimes \overline{Q_d}((e_1^{d-1}e_2)^{d-1} \cdot (e_1^{d-j}e_3{}^j)) \\
&= \frac{1}{d+1}\pi_{d,d} \otimes Id_{S^{d^2-d}V}((e_1^{d-j}e_3{}^j) \otimes Q_d(e_1^{d-1}e_2)) \\
&\quad + \frac{d}{d+1}\pi_{d,d} \otimes Id_{S^{d^2-d}V}((e_1^{d-1}e_2) \otimes \overline{Q_d}((e_1^{d-1}e_2)^{d-1} \cdot (e_1^{d-j}e_3{}^j)))
\end{aligned}$$

First, I compute and $\pi_{d,d} \otimes Id_{S^{d^2-d}V}((e_1^{d-j}e_3{}^j) \otimes Q_d(e_1^{d-1}e_2))$. By Lemma 2.4.4,

Proposition 3.4.1. $\pi_{d,d} \otimes Id_{S^{d^2-d}V}((e_1^{d-j}e_3{}^j) \otimes Q_d(e_1^{d-1}e_2))$ *is*

$$\begin{cases} d!(e_1 \wedge e_2)^{d-j}(e_3 \wedge e_2)^j \otimes e_1^{d^2-d} & j \neq d, \\ d!(e_3 \wedge e_2)^d \otimes (e_1^{d^2-d}) + (d-1)d! \otimes (e_3 \wedge e_1)^d \otimes e_1^{d^2-2d}e_2^d & j = d. \end{cases} \tag{3.7}$$

Next, I compute $\pi_{d,d} \otimes Id_{S^{d^2-d}V}((e_1^{d-1}e_2) \otimes \overline{Q_d}((e_1^{d-1}e_2)^{d-1}(e_1^{d-j}e_3{}^j)))$.

Lemma 3.4.2. *If $h \in S^dV$ is divisible by e_1^2, then $\pi_{d,d}(h, e_1^{d-1}e_2) = 0$.*

Lemma 3.4.3. *For any $f, g \in S^dV$, by polarizing (2.7),*

$$\begin{aligned}\overline{Q_d}(f^{d-1}g) &= (-1)^d \sum_{i_1+2i_2+\cdots+di_d=d} d(-1)^{i_1+i_2+\cdots+i_d}\frac{(i_1+i_2+\cdots+i_d-1)!}{i_1!\cdots i_d!} \\ &\quad (\sum_{s=1}^{d}\frac{i_s}{d}(\prod_{j=1,j\neq s}^{d} f_{j,d-j}^{i_j})\cdot(f_{s,d-s}^{i_s-1}g_{s,d-s})\cdot(1\otimes f^{d-(i_1+\cdots+i_d)})) \\ &\quad +(\frac{d-(i_1+\cdots+i_d)}{d}(\prod_{j=1}^{d} f_{j,d-j}^{i_j})\cdot(1\otimes f^{d-(i_1+\cdots+i_d)-1}g)).\end{aligned}$$

Now I use Lemma 3.4.3 to compute $\overline{Q_d}((e_1^{d-1}e_2)^{d-1}\cdot(e_1^{d-j}e_3{}^j))$. By lemma 3.4.2, terms of $\overline{Q_d}((e_1^{d-1}e_2)^{d-1}\cdot(e_1^{d-j}e_3{}^j))$ whose first components are divisible by e_1^2 are killed by $(e_1^{d-1}e_2)$ via $\pi_{d,d}$. Therefore, by Lemma 3.4.3, given $i_1,\ldots,i_d$ with $i_1+2i_2+\cdots+di_d=d$, we need $\#\{j\geq 3 | i_j\geq 1\}\leq 1$ so that the corresponding terms will not vanish. There are 2 possibilities, either some $i_s=1$ for some $s\geq 3$ or $i_s\equiv 0$ for all $s\geq 3$. More specifically, there are five cases for which $\overline{Q_d}((e_1^{d-1}e_2)^{d-1}(e_1^{d-j}e_3{}^j))$ may not vanish:

1. $i_s=1$ for some $s\geq 3$ and $i_2=0, i_1=d-s$;

2. $i_s=1$ for some $s\geq 3$ and $i_2=1, i_1=d-s-2$;

3. $i_s\equiv 0$ for all $s\geq 3$ and $i_2=0, i_1=d$;

4. $i_s\equiv 0$ for all $s\geq 3$ and $i_2=1, i_1=d-2$;

5. $i_s\equiv 0$ for all $s\geq 3$ and $i_2=2, i_1=d-4$.

I use the symbol $\equiv$ to omit those terms of $\overline{Q_d}((e_1^{d-1}e_2)^{d-1}\cdot(e_1^{d-j}e_3{}^j))$ whose first components are divisible by e_1^2. I use the notation I_1 to denote the terms of the first case in $\overline{Q_d}((e_1^{d-1}e_2)^{d-1}(e_1^{d-j}e_3{}^j))$. For the first case, $i_s = 1$ for some $s \geq 3$ and $i_2 = 0, i_1 = d-s$, so the coefficient of the terms of the first case is

$$(-1)^d(-1)^{i_1+i_2+\cdots i_d}\frac{(i_1+i_2+\cdots i_d-1)!}{i_1!\cdots i_d!} = (-1)^d d(-1)^{d-s-1}\frac{(d-s)!}{(d-s)!} = d(-1)^{s-1},$$

and the corresponding monomial in $Q_d(f)$ is

$$d(-1)^{s-1}f_{1,d-1}^{d-s}f_{s,d-s}\cdot(1\otimes f^{s-1}).$$

Since the first component of $(e_1^{d-1}e_2)_{s,d-s}$ is divisible by e_1^2, by lemma 3.4.2, in order that the terms will not be killed by $(e_1^{d-1}e_2)$ via $\pi_{d,d}$, $e_1^{d-j}e_3{}^j$ should replace $(e_1^{d-1}e_2)$ in the position $f_{s,d-s}$. By Lemma 3.4.3,

$$\begin{aligned}
I_1 &\equiv \sum_{s=3}^{d} d(-1)^{s-1}\frac{1}{d}(e_1^{d-1}e_2)_{1,d-1}^{d-s}(e_1^{d-j}e_3{}^j)_{s,d-s}\cdot[1\otimes(e_1^{d-1}e_2)^{s-1}] \\
&= \sum_{s=3}^{d}(-1)^{s-1}(e_1^{d-1}e_2)_{1,d-1}^{d-s}(e_1^{d-j}e_3{}^j)_{s,d-s}\cdot[1\otimes(e_1^{d-1}e_2)^{s-1}] \\
&\equiv \sum_{s=3}^{d}(-1)^{s-1}[(d-1)e_1\otimes e_1^{d-2}e_2+e_2\otimes e_1^{d-1}]^{d-s}[\binom{j}{s}e_3^s\otimes e_1^{d-j}e_3^{j-s} \\
&\quad +(d-j)\binom{j}{s-1}e_1e_3^{s-1}\otimes e_1^{d-j-1}e_3^{j-s+1}]\cdot[1\otimes(e_1^{d-1}e_2)^{s-1}] \\
&\equiv \sum_{s=3}^{d}(-1)^{s-1}[(d-1)(d-s)e_1e_2^{d-s-1}\otimes e_1^{d-2}e_2e_1^{(d-1)(d-s-1)}+e_2^{d-s}\otimes e_1^{(d-1)(d-s)}] \\
&\quad [\binom{j}{s}e_3^s\otimes e_1^{d-j}e_3^{j-s}+(d-j)\binom{j}{s-1}e_1e_3^{s-1}\otimes e_1^{d-j-1}e_3^{j-s+1}]\cdot[1\otimes(e_1^{d-1}e_2)^{s-1}] \\
&\equiv \sum_{s=3}^{d}(-1)^{s-1}[\binom{j}{s}e_2^{d-s}e_3^s\otimes e_1^{d^2-j-d+1}e_2^{s-1}e_3^{j-s} \\
&\quad +(d-j)\binom{j}{s-1}e_1e_2^{d-s}e_3^{s-1}\otimes e_1^{d^2-j-d}e_2^se_3^{j-s+1} \\
&\quad +(d-1)(d-s)\binom{j}{s}e_1e_2^{d-s-1}e_3^s\otimes e_1^{d^2-j-d}e_2^se_3^{j-s}].
\end{aligned}$$

Similarly for the other four cases,

$$
\begin{aligned}
I_2 &\equiv \sum_{s=3}^{d} d(-1)^s(d-s-1)\frac{1}{d}(e_1^{d-1}e_2)_{1,d-1}^{d-s-2}(e_1^{d-1}e_2)_{2,d-2}(e_1^{d-j}e_3{}^j)_{s,d-s}\cdot[1\otimes(e_1^{d-1}e_2)^s] \\
&\equiv \sum_{s=3}^{d}(-1)^s(d-s-1)[(d-1)e_1\otimes e_1^{d-2}e_2+e_2\otimes e_1^{d-1}]^{d-s-2}((d-1)e_1e_2\otimes e_1^{d-2}) \\
&\quad [\binom{j}{s}e_3^s\otimes e_1^{d-j}e_3^{j-s}+(d-j)\binom{j}{s-1}e_1e_3^{s-1}\otimes e_1^{d-j-1}e_3^{j-s+1}]\cdot[1\otimes(e_1^{d-1}e_2)^{s-1}] \\
&\equiv \sum_{s=3}^{d}(-1)^s(d-s-1)(d-1)\binom{j}{s}e_1e_2^{d-s-1}e_3^s\otimes e_1^{d^2-d-j}e_2^se_3^{j-s}.
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\equiv \frac{d(d-3)}{2}\frac{2}{d}(e_1^{d-1}e_2)_{1,d-1}^{d-4}(e_1^{d-1}e_2)_{2,d-2}(e_1^{d-j}e_3{}^j)_{2,d-2}\cdot[1\otimes(e_1^{d-1}e_2)^2] \\
&\equiv (d-3)[(d-1)e_1\otimes e_1^{d-2}e_2+e_2\otimes e_1^{d-1}]^{d-4}[(d-1)e_1e_2\otimes e_1^{d-2}] \\
&\quad [\binom{j}{2}e_3^2\otimes e_1^{d-j}e_3^{j-2}]\cdot[1\otimes(e_1^{d-1}e_2)^2] \\
&\equiv (d-3)(d-1)\binom{j}{s}e_1e_2^{d-3}e_3^2\otimes e_1^{d^2-d-j}e_2^2e_3^{j-2}.
\end{aligned}
$$

$$
\begin{aligned}
I_4 \equiv\ & (2-d)(e_1^{d-1}e_2)^{d-3}_{1,d-1}(e_1^{d-j}e_3{}^j)_{1,d-1}(e_1^{d-1}e_2)_{2,d-2}\cdot[1\otimes(e_1^{d-1}e_2)] \\
& -(e_1^{d-1}e_2)^{d-2}_{1,d-1}(e_1^{d-j}e_3{}^j)_{2,d-2}\cdot(1\otimes(e_1^{d-1}e_2)) \\
& -(e_1^{d-1}e_2)^{d-2}_{1,d-1}(e_1^{d-1}e_2)_{2,d-2}\cdot[1\otimes(e_1^{d-j}e_3{}^j)] \\
\equiv\ & -(d-2)[(d-1)e_1\otimes e_1^{d-2}e_2+e_2\otimes e_1^{d-1}]^{d-3}(je_3\otimes e_1^{d-j}e_3^{j-1}) \\
& [(d-1)e_1e_2\otimes e_1^{d-2}]\cdot[1\otimes(e_1^{d-1}e_2)] \\
& -[(d-1)e_1\otimes e_1^{d-2}e_2+e_2\otimes e_1^{d-1}]^{d-2} \\
& [(d-j)je_1e_3\otimes e_1^{d-j-1}e_3^{j-1}+\binom{j}{2}e_3^2\otimes e_1^{d-j}e_3^{j-2}]\cdot[1\otimes(e_1^{d-1}e_2)] \\
& -[(d-1)e_1\otimes e_1^{d-2}e_2+e_2\otimes e_1^{d-1}]^{d-2}[(d-1)e_1e_2\otimes e_1^{d-2}]\cdot[1\otimes(e_1^{d-j}e_3{}^j)] \\
\equiv\ & -(d-2)(e_2^{d-3}\otimes e_1^{(d-1)(d-3)})(je_3\otimes e_1^{d-j}e_3^{j-1})((d-1)e_1e_2\otimes e_1^{d-2})\cdot(1\otimes(e_1^{d-1}e_2)) \\
& -[e_2^{d-2}\otimes e_1^{(d-1)(d-2)}+(d-1)(d-2)e_1e_2^{d-2}\otimes e_1^{d^2-3d+1}e_2] \\
& [(d-j)je_1e_3\otimes e_1^{d-j-1}e_3^{j-1}+\binom{j}{2}e_3^2\otimes e_1^{d-j}e_3^{j-2}]\cdot[1\otimes(e_1^{d-1}e_2)] \\
& -(e_2^{d-2}\otimes e_1^{(d-1)(d-2)})[(d-1)e_1e_2\otimes e_1^{d-2}]\cdot(1\otimes[e_1^{d-j}e_3{}^j)] \\
\equiv\ & [-(d-2)(d-1)j-j(d-j)]e_1e_2^{d-2}e_3\otimes e_1^{d^2-d-j}e_2e_3^{j-1} \\
& -\binom{j}{2}e_2^{d-2}e_3^2\otimes e_1^{d^2-j-d+1}e_2e_3^{j-2}-(d-2)(d-1)\binom{j}{2}e_1e_2^{d-3}e_3^2\otimes e_1^{d^2-j-d}e_2^2e_3^{j-2} \\
& -(d-1)e_1e_2^{d-1}\otimes e_1^{d^2-j-d}e_3^j.
\end{aligned}
$$

$$
\begin{aligned}
I_5 &\equiv (e_1^{d-1}e_2)^{d-1}_{1,d-1}(e_1^{d-j}e_3{}^j)_{1,d-1} \\
&\equiv [(d-1)e_1\otimes e_1^{d-2}e_2+e_2\otimes e_1^{d-1})]^{d-1}[je_3\otimes e_1^{d-j}e_3^{j-1}+(d-j)e_1\otimes e_3^{j}e_1^{d}] \\
&\equiv [e_2^{d-1}\otimes e_1^{(d-1)(d-1)}+(d-1)^2e_1^{d-2}e_2\otimes e_1^{(d-2)^2+d-1}e_2^{d-2}] \\
&\quad [je_3\otimes e_1^{d-j}e_3^{j-1}+(d-j)e_1\otimes e_3^{j}e_1^{d}] \\
&\equiv je_2^{d-1}e_3\otimes e_1^{d^2-j}e_2e_3^{j-1}+(d-j)e_1e_2^{d-1}\otimes e_1^{d^2-j-d}e_3^{j} \\
&\quad +j(d-1)^2e_1e_2^{d-2}e_3\otimes e_1^{d^2-j-d}e_2e_3^{j-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\overline{Q_d}((e_1^{d-1}e_2)^{d-1}(e_1^{d-j}e_3{}^j)) \quad &\equiv \quad \sum_{s=0}^{min\{j,d-1\}}(-1)^s\binom{j}{s}(1-j)e_1e_2^{d-s-1}e_3^{s}\otimes e_1^{d^2-d-j}e_2^{s}e_3^{j-s} \\
&\quad +\sum_{s=1}^{j}(-1)^{s-1}\binom{j}{s}e_2^{d-s}e_3^{s}\otimes e_1^{d^2-d-j+1}e_2^{s-1}e_3^{j-s}.
\end{aligned}
$$

This implies

Proposition 3.4.4.

$$
\begin{aligned}
&\pi_{d,d}\otimes Id_{S^{d^2-d}V}((e_1^{d-1}e_2)\otimes\overline{Q_d}((e_1^{d-1}e_2)^{d-1}\cdot(e_1^{d-j}e_3{}^j))= \\
&\sum_{s=0}^{min\{j,d-1\}}(-1)^{s-1}\binom{j}{s}(1-j)(d-1)!(e_1\wedge e_2)^{d-s}(e_1\wedge e_3)^s\otimes e_1^{d^2-d-j}e_2^{s}e_3^{j-s} \\
&+\sum_{s=1}^{j}(-1)^{s-1}\binom{j}{s}s(d-1)!(e_1\wedge e_2)^{d-s}(e_1\wedge e_3)^{s-1}(e_2\wedge e_3)\otimes e_1^{d^2-d-j+1}e_2^{s-1}e_3^{j-s}.
\end{aligned}
$$

Proposition 3.4.1 and Proposition 3.4.4 imply:

Proposition 3.4.5.

$$\bar{\mathfrak{B}}((e_1^{d-1}e_2)^d \cdot (e_1^{d-j}e_3{}^j)) = \frac{d!}{d+1}(e_1 \wedge e_2)^{d-j}(e_3 \wedge e_2)^j \otimes (e_1^{d^2-d})$$
$$+\frac{d!}{d+1}\sum_{s=0}^{j}(-1)^{s-1}\binom{j}{s}(1-j)(e_1 \wedge e_2)^{d-s}(e_1 \wedge e_3)^s \otimes e_1^{d^2-d-j}e_2^s e_3^{j-s}$$
$$+\frac{d!}{d+1}\sum_{s=1}^{j}(-1)^{s-1}\binom{j}{s}s(e_1 \wedge e_2)^{d-s}(e_1 \wedge e_3)^{s-1}(e_2 \wedge e_3) \otimes e_1^{d^2-d-j+1}e_2^{s-1}e_3^{j-s}.$$

3.5 Orthogonal decomposition of $S_{(d,d)V} \otimes S^{d^2-d}V$

Let e_1, e_2, e_3 be a basis of V and define a Hermitian inner product on V such that

$$< e_i, e_j >= \delta_{i,j}.$$

Extend the Hermitian inner product to $V^{\otimes(d^2+d)}$ naturally by

$$< e_{i_1} \otimes \cdots \otimes e_{i_{d^2+d}}, e_{j_1} \otimes \cdots \otimes e_{j_{d^2+d}} >= \delta_{i_1,j_1} \cdots \delta_{i_{d^2+d},j_{d^2+d}}.$$

One can decompose $V^{\otimes(d^2+d)}$ into direct sum of isotypic components as a $GL(V)$-module. Since the Hermitian inner product is unitary invariant, distinct isotypic components of $V^{\otimes(d^2+d)}$ are orthogonal (see e.g. [16]).

Consider $S_{(d,d)}V \otimes S^{d^2-d}V = S^d(\wedge^2 V) \otimes S^{d^2-d}V$ as a subspace of $V^{\otimes(d^2+d)}$, the decomposition $S_{(d,d)}V \otimes S^{d^2-d}V = \bigoplus_{j=0}^{d} S_{(d^2-j,d,j)}V$ is an orthogonal decomposition with respect to the Hermitian inner product, therefore

Proposition 3.5.1. *The projection of $\bar{\mathfrak{B}}(v_j)$ on $S_{(d^2-j,d,j)}V \subset S_{(d,d)}V \otimes S^{d^2-d}V$ is not 0 if and only if $< B(v_j), \tilde{v_j} >\neq 0$, where $\tilde{v_j}$ is defined in Lemma 3.3.4.*

Lemma 3.5.2. *If $a_1 + a_2 + a_3 = d$ and $b_1 + b_2 + b_3 = d^2 - d$, then*

$$< (e_1 \wedge e_2)^{a_1} (e_1 \wedge e_3)^{a_2} (e_2 \wedge e_3)^{a_3} \otimes e_1^{b_1} e_2^{b_2} e_3^{b_3}, (e_1 \wedge e_2)^{a_1} (e_1 \wedge e_3)^{a_2} (e_2 \wedge e_3)^{a_3}$$
$$\otimes e_1^{b_1} e_2^{b_2} e_3^{b_3} >= (\frac{1}{2})^d \frac{a_1! a_2! a_3!}{d!} \frac{b_1! b_2! b_3!}{(d^2 - d)!}$$

Recall by Lemma 3.3.4,

$$\tilde{v_j} = \sum_{s=0}^{j} \sum_{t=0}^{s} (-1)^t \binom{j}{s} \binom{s}{t} (e_1 \wedge e_2)^{d+s-j-t} (e_1 \wedge e_3)^t (e_1 \wedge e_2)^{j-s} \otimes e_1^{d^2-d-s} e_2^t e_3^{s-t}.$$

and by Proposition 3.4.5,

$$\bar{\mathfrak{B}}((e_1^{d-1} e_2)^d \cdot (e_1^{d-j} e_3{}^j)) = \frac{d!}{d+1} (-1)^j (e_1 \wedge e_2)^{d-j} (e_2 \wedge e_3)^j \otimes (e_1^{d^2-d})$$
$$+ \frac{d!}{d+1} \sum_{t=0}^{j} (-1)^t \binom{j}{t} (j-1) (e_1 \wedge e_2)^{d-t} (e_1 \wedge e_3)^t \otimes e_1^{d^2-d-j} e_2^t e_3^{j-t}$$
$$+ \frac{d!}{d+1} \sum_{t=1}^{j} (-1)^{t-1} \binom{j}{t} t (e_1 \wedge e_2)^{d-t} (e_1 \wedge e_3)^{t-1} (e_2 \wedge e_3) \otimes e_1^{d^2-d-j+1} e_2^{t-1} e_3^{j-t}.$$

By Lemma 3.5.2,

Proposition 3.5.3. *For any fixed $j \in \{0, 1, \ldots, d\}$,*

$$\begin{aligned} < B(v_j), \tilde{v_j} > &= \frac{(\frac{1}{2})^d}{(d+1)(d^2-d)!} (\sum_{t=0}^{j} \frac{(j!)^2 (j-1)(d-t)!(d^2-d-j)!}{(j-t)!} \\ &+ \sum_{t=0}^{j-1} \frac{(j!)^2 (d-t-1)!(d^2-d-j+1)!}{(j-t-1)!} + (-1)^j (d-j)! j! (d^2-d)!). \end{aligned}$$

$< B(v_j), \tilde{v_j} >= 0$ only when

1. *$j = 0, 1$ for all $d \geq 2$;*

2. *$j = 3$ and $d = 3$.*

Proof. The ratio of $(d-j)!j!(d^2-d)!$ and $\frac{(j!)^2(j-1)(d-t)!(d^2-d-j)!}{(j-t)!}$ is $\frac{\binom{d^2-d}{j}}{(j-1)\binom{d-t}{j-t}}$, and the ratio of $(d-j)!j!(d^2-d)!$ and $\frac{(j!)^2(d-t-1)!(d^2-d-j+1)!}{(j-t-1)!}$ is $\frac{\binom{d^2-d}{j-1}}{j\binom{d-t-1}{j-t-1}}$. Therefore when d is large enough and $j \geq 2$, the term $(-1)^j(d-j)!j!(d^2-d)!$ dominates. For small cases, one can check directly. $\square$

Combining all the results above, we prove Theorem 3.2.1 to prove Theorem 1.2.1.

Proof of Theorem 3.2.1. First, for $j=0$ and $d \geq 2$, $S_{(d^2,d)}V$ is not in the image of Brill's map because $Ch_d(\mathbb{C}^2) = S^d\mathbb{C}^2$.

Second, for $j=1$ and all $d \geq 2$, $S_{(d^2-1,d,1)}V$ is not in the image of Brill's map. If it were in the image of Brill's map, then $\mathfrak{B}(v_1) = \mathfrak{B}(v_{13}) = C\tilde{v_1} \neq 0$ (where v_{13} is defined in the proof of Proposition 3.3.5), so $< B(v_1), \tilde{v_1} >=< C\tilde{v_1}, \tilde{v_1} >\neq 0$, contradiction.

Third, when $d=3$ and $j=3$, the module $S_{(6,3,3)}V$ is not in the decomposition of $S^4(S^3V)$, so it is not in the image.

Finally, for other cases, $< B(v_j), \tilde{v_j} >\neq 0$, by Proposition 3.3.5 and Proposition 3.5.1, $S_{(d^2-j,d,j)}V$ is in the image of Brill's map. $\square$

4. DETERMINANTAL EQUATIONS FOR VARIETIES ARISING IN COMPLEXITY THEORY

This chapter is based on [20] and consists of two sections. In the first section, I compute Koszul Young Flattenings of Chow varieties and their secant varieties and obtain equations for these varieties, which enables a new lower bound for symmetric border rank of $x_1x_2\cdots x_d$ when d is odd. In the second section, I compute flattenings of Veronese reembeddings of secant varieties of Veronese varieties and obtain a new complexity lower bound for the permanent.

4.1 Koszul Young Flattenings of Chow varieties and their secant varieties

4.1.1 Disjoint linear maps

Definition 4.1.1. *Let V and W to be two finite dimensional complex vector spaces and let $f : V \to W$ be a linear map. Let $V_1, V_2, \dots V_m$ be subspaces of V such that $V = \bigoplus_{i=1}^m V_i$. The map f is called a disjoint map with respect to the decomposition $V = \bigoplus_{i=1}^m V_i$ if $f(V) = \bigoplus_{i=1}^m f(V_i)$.*

Note that if $f : V \to W$ is a disjoint linear map with respect to the decomposition $V = \bigoplus_{i=1}^m V_i$, then $\mathrm{rank}(f) = \sum_{i=1}^m \mathrm{rank}(f|_{V_i})$.

4.1.2 Koszul Young flattenings of Chow varieties

Let $V = \mathbb{C}^d$ with basis $\{x_1, \dots, x_d\}$. Let $P = x_1 \cdots x_d$ and write $[d] = \{1, \dots, d\}$. The following proposition is standard:

Proposition 4.1.2. *Let $k \le \lfloor \frac{d}{2} \rfloor$, then the image of*

$$P_{k,d-k} : S^kV^* \to S^{d-k}V$$

is $(S^{d-k}V)\mathrm{reg} := \mathrm{span}\{x_{i_1}x_{i_2}\cdots x_{i_{d-k}}\}_{1\le i_1<i_2<\cdots<i_{d-k}\le d}$.

By Proposition 4.1.2,

Corollary 4.1.3. *Let* $k \le \lfloor \frac{d}{2} \rfloor$, $p \le \lfloor \frac{d}{2} \rfloor$. *The image of*

$$P^{\wedge p}_{k,d-k} : S^kV^* \otimes \Lambda^pV \to S^{d-k-1}V \otimes \Lambda^{p+1}V$$

is the image of $(S^{d-k}V)_{\mathrm{reg}} \otimes \Lambda^pV$ *under the map*

$$\wedge_{d-k,p} : S^{d-k}V \otimes \Lambda^pV \to S^{d-k-1}V \otimes \Lambda^{p+1}V.$$

Since the map $\wedge_{d-k,p}|_{(S^{d-k}V)_{\mathrm{reg}}\otimes\Lambda^pV}$ preserves weights, it is helpful for us to decompose $(S^{d-k}V)_{\mathrm{reg}} \otimes \Lambda^pV$ into a direct sum of weight spaces.

Lemma 4.1.4. *Let* $W_{k_1,..,k_s;j_1,j_2,\cdots,j_{d-k+p-2s}}$ *be the span of* $\{x_{k_1}\cdots x_{k_s}x_{m_1}\cdots x_{m_{d-k-s}} \otimes x_{k_1} \wedge \cdots x_{k_s} \wedge x_{n_1} \wedge \cdots \wedge x_{n_{p-s}}\}_{\{m_1,\ldots,m_{d-k-s},n_1,\ldots,n_{p-s}\}=\{j_1,j_2,\ldots,j_{d-k+p-2s}\}}$. *Then*

$$(S^{d-k}V)_{\mathrm{reg}} \otimes \Lambda^pV = \bigoplus_{\{k_1,..,k_s\}\subset[d]} \bigoplus_{\max\{0,p-k\}\le s\le\min\{p,d-k\}} \bigoplus_{\{j_1,j_2,\ldots,j_{d-k+p-2s}\}\subset[d]-\{k_1,\ldots,k_s\}} W_{k_1,\ldots,k_s;j_1,j_2,\ldots,j_{d-k+p-2s}}. \tag{4.1}$$

Moreover $\wedge_{d-k,p}|_{(S^{d-k}V)_{\mathrm{reg}}\otimes\Lambda^pV}$ *is a disjoint map with respect to this decomposition.*

Example 4.1.5. *Let* $d = 3$ *and* $k = p = 1$, *then*

$$(S^2V)_{\mathrm{reg}} \otimes V = \bigoplus_{1\le i,j\le 3 \ i\ne j} W_{i;j} \bigoplus W_{;1,2,3}.$$

Where $W_{i;j} = \mathrm{span}\{x_ix_j \otimes x_i\}$, *and* $W_{;1,2,3} = \mathrm{span}\{x_1x_2 \otimes x_3, x_1x_3 \otimes x_2, x_2x_3 \otimes x_1\}$.

Lemma 4.1.6. *Let* $Q = \operatorname{span}\{y_1, ..., y_{u+v}\}$ *and let*

$$A_{u,v}Q = \operatorname{span}\{y_{m_1} \cdots y_{m_u} \otimes y_{n_1} \wedge \cdots \wedge y_{n_v}\}_{\{m_1,...,m_u,n_1,...,n_v\}=[u+v]}.$$

Then the rank of $\wedge_{u,v}|_{A_{u,v}Q} : A_{u,v}Q \to A_{u-1,v+1}Q$ *is* $\binom{u+v-1}{v}$*. Moreover*

$$\begin{aligned} \operatorname{rank}(\wedge_{d-k,p}|_{W_{k_1,...,k_s;j_1,j_2,...,j_{d-k+p-2s}}}) &= \operatorname{rank}(\wedge_{d-k-s,p-s}|_{A_{d-k-s,p-s}Q}) \\ &= \binom{d-k+p-2s-1}{p-s}. \end{aligned}$$

Proof. By [16] Exercise 4.6, $[u+v-1,1]$ is the standard representation of $\mathfrak{S}_{u+v}$ and $\Lambda^s[u+v-1,1] = [u+v-s,1^s]$. As a $\mathfrak{S}_{u+v}$-module, $A_{u,v}Q = \Lambda^u([u+v-1,1] + \mathbb{C}) = \Lambda^u[u+v-1,1] + \Lambda^{u-1}[u+v-1,1] = [u,1^v] \oplus [u+1,1^{v-1}]$ and $A_{u-1,v+1}Q = [u-1,1^{v+1}] \oplus [u,1^v]$, since $\wedge_{u,v}|_{A_{u,v}Q}$ is a $\mathfrak{S}_{u+v}$-module map, by Schur's lemma, $\operatorname{image}(\wedge_{u,v}|_{A_{u,v}Q}) = [u,1^v]$, with dimension $\binom{u+v-1}{v}$. Notice $\wedge_{d-k,p}|_{W_{k_1,...,k_s;j_1,j_2,...,j_{d-k+p-2s}}}$ is essentially the same map as $\wedge_{d-k-s,p-s}|_{A_{d-k-s,p-s}Q}$, so

$$\begin{aligned} \operatorname{rank}(\wedge_{d-k,p}|_{W_{k_1,..,k_s;j_1,j_2,\cdots,j_{d-k+p-2s}}}) &= \operatorname{rank}(\wedge_{d-k-s,p-s}|_{A_{d-k-s,p-s}Q}) \\ &= \binom{d-k+p-2s-1}{p-s}. \end{aligned}$$

□

Proof of Theorem 1.3.1. By Corollary 4.1.3, we only need to compute the rank of

$$\wedge_{d-k,p}|_{(S^{d-k}V)_{\text{reg}} \otimes \Lambda^p V} : (S^{d-k}V)_{\text{reg}} \otimes \Lambda^p V \to S^{d-k-1}V \otimes \Lambda^{p+1}V.$$

By Lemma 4.1.6, $\operatorname{rank}(\wedge_{d-k,p}|_{W_{k_1,...,k_s;j_1,j_2,...,j_{d-k+p-2s}}})$ depends only on s. Consider the decomposition of $(S^{d-k}V)_{\text{reg}} \otimes \Lambda^p V$ in (4.1), for any given s, the number of subspaces

$W_{k_1,\dots,k_s;j_1,j_2,\dots,j_{d-k+p-2s}}$ is $\binom{d}{s}\binom{d-s}{d-k+p-2s}$. Therefore, by Lemma 4.1.4 and Lemma 4.1.6.

$$\begin{aligned}
\mathbf{S(p,d,k)} &= \sum_{s=\max\{0,p-k\}}^{\min\{p,d-k\}} \binom{d}{s}\binom{d-s}{d-k+p-2s}\operatorname{rank}(\wedge_{d-k,p}|_{W_{k_1,\dots,k_s;j_1,j_2,\dots,j_{d-k+p-2s}}}) \\
&= \sum_{s=\max\{0,p-k\}}^{\min\{p,d-k-1\}} \binom{d}{s}\binom{d-s}{d-k+p-2s}\binom{d-k+p-2s-1}{p-s} \qquad (4.2) \\
&= \frac{d!}{p!(d-p-1)!}\sum_{s=\max\{0,p-k\}}^{\min\{p,d-k-1\}} \frac{\binom{p}{s}\binom{d-1-p}{s+k-p}}{d-k+p-2s}. \qquad (4.3)
\end{aligned}$$

□

Lemma 4.1.7. *Let $P = l^d$ for some linear form $l \in V$, then* $\operatorname{rank}(P^{\wedge p}_{k,d-k}) = \binom{d-1}{p}$.

Proof. Let $P = l^d$, the image of

$$P^{\wedge p}_{k,d-k} : S^k V^* \otimes \Lambda^p V \to S^{d-k-1} V \otimes \Lambda^{p+1} V$$

is $\operatorname{span}\{l^{d-k-1} \otimes (l \wedge x_{i_1} \wedge \cdots \wedge x_{i_p})\}$. We can assume $l = x_1$, then $\operatorname{rank}(P^{\wedge p}_{k,d-k}) = \binom{d-1}{p}$. □

Proof of Theorem 1.2.3. When $d = 2n+1$ and $k = p = n$,

$$\mathbf{S(n,2n+1,n)} = \frac{(2n+1)!}{(n!)^2}\sum_{s=0}^{n} \frac{\binom{n}{s}^2}{1+2s}. \qquad (4.4)$$

and by Lemma 4.1.7, $\underline{\mathbf{R}}_S(x_1 \cdots x_{2n}) \geq \frac{\mathbf{S(n,2n+1,n)}}{\binom{2n}{n}}$.

Let

$$\mathbf{A} = \sum_{s=0}^{n} \frac{\binom{n}{s}^2}{1+2s}.$$

Then

$$
\begin{aligned}
2\mathbf{A} &= \sum_{s=0}^{n} \frac{\binom{n}{s}^2}{1+2s} + \sum_{s=0}^{n} \frac{\binom{n}{s}^2}{1+2n-2s} \\
&= \sum_{s=0}^{n} \binom{n}{s}^2 \left(\frac{1}{1+2s} + \frac{1}{1+2n-2s}\right) \\
&= \sum_{s=0}^{n} \binom{n}{s}^2 \frac{2+2n}{(1+2s)(1+2n-2s)}.
\end{aligned}
$$

Notice that

$$(1+2s)(1+2n-2s) \le (1+n)^2.$$

So

$$
\begin{aligned}
\mathbf{A} &= \sum_{s=0}^{n} \binom{n}{s}^2 \frac{1+n}{(1+2s)(1+2n-2s)} \\
&= \sum_{s=0}^{n} \binom{n}{s}^2 \left(\frac{1+n}{(1+2s)(1+2n-2s)} - \frac{1}{1+n} + \frac{1}{1+n}\right) \\
&= \sum_{s=0}^{n} \binom{n}{s}^2 \left(\frac{n^2-4ns+4s^2}{(n+1)(1+2s)(1+2n-2s)} + \frac{1}{1+n}\right) \\
&\ge \sum_{s=0}^{n} \binom{n}{s}^2 \left(\frac{n^2-4ns+4s^2}{(n+1)^3} + \frac{1}{1+n}\right) \\
&= \sum_{s=0}^{n} \binom{n}{s}^2 \frac{n^2-4ns+4s^2}{(n+1)^3} + \binom{2n}{n} \frac{1}{1+n}.
\end{aligned}
$$

Since

$$\sum_{s=0}^{n} s\binom{n}{s}^2 = n\binom{2n-1}{n},$$

and

$$\sum_{s=0}^{n} s(s-1)\binom{n}{s}^2 = n(n-1)\binom{2n-2}{n},$$

we have

$$\sum_{s=0}^{n} s^2\binom{n}{s}^2 = n(n-1)\binom{2n-2}{n} + n\binom{2n-1}{n}.$$

This implies

$$\begin{aligned}
\mathbf{A} &\geq \sum_{s=0}^{n}\binom{n}{s}^2\frac{n^2-4ns+4s^2}{(n+1)^3} + \binom{2n}{n}\frac{1}{1+n} \\
&= \frac{n^2\binom{2n}{n} + 4n(n-1)\binom{2n-2}{n} + 4n\binom{2n-1}{n} - 4n^2\binom{2n-1}{n}}{(n+1)^3} + \binom{2n}{n}\frac{1}{1+n} \\
&= \frac{\binom{2n}{n}}{(n+1)^3}[n^2 + 4n(n-1)\frac{n(n-1)}{2n(2n-1)} + (4n-4n^2)\frac{n}{2n}] + \binom{2n}{n}\frac{1}{1+n} \\
&= \frac{\binom{2n}{n}}{(n+1)^3}(\frac{2n(n-1)^2}{2n-1} - n^2 + 2n) + \binom{2n}{n}\frac{1}{1+n} \\
&= \frac{\binom{2n}{n}}{(n+1)^3}\frac{2n(n-1)^2 - (n^2-2n)(2n-1)}{2n-1} + \binom{2n}{n}\frac{1}{1+n} \\
&= \frac{\binom{2n}{n}}{(n+1)^3}\frac{n^2}{2n-1} + \binom{2n}{n}\frac{1}{1+n}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\mathbf{S(n,2n+1,n)}}{\binom{2n}{n}\binom{2n+1}{n}} &= \frac{(2n+1)!}{(n!)^2}\frac{\mathbf{A}}{\binom{2n}{n}\binom{2n+1}{n}} \\
&\geq \frac{(2n+1)!}{(n!)^2}\frac{\frac{\binom{2n}{n}}{(n+1)^3}\frac{n^2}{2n-1}+\binom{2n}{n}\frac{1}{1+n}}{\binom{2n}{n}\binom{2n+1}{n}} \\
&= \frac{(2n+1)!}{(n!)^2}\frac{1}{\binom{2n+1}{n}}(\frac{1}{1+n}+\frac{n^2}{(n+1)^3(2n-1)}) \\
&= (n+1)(\frac{1}{1+n}+\frac{n^2}{(n+1)^3(2n-1)}) \\
&= 1+\frac{n^2}{(n+1)^2(2n-1)}.
\end{aligned}$$

Therefore $\underline{\mathbf{R}}_S(x_1\cdots x_{2n}) \geq \frac{\mathbf{S(n,2n+1,n)}}{\binom{2n}{n}} \geq \binom{2n+1}{n}(1+\frac{n^2}{(n+1)^2(2n-1)})$. □

4.1.3 Koszul Young flattenings of secant varieties of Chow varieties

I study equations for secant varieties of Chow varieties from the perspective of Koszul Young flattenings.

Proposition 4.1.8. *Let $V=\mathbb{C}^{dr}$ with a basis $\{x_1,\ldots,x_{dr}\}$. Let $P=x_1\cdots x_d+x_{d+1}\cdots x_{2d}+\cdots+x_{(r-1)d+1}\cdots x_{rd}$. For any $k\leq\lfloor\frac{d}{2}\rfloor$, the map $P_{k,d-k}:S^kV^*\to S^{d-k}V$ has rank*

$$\mathrm{rank}(P_{k,d-k})=r\binom{d}{k}.$$

Therefore the $(r\binom{d}{k}+1)\times(r\binom{d}{k}+1)$ minors of the linear map $P_{k,d-k}$ are in the ideal of $\sigma_r(Ch_d(V))$. In particular, when $k=2$ and $d\geq 4$, $r\binom{d}{2}<\binom{rd+1}{2}$, so we obtain equations for $\sigma_r(Ch_d(V))$ of degree $r\binom{d}{2}+1$.

While we can not obtain equations of $\sigma_r(Ch_3(\mathbb{C}^{3r}))$ just by usual flattenings, we can obtain equations for $\sigma_r(Ch_3\mathbb{C}^{3r})$ by Koszul Young flattenings in Theorem 1.3.2.

Proof of Theorem 1.3.2. Let $V_i = \text{span}\{x_{(i-1)d+1}, \ldots, x_{(i-1)d+d}\}$, for $i = 1, 2, \ldots, r$. Then $V = \oplus_{i=1}^r V_i$. The image of the map

$$P_{k,d-k}^{\wedge p} : S^k V^* \otimes \Lambda^p V \to S^{d-k-1} V \otimes \Lambda^{p+1} V$$

is the image of the map

$$\wedge_{d-k,p} : \bigoplus_{i=1}^{r} (S^{d-k} V_i)_{\text{reg}} \otimes \Lambda^p V \to S^{d-k-1} V \otimes \Lambda^{p+1} V.$$

Write $\Lambda^p V = \Lambda^p V_i \oplus W_i$, where W_i is the complement of $\Lambda^p V_i$ with respect to the basis $\{x_{i_1} \wedge \cdots \wedge x_{i_p}\}_{1 < i_1 < i_2 < \cdots < i_p \leq dr}$. Rewrite the map as

$$\wedge_{d-k,p} : \bigoplus_{i=1}^{r} (S^{d-k} V_i)_{\text{reg}} \otimes (\Lambda^p V_i \oplus W_i) \to S^{d-k-1} V \otimes \Lambda^{p+1} V.$$

Then

$$\begin{aligned}
\text{rank}(P_{k,d-k}^{\wedge p}) &\leq \sum_{i=1}^{r} \text{rank}(\wedge_{d-k,p}|_{(S^{d-k}V_i)_{\text{reg}} \otimes \Lambda^p V_i}) + \text{rank}(\wedge_{d-k,p}|_{(S^{d-k}V_i)_{\text{reg}} \otimes W_i}) \\
&\leq r[\mathbf{S}(\mathbf{p}, \mathbf{k}, \mathbf{d}) + \binom{d}{k}(\binom{dr}{p} - \binom{d}{p})].
\end{aligned}$$

In particular, when $d \geq 2$, and $p = k = 1$,

$$\text{rank}(P_{1,d-1}^{\wedge 1}) \leq d^2 r^2 - r.$$

Therefore the $(d^2r^2 - r + 1) \times (d^2r^2 - r + 1)$ minors of $P_{1,d-1}^{\wedge 1}$ are in the ideal of $\sigma_r(Ch_d(V))$. $\square$

4.2 Flattenings of Veronese reembeddings of secant varieties of Veronese varieties

4.2.1 Flattenings of Veronese reembeddings of secant varieties of Veronese varieties

Let $\{x_1, \ldots, x_r\}$ be a basis of V, and $\{y_1, \ldots, y_r\}$ be the dual basis of V^*. Let $P = (x_1^{\delta_2} + \cdots + x_r^{\delta_2})^{\delta_1} \in S^d V$, where $d = \delta_1\delta_2$, note that $[P]$ is a generic polynomial in $v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}^{r-1})))$. The goal is to compute the rank of its $(k, d-k)$-flattening $P_{k,d-k} : S^k V^* \to S^{d-k} V$, where $k \leq \lfloor \frac{d}{2} \rfloor$.

Definition 4.2.1. *Let $y_1^{\alpha_1} \cdots y_r^{\alpha_r} \in S^k V^*$, the support of $P_{k,d-k}(y_1^{\alpha_1} \cdots y_r^{\alpha_r})$ is the set of all monomials appearing in $P_{k,d-k}(y_1^{\alpha_1} \cdots y_r^{\alpha_r})$.*

Example 4.2.2. *Consider $\delta_1 = \delta_2 = r = k = 2$, then $P = (x_1^2 + x_2^2)^2$, $P_{2,2}(y_1^2) = 12x_1^2 + x_2^2$, the support of $P_{2,2}(y_1^2)$ is $\{x_1^2, x_2^2\}$, Similarly the support of $P_{2,2}(y_2^2)$ is $\{x_1^2, x_2^2\}$, and the support of $P_{2,2}(y_1y_2)$ is $\{x_1x_2\}$.*

Proposition 4.2.3. *Let $y_1^{\alpha_1} \cdots y_r^{\alpha_r}$ and $y_1^{\eta_1} \cdots y_r^{\eta_r} \in S^k V^*$, then $P_{k,d-k}(y_1^{\alpha_1} \cdots y_r^{\alpha_r})$ and $P_{k,d-k}(y_1^{\eta_1} \cdots y_r^{\eta_r})$ have the same support in $S^{d-k}V$ if and only if $\alpha_i - \eta_i = n_i\delta_2$ for some integers n_i and for each $i = 1, \ldots, r$.*

Proof.

$$
\begin{aligned}
P &= (x_1^{\delta_2} + \cdots + x_r^{\delta_2})^{\delta_1} \\
&= \sum_{t_1+\cdots+t_r=\delta_1} \binom{\delta_1}{t_1, \cdots, t_r} x_1^{t_1\delta_2} x_2^{t_2\delta_2} \cdots x_r^{t_r\delta_2}.
\end{aligned}
$$

Therefore

$$
P_{k,d-k}(y_1^{\alpha_1} \cdots y_r^{\alpha_r}) = \sum_{t_1+\cdots+t_r=\delta_1} C(t_1, \ldots, t_r; \alpha_1, \ldots, \alpha_r) x_1^{t_1\delta_2-\alpha_1} x_2^{t_2\delta_2-\alpha_2} \cdots x_r^{t_r\delta_2-\alpha_r}.
$$

Where $C(t_1, \ldots, t_r; \alpha_1, \ldots, \alpha_r)$ depends on $(t_1, \ldots, t_r)$ and $(\alpha_1, \ldots, \alpha_r)$.

$$P_{k,d-k}(y_1^{\eta_1} \cdots y_r^{\eta_r}) = \sum_{t_1+\cdots+t_r=\delta_1} C(t_1, \ldots, t_r; \eta_1, \ldots, \eta_r) x_1^{t_1\delta_2-\eta_1} x_2^{t_2\delta_2-\eta_2} \cdots x_r^{t_r\delta_2-\eta_r}$$

Therefore $P_{k,d-k}(y_1^{\alpha_1} \cdots y_r^{\alpha_r})$ and $P_{k,d-k}(y_1^{\eta_1} \cdots y_r^{\eta_r})$ have the same support in $S^{d-k}V$ if and only if $\alpha_i - \eta_i = n_i\delta_2$ for some integers n_i and for each $i = 1, \ldots, r$. $\square$

Definition 4.2.4. *For any $\alpha_1 + \cdots + \alpha_r = k$, define the subspace $A[\alpha_1, \ldots, \alpha_r] \subset S^kV^*$ by*

$$A[\alpha_1, \ldots, \alpha_r] =$$
$$\operatorname{span}\{y_1^{\eta_1} \cdots y_r^{\eta_r} \in S^kV^* | \alpha_i - \eta_i = n_i\delta_2 \text{ for some integer } n_i \text{ and for each } i = 1, \cdots, r\}.$$

Define $B[\alpha_1, \ldots, \alpha_r]$ to be the subspace of $S^{d-k}V$ spanned by the support of $P_{k,d-k}(y_1^{\alpha_1} \cdots y_r^{\alpha_r})$.

Remark 4.2.5. *With the notation above, S^kV^* can be decomposed as a direct sum of subspaces $A[\alpha_1, \cdots, \alpha_r]$ and by Proposition 4.2.3 $P_{k,d-k}$ is a disjoint map with respect to this decomposition.*

Definition 4.2.6. *For each subspace $A[\alpha_1, \ldots, \alpha_r]$, write $\alpha_i = \beta_i + s_i\delta_2$, where $0 \leq \beta_i < \delta_2$, easy to see each β_i and $\sum_{i=1}^r s_i$ are invariant for the basis vectors. Define two functions*

$$A(\alpha_1, \ldots, \alpha_r) = \sum_{i=1}^r s_i = \sum_{i=1}^r \lfloor \frac{\alpha_i}{\delta_2} \rfloor. \tag{4.5}$$

and

$$B(\alpha_1, \ldots, \alpha_r) = \delta_1 - \sum_{i=1}^r \lceil \frac{\alpha_i}{\delta_2} \rceil. \tag{4.6}$$

One can check $0 \leq A(\alpha_1, \cdots, \alpha_r) \leq \lfloor \frac{k}{\delta_2} \rfloor$ and

$$B(\alpha_1, \ldots, \alpha_r) = \delta_1 - A(\alpha_1, \ldots, \alpha_r) - \#\{\beta_i > 0\} \tag{4.7}$$

Proposition 4.2.7. *Consider*

$$P_{k,d-k}|_{A[\alpha_1,\ldots,\alpha_r]} : A[\alpha_1,\ldots,\alpha_r] \to B[\alpha_1,\ldots,\alpha_r],$$

then $\dim A[\alpha_1,\ldots,\alpha_r] = \binom{A(\alpha_1,\ldots,\alpha_r)+r-1}{A(\alpha_1,\ldots,\alpha_r)}$ *and* $\dim B[\alpha_1,\ldots,\alpha_r] = \binom{B(\alpha_1,\ldots,\alpha_r)+r-1}{B(\alpha_1,\ldots,\alpha_r)}$.

Proof. Write $\alpha_i = \beta_i + s_i\delta_2$, where $0 \leq \beta_i < \delta_2$. Then

$$\begin{aligned}
&A[\alpha_1,\ldots,\alpha_r] = \text{span}\{y_1^{\eta_1}\cdots y_r^{\eta_r} \in S^kV^* | \eta_i = n_i\delta_2 + \beta_i \\
&\text{for some nonegetive integer } n_i \text{ such that } n_1 + \cdots + n_r = \sum_{i=1}^r s_i = A(\alpha_1,\ldots,\alpha_r)\}.
\end{aligned}$$

Therefore $\dim A[\alpha_1,\ldots,\alpha_r] = \binom{A(\alpha_1,\ldots,\alpha_r)+r-1}{A(\alpha_1,\ldots,\alpha_r)}$.

Similarly let

$$\theta_i = \begin{cases} \delta_2 - \beta_i & \beta_i \neq 0 \\ 0 & \beta_i = 0. \end{cases}$$

Let $x_1^{\zeta_1}\cdots x_r^{\zeta_r} \in B[\alpha_1,\ldots,\alpha_r]$, then $\zeta_i = m_i\delta_2 + \theta_i$ for some nonnegative integer m_i,

and

$$\begin{aligned}
\sum_{i=1}^{r}(m_i\delta_2+\theta_i) &= \delta_1\delta_2 - k \\
&= \delta_1\delta_2 - \sum_{i=1}^{r}\alpha_i \\
&= \delta_1\delta_2 - \sum_{i=1}^{r}(\beta_i + s_i\delta_2) \\
&= \delta_1\delta_2 - A(\alpha_1,\ldots,\alpha_r)\delta_2 - \sum_{i=1}^{r}\beta_i \\
&= (\delta_1 - A(\alpha_1,\ldots,\alpha_r) - \#\{\beta_i > 0\})\delta_2 + \sum_{i=1}^{r}\theta_i.
\end{aligned}$$

So

$$m_1+\cdots+m_r = \delta_1 - A(\alpha_1,\ldots,\alpha_r) - \#\{\beta_i>0\} = B(\alpha_1,\ldots,\alpha_r).$$

Therefore

$$B[\alpha_1,\ldots,\alpha_r] = \text{span}\{x_1^{\zeta_1}\cdots x_r^{\zeta_r} \in S^{d-k}V|\zeta_i = m_i\delta_2+\theta_i$$

$$\text{for some nonnegative integer } m_i \text{ such that } m_1+\cdots+m_r = B(\alpha_1,\ldots,\alpha_r)\}.$$

and dim $B[\alpha_1,\ldots,\alpha_r] = \binom{B(\alpha_1,\ldots,\alpha_r)+r-1}{B(\alpha_1,\ldots,\alpha_r)}$. □

Remark 4.2.8. *Spaces $A[\alpha_1,\ldots,\alpha_r]$ and $B[\alpha_1,\ldots,\alpha_r]$ are essentially determined by $(\beta_1,\ldots,\beta_r)$ with $0\le\beta_i<\delta_2$ $(i=1,\ldots,r)$.*

Definition 4.2.9. *Define $\mathrm{NUM}(A,B)$ to be the number of subspaces $A[\alpha_1,\ldots,\alpha_r]$ of S^kV^* such that $A(\alpha_1,\ldots,\alpha_r)=A$ and $B(\alpha_1,\ldots,\alpha_r)=B$.*

One can decompose

$$S^kV^* = \bigoplus_{A=0}^{\lfloor \frac{k}{\delta_2} \rfloor} \bigoplus_{B \le \delta_1 - A} W(A,B)^{\oplus \mathrm{NUM}(A,B)} \tag{4.8}$$

such that $W(A,B)$ is a copy of subspace $A[\alpha_1,\ldots,\alpha_r]$ in S^kV^* with $A(\alpha_1,\ldots,\alpha_r) = A$ and $B(\alpha_1,\ldots,\alpha_r) = B$. Furthermore $P_{k,d-k}$ is a disjoint map with respect to this decomposition.

Proposition 4.2.10. $\mathrm{NUM}(A,B) = \#\{(\beta_1,\ldots,\beta_r) | \beta_1 + \cdots + \beta_r = k - A\delta_2, \#\{\beta_i > 0\} = \delta_1 - B - A$ and $0 \le \beta_i < \delta_2$ for $i = 1,\ldots,r\}$. *Moreover we have the following bounds for the rank of* $P_{k,d-k}$,

$$\begin{aligned} \mathrm{NUM}(0,0) &\le \mathrm{rank}(P_{k,d-k}) && (4.9)\\ &\le \sum_{A=0}^{\lfloor \frac{k}{\delta_2} \rfloor} \sum_{B=0}^{\delta_1 - A} \min\{\binom{A+r-1}{A}, \binom{B+r-1}{B}\}\mathrm{NUM}(A,B). && (4.10) \end{aligned}$$

Proof. First let $A[\alpha_1,\ldots,\alpha_r]$ be a subspace of S^kV^* such that $A(\alpha_1,\ldots,\alpha_r) = A$ and $B(\alpha_1,\ldots,\alpha_r) = B$. Write $\alpha_i = \beta_i + n_i\delta_2$, where $0 \le \beta_i < \delta_2$, then

$$\begin{aligned} k &= \sum_{i=1}^{r} n_i\delta_2 + \beta_i \\ &= A\delta_2 - \sum_{i=1}^{r} \beta_i \end{aligned}$$

Therefore $\beta_1 + \cdots + \beta_r = k - A\delta_2$. By (4.7), $\#\{\beta_i > 0\} = \delta_1 - B - A$ *and* $0 \le \beta_i < \delta_2$. On the other hand, since $(\beta_1,\ldots,\beta_r)$ determine the class $A[\alpha_1,\ldots,\alpha_r]$ uniquely, $\mathrm{NUM}(A,B) = \#\{(\beta_1,\ldots,\beta_r) | \beta_1 + \cdots + \beta_r = k - A\delta_2, \#\{\beta_i > 0\}) = \delta_1 - B - A$ and $0 \le \beta_i < \delta_2$ for $i = 1,\ldots,r\}$.

Second, by (4.8) one can decompose $S^kV^* = \bigoplus_{A=0}^{\lfloor \frac{k}{\delta_2} \rfloor} \bigoplus_{B \le \delta_1 - A} W(A,B)^{\oplus \mathrm{NUM}(A,B)}$

such that $W(A,B)$ is a copy of subspace $A[\alpha_1,\ldots,\alpha_r]$ in S^kV^* with $A(\alpha_1,\ldots,\alpha_r) = A$ and $B(\alpha_1,\ldots,\alpha_r) = B$, and $P_{k,d-k}$ is a disjoint map with respect to the decomposition. Since when $B < 0$, $P_{k,d-k}|_{W(A,B)}$ is the zero map,

$$\begin{aligned}
\mathrm{rank}(P_{k,d-k}) &= \sum_{A=0}^{\lfloor \frac{k}{\delta_2} \rfloor} \sum_{B=0}^{\delta_1 - A} \mathrm{rank}(P_{k,d-k}|W(A,B))\mathrm{NUM}(A,B) \\
&\leq \sum_{A=0}^{\lfloor \frac{k}{\delta_2} \rfloor} \sum_{B=0}^{\delta_1 - A} \min\{\binom{A+r-1}{A}, \binom{B+r-1}{B}\}\mathrm{NUM}(A,B).
\end{aligned}$$

On the other hand, consider the subspaces $A[\alpha_1,\ldots,\alpha_r]$ such that $A(\alpha_1,\ldots,\alpha_r) = 0$ and $B(\alpha_1,\ldots,\alpha_r) = 0$, the map $P_{k,d-k}|_{A[\alpha_1,\ldots,\alpha_r]} : A[\alpha_1,\ldots,\alpha_r] \to B[\alpha_1,\ldots,\alpha_r]$ is just a linear map from a one-dimensional space to another one-dimensional space. Therefore

$$\begin{aligned}
\mathrm{rank}(P_{k,d-k}) &= \sum_{A=0}^{\lfloor \frac{k}{\delta_2} \rfloor} \sum_{B=0}^{\delta_1 - A} \mathrm{rank}(P_{k,d-k}|W(A,B))\mathrm{NUM}(A,B) \\
&\geq \mathrm{NUM}(0,0).
\end{aligned}$$

□

Corollary 4.2.11. *Let $\delta_1\delta_2 = d$ with $\delta_1, \delta_2 \sim \sqrt{d}$, $k = \lfloor \frac{d}{2} \rfloor$ and $r \geq 2\delta_1$, then*

$$\binom{r}{\delta_1} \leq \mathrm{rank}(P_{k,d-k}) \leq 2\binom{r}{\delta_1}\binom{k}{\delta_1}. \tag{4.11}$$

Proof. By Proposition 4.2.10,

$$\begin{aligned}
\operatorname{rank}(P_{k,d-k}) &\geq \mathrm{NUM}(0,0) \\
&= \#\{(\beta_1,\ldots,\beta_r)|\beta_1+\cdots+\beta_r=k, \#\{\beta_i>0\}) = \delta_1 \\
&\quad \text{and } 0\leq\beta_i<\delta_2 \text{ for } i=1,\ldots,r\} \\
&= \binom{r}{\delta_1}\#\{(\beta_1,\ldots,\beta_{\delta_1})|\beta_1+\cdots+\beta_{\delta_1}=k,\ 0<\beta_i<\delta_2 \text{ for } i=1,\ldots,\delta_1\} \\
&\geq \binom{r}{\delta_1}.
\end{aligned}$$

To prove the second inequality, notice that each monomial of $P=(x_1^{\delta_2}+\cdots+x_r^{\delta_2})^{\delta_1}$ is of the form $x_{i_1}^{\delta_2}\cdots x_{i_{\delta_1}}^{\delta_2}$. Let $y_1^{\alpha_1}\cdots y_r^{\alpha_r}\in S^kV^*$, if the number of positive α_i is greater than δ_1, then each monomial of P vanishes, i.e. $x_{i_1}^{\delta_2}\cdots x_{i_{\delta_1}}^{\delta_2}(y_1^{\alpha_1}\cdots y_r^{\alpha_r})=0$, so $P_{k,d-k}(y_1^{\alpha_1}\cdots y_r^{\alpha_r})=0$ under this condition. Therefore

$$\operatorname{rank}(P_{k,d-k})\leq\#\{(\alpha_1,\ldots,\alpha_r)|\text{ where } \alpha_1+\cdots+\alpha_r=k \text{ and } \#\{\alpha_i>0\}\leq\delta_1\}.$$

So

$$\begin{aligned}
\operatorname{rank}(P_{k,d-k}) &\leq \sum_{s=1}^{\delta_1}\binom{r}{s}\binom{k-1}{s-1} \\
&\leq \delta_1\binom{r}{\delta_1}\binom{k-1}{\delta_1-1} \\
&\sim 2\binom{r}{\delta_1}\binom{k}{\delta_1}
\end{aligned}$$

□

Remark 4.2.12. *While we can get better bounds for* $\operatorname{rank}(P_{k,d-k})$*, it is always the case that r is much bigger than $d=\delta_1\delta_2$, therefore the term $\binom{r}{\delta_1}$ dominates.*

4.2.2 Comparison with the permanent

Let $\{x_1, ..., x_{n^2}\}$ be a basis of V and $r \geq n$, and let $[P] = [(l_1^{\delta_2} + \cdots + l_r^{\delta_2})^{\delta_1}] \in v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}V))) \subset \mathbb{P}S^nV$ be generic, where $\delta_1, \delta_2 \sim \sqrt{n}$ and $k = \lfloor \frac{n}{2} \rfloor$. I compute the rank of the flattening $P_{k,n-k} : S^kV^* \to S^{n-k}V$ and compare it with $\text{rank}((\text{perm}_n)_{k,n-k})$.

For any $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $\text{rank}(\text{perm}_n)_{k,n-k} = \binom{n}{k}^2$.

Lemma 4.2.13. *Let $\tilde{A}, \tilde{B}$ be two complex vector spaces, let A be a subspace of $\tilde{A}$ and B be a subspace of $\tilde{B}$. If $\widetilde{T} \in \tilde{A} \otimes \tilde{B}$ and $T \in A \otimes B$ is the linear projection of $\tilde{T}$, then $R(\tilde{T}) \geq R(T)$.*

Corollary 4.2.14. *Let $[P] = [(l_1^{\delta_2} + \cdots + l_r^{\delta_2})^{\delta_1}] \in v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}V))) \subset \mathbb{P}S^nV$, where $\delta_1, \delta_2 \sim \sqrt{n}$ and $k = \lfloor \frac{n}{2} \rfloor$, then* $\text{rank}(P_{k,n-k}) \leq 2\binom{r}{\delta_1}\binom{k}{\delta_1} \leq (rk)^{\delta_1}$.

Proof. Assume that $l_1, ..., l_r$ are linearly independent and let $W = \text{span}\{l_1, ..., l_r\}$, By Corollary 4.2.11 the rank of the new flattening $\tilde{P}_{k,n-k} : S^kW^* \to S^{n-k}W$. By Lemma 4.2.13 $\text{rank}(P_{k,n-k}) \leq \text{rank}(\tilde{P}_{k,n-k})$.

By Corollary 4.2.11,

$$\begin{aligned} \text{rank}(P_{k,n-k}) &\leq \text{rank}(\tilde{P}_{k,n-k}) \\ &\leq 2\binom{r}{\delta_1}\binom{k}{\delta_1} \\ &\leq (rk)^{\delta_1} \end{aligned}$$

$\square$

Proof of Theorem 1.2.5. Let $k = \lfloor \frac{n}{2} \rfloor$, by Corollary 4.2.14,

$$
\begin{aligned}
\frac{\operatorname{rank}(\operatorname{perm}_n)_{k,n-k}}{\operatorname{rank}(P_{k,n-k})} &\geq \frac{\binom{n}{\lfloor \frac{n}{2} \rfloor}^2}{(rk)^{\delta_1}} \\
&\sim \frac{2^{2n}}{2^{\sqrt{n}\log(rn/2)}} \\
&= 2^{2n-\sqrt{n}\log(r)-\sqrt{n}\log(n/2)} \\
&= 2^{\sqrt{n}\log(n)\omega(1)-\sqrt{n}\log(n/2)} \\
&= 2^{\sqrt{n}\log(n)\omega(1)}.
\end{aligned}
$$

□

5. EQUATIONS FOR SECANT VARIETIES OF CHOW VARIETIES

This chapter is based on [21], I use the method of prolongation to obtain equations for secant varieties of Chow varieties as $GL(V)$-modules.

5.1 Prolongations, multiprolongations and partial derivatives

5.1.1 Prolongations, multiprolongations and ideals of secant varieties

I study prolongations, multiprolongations and how they relate to ideals of secant varieties. Let W be a complex vector space with a basis $\{e_1, \ldots, e_n\}$. I follow the notation in [31].

Definition 5.1.1. *For $A \subset S^d W$, define the p-th prolongation of A to be:*

$$A^{(p)} = (A \otimes S^p W) \cap S^{p+d} W.$$

It is equivalent to saying that

$$A^{(p)} = \{f \in S^{p+d} W | \frac{\partial^p f}{\partial e^\beta} \in A \text{ any } \beta \in \mathbb{N}^n \text{ with } |\beta| = p\}.$$

Here are properties of prolongation:

Proposition 5.1.2. *For $A \subset S^d W$, $A^{(p)}$ is the inverse image of $A \otimes S^p W$ under the polarization map $F_{d,p} : S^{d+p} W \to S^d W \otimes S^p W$.*

Proof. For any $f \in S^{(p+d)} W$,

$$F_{d,p}(f) = \sum_{|\alpha|=p} \frac{\partial^p f}{\partial e^\alpha} \otimes e^\alpha.$$

Hence

$$F_{d,p}(f) = \sum_{|\alpha|=p} \frac{\partial^p f}{\partial e^\alpha} \otimes e^\alpha \in A \otimes S^p W \Leftrightarrow \frac{\partial^p f}{\partial e^\alpha} \in A \text{ for any } |\alpha| = p \Leftrightarrow f \in A^{(p)}.$$

$\square$

Theorem 5.1.3. (J. Sidman, S. Sullivant [43]) *Let $X \in \mathbb{P}W^*$ be an algebraic variety and let d be the integer such that $I_{d-1}(X) = 0$ and $I_d(X) \neq 0$. Then $I_{r(d-1)}(\sigma_r(X)) = 0$ and $I_{r(d-1)+1}(\sigma_r(X)) = I_d(X)^{(r-1)(d-1)}$.*

Remark 5.1.4. *Theorem 5.1.3 bounds the lowest degree of an element in the ideal of $\sigma_r(X)$ if we know generators of the ideal of X.*

Proposition 5.1.5. *Let $X \subset \mathbb{P}W^*$ be a variety, then $I_d(X)^{(p)} \subset I_{d+1}(X)^{(p-1)}$.*

Proof. Let $f \in I_d(X)^{(p)} \subset S^{d+p}W$, consider $\frac{\partial^{p-1} f}{\partial e^\alpha}$ with $|\alpha| = p-1$,

$$\frac{\partial^{p-1} f}{\partial e^\alpha} = \sum_{i=1}^{n} \frac{\partial^p f}{\partial (e^\alpha e_i)} e_i \in I_{d+1}(X).$$

$\square$

Example 5.1.6. *Consider $Ch_3(V^*)$ with $\dim V \geq 4$, by Proposition 2.5.4 and Proposition 2.5.5, $I_3(Ch_3(V^*)) = 0$ and*

$$I_4(Ch_3(V^*)) = S^4(S^3V) - S^3(S^4V) = S_{(7,3,2)}V + S_{(6,2,2,2)}V + S_{(5,4,2,1)}V. \tag{5.1}$$

Therefore by Theorem 5.1.3 $I_6(\sigma_2(X)) = 0$ and $I_7(\sigma_2(X)) = I_4(X)^{(3)}$.

The following proposition is about multiprolongations:

Proposition 5.1.7. (Multiprolongation [31]) *Let $X \subset PW^*$ be an algebraic variety, a polynomial $P \in S^\delta W$ is in $I_\delta(\sigma_r(X))$ if and only if for any nonnegative decreasing* *sequence* $(\delta_1, \delta_2, \ldots, \delta_r)$ *with* $\delta_1 + \delta_2 + \cdots + \delta_r = \delta$,

$$\bar{P}(v_1, \ldots, v_1, v_2, \ldots, v_2, \ldots, v_r, \ldots, v_r) = 0$$

for all $v_i \in \hat{X}$, *where the number of* v_i'*s appearing in the formula is* m_i.

The following proposition rephrases multiprolongations.

Proposition 5.1.8. *Let* $X \subset PW^*$ *be an algebraic variety, for any positive integer* δ *and* r, *and for any decreasing sequence* $\vec{\delta} = (\delta_1, \delta_2, \ldots, \delta_r)$ *with* $\delta_1 + \delta_2 + \cdots + \delta_r = \delta$, *consider the following polarization maps*

$$F_{\delta_1, \delta_2, \cdots, \delta_r} : S^\delta W \to S^{\delta_1} W \otimes S^{\delta_2} W \otimes \cdots \otimes S^{\delta_r} W.$$

Let $A_{\vec{\delta},i} = S^{\delta_1} W \otimes \cdots \otimes S^{\delta_{i-1}} W \otimes I_{\delta_i}(X) \otimes S^{\delta_{i+1}} W \otimes \cdots \otimes S^{\delta_r} W \subset S^{\delta_1} W \otimes S^{\delta_2} W \otimes \cdots \otimes S^{\delta_r} W$, *then*

$$I_\delta(\sigma_r(X)) = \bigcap_{\delta_1 + \delta_2 + \cdots + \delta_r = \delta} F^{-1}_{\delta_1, \delta_2, \ldots, \delta_r}(A_{\vec{\delta},1} + \cdots + A_{\vec{\delta},r})$$

Corollary 5.1.9. $I_d(X)^{((r-1)(d-1))} \subset I_{r(d-1)+1}(\sigma_r(X))$.

Proof. By Proposition 5.1.8,

$$\begin{aligned} I_{r(d-1)+1}(\sigma_r(X)) &= \bigcap_{\delta_1 + \delta_2 + \cdots + \delta_r = r(d-1)+1,\ \delta_1 \geq \delta_2 \geq \cdots \geq \delta_r} F^{-1}_{\delta_1, \delta_2, \ldots, \delta_r}(A_{\vec{\delta},1} + \cdots + A_{\vec{\delta},r}) \\ &\supset \bigcap_{\delta_1 + \delta_2 + \cdots + \delta_r = r(d-1)+1,\ \delta_1 \geq \delta_2 \geq \cdots \geq \delta_r} F^{-1}_{\delta_1, \delta_2, \ldots, \delta_r}(A_{\vec{\delta},1}). \end{aligned}$$

By similar arguments as Proposition 5.1.2, $F^{-1}_{\delta_1, \delta_2, \ldots, \delta_r}(A_{\vec{\delta},1}) = I_{\delta_1}(X)^{(r(d-1)+1-\delta_1)}$. Since $\delta_1 \geq d$, by Proposition 5.1.5, $I_d(X)^{((r-1)(d-1))} \subset I_{\delta_1}(X)^{(r(d-1)+1-\delta_1)}$, therefore

$I_d(X)^{((r-1)(d-1))} \subset I_{r(d-1)+1}(\sigma_r(X))$. □

A new proof of Theorem 5.1.3. First, by Proposition 5.1.8,

$$I_{r(d-1)}(\sigma_r(X)) \;=\; \bigcap_{\delta_1+\delta_2+\cdots+\delta_r=r(d-1)} F^{-1}_{\delta_1,\delta_2,\ldots,\delta_r}(A_{\vec{\delta},1}+\cdots+A_{\vec{\delta},r}).$$

In particular, when $\delta_1 = \delta_2 = \cdots = \delta_r = (d-1)$, $A_{\vec{\delta},i} = 0$ for $i = 1,\ldots,r$, so $F^{-1}_{\delta_1,\delta_2,\cdots,\delta_r}(A_{\vec{\delta},1}+\cdots+A_{\vec{\delta},r}) = 0$. Therefore $I_{r(d-1)}(\sigma_r(X)) = 0$.
Second, by Proposition 5.1.8,

$$I_{r(d-1)+1}(\sigma_r(X)) \;=\; \bigcap_{\delta_1+\delta_2+\cdots+\delta_r=r(d-1)+1} F^{-1}_{\delta_1,\delta_2,\cdots,\delta_r}(A_{\vec{\delta},1}+\cdots+A_{\vec{\delta},r}).$$

In particular, when $\delta_1 = d,\ \delta_2 = \cdots = \delta_r = d-1$, $A_{\vec{\delta},i} = 0$ for $i = 2,\ldots,r$. so

$$F^{-1}_{\delta_1,\delta_2,\cdots,\delta_r}(A_{\vec{\delta},1}+\cdots+A_{\vec{\delta},r}) = F^{-1}_{\delta_1,\delta_2,\cdots,\delta_r}(A_{\vec{\delta},1}) = I_d(X)^{((r-1)(d-1))}.$$

Therefore $I_{r(d-1)+1}(\sigma_r(X)) \subset I_d(X)^{((r-1)(d-1))}$.
On the other hand, by Corollary 5.1.9, $I_d(X)^{((r-1)(d-1))} \subset I_{r(d-1)+1}(\sigma_r(X))$, so equality holds. □

Theorem 5.1.3, small examples and intuition lead to the following conjecture:

Conjecture 5.1.10. *Let $X \in PW^*$ be an algebraic variety, and $\delta = kr + l$ with $0 \le l < r$, take $\vec{\delta}$ such that $\delta_1 = \cdots = \delta_l = k+1$ and $\delta_{l+1} = \cdots = \delta_r = k$, then*

$$I_\delta(\sigma_r(X)) = F^{-1}_{\delta_1,\delta_2,\ldots,\delta_r}(A_{\vec{\delta},1}+\cdots+A_{\vec{\delta},r}).$$

Example 5.1.11. *Consider the variety $Ch_3(V^*)$, by Example 5.1.6, $I_3(Ch_3(V^*)) = 0$*

and $I_4(Ch_3(V^*))= S_{(7,3,2)}V + S_{(6,2,2,2)}V + S_{(5,4,2,1)}V$. *Consider the polarization maps*

$$F_{\delta,8-\delta} : S^8(S^3V) \to S^\delta(S^3V) \otimes S^{8-\delta}(S^3V).$$

By Propositions 5.1.8 and 5.1.5,

$$\begin{aligned} I_8(\sigma_2(Ch_3(V^*))) &= \bigcap_{\delta=4}^{8} F^{-1}_{\delta,8-\delta}[S^\delta(S^3V) \otimes I_{8-\delta}(Ch_3(V^*)) + I_\delta(Ch_3(V^*)) \otimes S^{8-\delta}(S^3V)] \\ &= \bigcap_{\delta=5}^{8} I_\delta(Ch_3(V^*))^{(8-\delta)} \bigcap F^{-1}_{4,4}[I_4(Ch_3(V^*)) \otimes S^4(S^3V) \\ &\quad +S^4(S^3V) \otimes I_4(Ch_3(V^*))] \\ &= I_5(Ch_3(V^*))^{(3)} \bigcap F^{-1}_{4,4}[I_4(Ch_3(V^*)) \otimes S^4(S^3V) \\ &\quad +S^4(S^3V) \otimes I_4(Ch_3(V^*))]. \end{aligned} \tag{5.2}$$

5.1.2 Partial derivatives and prolongations

Let $V = \text{span}\{e_1, \ldots, e_n\}$, S^dV has a natural basis $\{e_1^{\alpha_1} \cdots e_n^{\alpha_n} := e^\alpha\}_{\alpha_1+\cdots+\alpha_n=d}$. Assume $e_1 > e_2 > \cdots > e_n$. Define the *dominance partial order* on the natural basis of S^dV such that

$$e^\alpha > e^\beta \Leftrightarrow \alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i \text{ for each } i.$$

It is equivalent to saying

$$e^\alpha > e^\beta \Leftrightarrow \text{ one can get } e^\alpha \text{ from } e^\beta \text{ via raising operators.}$$

Let $f \in W_{(a_1,\cdots,a_n)} \subset S^k(S^dV)$, let α be the index of the last d elements in $(a_1, \cdots, a_n)$, then $\frac{\partial}{\partial e^\alpha}$ is the lowest possible partial derivative of f with respect to the dominance partial order.

Example 5.1.12. *Let $f \in W_{(5,4,4,2)} \subset S^5(S^3V)$, then $\alpha = (0,0,1,2)$ and the lowest possible partial derivative of f is $\frac{\partial f}{\partial e_3 e_4^2}$.*

Definition 5.1.13. *Let $e^\alpha = e_1^{\alpha_1} \cdots e_j^{\alpha_j} e_{j+1}^{\alpha_{j+1}} \cdots e_n^{\alpha_n}$, for $j = 1, \ldots, n-1$, define the normalized lowering operators*

$$\tilde{E}_j^{j+1} e^\alpha = e_1^{\alpha_1} \cdots e_j^{\alpha_j - 1} e_{j+1}^{\alpha_{j+1}+1} \cdots e_n^{\alpha_n}.$$

The following proposition gives the relationship between raising operators and partial derivatives of polynomials in $S^k(S^dV)$.

Proposition 5.1.14. *Let $f \in S^k(S^dV)$ and e^α be a basis vector of S^dV, then*

$$[\frac{\partial}{\partial e^\alpha}, E_{j+1}^j] f = (1 + \alpha_{j+1}) \frac{\partial f}{\partial (\tilde{E}_j^{j+1} e^\alpha)}.$$

Where $\tilde{E}_j^{j+1} (j = 1, \cdots, n-1)$ are the normalized lowering operators.

Proof. Since all the operators here are linear, we only to prove the case when f is a monomial. Let $e^\alpha = e_1^{\alpha_1} \cdots e_j^{\alpha_j} e_{j+1}^{\alpha_{j+1}} \cdots e_n^{\alpha_n}$, so $\tilde{E}_j^{j+1} e^\alpha = e_1^{\alpha_1} \cdots e_j^{\alpha_j - 1} e_{j+1}^{\alpha_{j+1}+1} \cdots e_n^{\alpha_n} = e^\beta$. Write $f = g(e^\alpha)^m (e^\beta)^n$, where g is not divisible by e^α or e^β. Then

$$\begin{aligned} E_j^{j+1} f &= (E_j^{j+1} g)(e^\alpha)^m (e^\beta)^n + g E_j^{j+1}((e^\alpha)^m)(e^\beta)^n + g(e^\alpha)^m E_j^{j+1}((e^\beta)^n) \\ &= (E_j^{j+1} g)(e^\alpha)^m (e^\beta)^n + m g (e^\alpha)^{m-1} E_j^{j+1}(e^\alpha)(e^\beta)^n \\ &\quad + n(1 + \alpha_{j+1}) g (e^\alpha)^{m+1} (e^\beta)^{n-1}. \end{aligned}$$

So

$$\begin{aligned} \frac{\partial (E_j^{j+1} f)}{\partial e^\alpha} &= m(E_j^{j+1} g)(e^\alpha)^{m-1}(e^\beta)^n + m(m-1) g (e^\alpha)^{m-2} E_j^{j+1}(e^\alpha)(e^\beta)^n + \\ &\quad n(m+1)(1 + \alpha_{j+1}) g (e^\alpha)^m (e^\beta)^{n-1}. \end{aligned} \tag{5.3}$$

On the other hand

$$\frac{\partial f}{\partial e^{\alpha}} = mg(e^{\alpha})^{m-1}(e^{\beta})^{n}.$$

$$\begin{aligned} E_j^{j+1}(\frac{\partial f}{\partial e^{\alpha}}) &= m(E_j^{j+1}g)(e^{\alpha})^{m-1}(e^{\beta})^{n} + m(m-1)g(e^{\alpha})^{m-2}E_j^{j+1}(e^{\alpha})(e^{\beta})^{n} + \\ & nm(1+\alpha_{j+1})g(e^{\alpha})^{m}(e^{\beta})^{n-1}. \end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4), we conclude:

$$\frac{\partial(E_j^{j+1}f)}{\partial e^{\alpha}} - E_{j+1}^{j}(\frac{\partial f}{\partial e^{\alpha}}) = n(1+\alpha_{j+1})g(e^{\alpha})^{m}(e^{\beta})^{n-1} = (1+\alpha_{j+1})\frac{\partial f}{\partial(\tilde{E}_j^{j+1}e^{\alpha})}.$$

□

In particular if $f \in S^k(S^dV)$ is a highest weight vector of some $GL(V)$-module, then

$$E_{j+1}^{j}(\frac{\partial f}{\partial e^{\alpha}}) = -(1+\alpha_{j+1})\frac{\partial f}{\partial(\tilde{E}_j^{j+1}e^{\alpha})}. \tag{5.5}$$

Therefore

Lemma 5.1.15. *If $f \in S^{k+1}(S^dV)$ is a highest weight vector for some $GL(V)$ module $S_{(a_1,\cdots,a_n)}V = S_aV$, then the lowest possible partial derivative $\frac{\partial f}{\partial e^{\alpha}}$ is killed by all the raising operators, i.e. either $\frac{\partial f}{\partial e^{\alpha}}$ is 0 or a highest weight vector of $S_{a-\alpha}V \subset S^k(S^dV)$.*

By induction on dominance partial order, I conclude

Proposition 5.1.16. *If $f \in S^{k+1}(S^dV)$ is a highest weight vector for some module $S_{(a_1,\cdots,a_n)}V = S_aV$, then there exists a basis vector e^{β} of S^dV such that $\frac{\partial f}{\partial e^{\beta}}$ is a highest vector of $S_{a-\beta}V \subset S^k(S^dV)$.*

By Proposition 5.1.16,

Corollary 5.1.17. *Let $f \in S^{k+1}(S^dV)$ be a highest weight vector for some module $S_{(a_1,\cdots,a_n)}V = S_aV$, if we can find all the e^β such that $\frac{\partial f}{\partial e^\beta}$ is a highest vector of $S_{a-\beta}V \subset S^k(S^dV)$, the sum of all these modules is the smallest possible module such that S_aV lies in its first prolongation.*

For simplicity, write $\frac{\partial f}{\partial e^\beta} = f_{e^\beta}$ from now on.

Example 5.1.18. *Let f be the highest weight vector of $S_{(7,3,2)}V \subset S^4(S^3V)$ in Example 2.3.12, then $f_{e_2e_3^2} = (e_1^3)^2(e_1e_2^2) - e_1^3(e_1^2e_2)^2$, which is a highest weight vector of $S_{(7,2)}V \subset S^3(S^3V)$.*

The following proposition, tells us which prolongation a given module lies in.

Proposition 5.1.19. *If $S_aV \subset S^{k+1}(S^dV)$ with multiplicity $m_a > 0$, let*

$$\begin{aligned} M_a &= \{b | S_aV \subset S_bV \otimes S^dV \text{ as abstract modules by Pieri's rule} \\ &\quad \text{and } S_bV \subset S^k(S^dV) \text{ with multiplicity } m_b > 0\}. \end{aligned}$$

then

$$(S_aV)^{\oplus m_a} \subset (\bigoplus_{b \in M_a} (S_bV)^{\oplus m_b})^{(1)}.$$

In particular,

$$m_a \leq \sum_{b \in M_a} m_b.$$

Proof. Consider the polarization map

$$P_{k,1} : S^{k+1}(S^dV) \to S^k(S^dV) \otimes S^dV.$$

By Schur's lemma

$$P_{k,1}((S_aV)^{\oplus m_a}) \subset (\bigoplus_{b\in M_a}(S_bV)^{\oplus m_b}) \otimes S^dV.$$

By Proposition 5.1.2

$$(S_aV)^{\oplus m_a} \subset (\bigoplus_{b\in M_a}(S_bV)^{\oplus m_b})^{(1)}.$$

Since $P_{k,1}$ is injective,

$$m_a \leq \sum_{b\in M_a} m_b.$$

□

Proposition 5.1.20. *The module $S_{(5,4,4,2)}V \subset S^5(S^3V)$ is contained in $(S_{(5,4,2,1)}V \oplus S_{(4,4,4)}V)^{(1)}$. Let $f \in S_{(5,4,4,2)}V \subset S^5(S^3V)$ be a highest weight vector, then $f_{e_1e_4^2}$ is a highest weight vector of $S_{(4,4,4)}V \subset S^4(S^3V)$ and $f_{e_3^2e_4}$ is a highest weight vector of $S_{(5,4,2,1)}V \subset S^4(S^3V)$. Therefore $S_{(5,4,4,2)}V$ is not contained in the first prolongation of $S_{(4,4,4)}V$ or $S_{(5,4,2,1)}V$.*

Proof. Since

$$\begin{aligned} S^4(S^3V) \quad = \quad & S_{(12)}V + S_{(10,2)}V + S_{(9,3)}V + S_{(8,4)}V + \\ & S_{(8,2,2)}V + S_{(7,4,1)}V + S_{(7,3,2)}V + S_{(6,6)}V + \\ & S_{(6,4,2)}V + S_{(6,2,2,2)}V + S_{(5,4,2,1)}V + S_{(4,4,4)}V. \end{aligned}$$

By Proposition 5.1.19, $S_{(5,4,4,2)} \subset (S_{(5,4,2,1)}V \oplus S_{(4,4,4)}V)^{(1)}$. By induction on the dominance partial order, $f_{e_1e_4^2}$ and $f_{e_3^2e_4}$ are killed by all raising operators. Let h_1 be a highest weight vector of $S_{(4,4,4)}V \subset S^4(S^3V)$ and h_2 be a highest weight vector of $S_{(5,4,2,1)}V \subset S^4(S^3V)$. Set $f_{e_1e_4^2} = c_1h_1$ and $f_{e_3^2e_4} = c_2h_2$, where c_1 and c_2 are constants, then c_1, c_2 can not be both 0 by Proposition 5.1.16.

Since $f_{e_3^3} \in S_{(5,4,2,1)}V \oplus S_{(4,4,4)}V$ with weight (5,4,1,2), $f_{e_3^3} = c_3 E_3^4 f_{e_3^2 e_4}$, where c_3 is a constant. By (5.5), $E_4^3 f_{e_3^3} = -f_{e_3^2 e_4}$, so $c_3 E_4^3 E_3^4 f_{e_3^2 e_4} = -f_{e_3^2 e_4}$, which implies $c_3(E_3^3 - E_4^4) f_{e_3^2 e_4} = -f_{e_3^2 e_4}$, so $c_3 = -1$. Since $(f_{e_1 e_4^2})_{e_3^3} = (f_{e_3^3})_{e_1 e_4^2}$,

$$\begin{aligned} c_1(h_1)_{e_3^3} &= (-E_3^4 f_{e_3^2 e_4})_{e_1 e_4^2} \\ &= -c_2(E_3^4 h_2)_{e_1 e_4^2} \\ &= -c_2(E_3^4 (h_2)_{e_1 e_4^2} - (h_2)_{e_1 e_3 e_4}) \\ &= c_2(h_2)_{e_1 e_3 e_4} \end{aligned}$$

By Proposition 2.3.14, $(h_1)_{e_3^3}$ and $(h_2)_{e_1 e_3 e_4}$ are both highest weight vectors of $S_{(4,4,1)}V \subset S^4(S^3V)$, by rescaling, we may assume they are equal, so $c_1 = c_2$, so c_1 and c_2 are both nonzero, therefore $f_{e_1 e_4^2}$ is a highest weight vector of $S_{(4,4,4)}V \subset S^4(S^3V)$ and $f_{e_3^2 e_4}$ is a highest weight vector of $S_{(5,4,2,1)}V \subset S^4(S^3V)$. $\square$

5.2 The case when the degree is 3

Consider $\sigma_2(Ch_3(V^*))$, without loss of generality we assume dim $V = 6$.

Proposition 5.2.1.

$$\begin{aligned} I_4(Ch_3(V^*))^{(1)} &= S_{(7,2,2,2,2)}V \oplus S_{(6,4,2,2,1)}V \oplus S_{(5,5,3,1,1)}V, \\ I_4(Ch_3(V^*))^{(2)} &= S_{(8,2,2,2,2,2)}V \oplus S_{(7,4,2,2,2,1)}V \oplus S_{(6,5,3,2,2,1)}V \oplus S_{(5,5,5,1,1,1)}V, \\ I_4(Ch_3(V^*))^{(3)} &= 0. \end{aligned}$$

Proof. First we claim

$$I_4(Ch_3(V^*))^{(1)} = S_{(7,2,2,2,2)}V \oplus S_{(6,4,2,2,1)}V \oplus S_{(5,5,3,1,1)}V. \tag{5.6}$$

By (5.1),

$$I_4(Ch_3(V^*)) = S_{(7,3,2)}V + S_{(6,2,2,2)}V + S_{(5,4,2,1)}V.$$

By computer softwares (e.g. Lie),

$$\begin{aligned} S^4(S^3V) &= S_{(12)}V + S_{(10,2)}V + S_{(9,3)}V + S_{(8,4)}V + \\ &\quad S_{(8,2,2)}V + S_{(7,4,1)}V + S_{(7,3,2)}V + S_{(6,6)}V + \\ &\quad S_{(6,4,2)}V + S_{(6,2,2,2)}V + S_{(5,4,2,1)}V + S_{(4,4,4)}V. \end{aligned}$$

and

$$\begin{aligned} S^5(S^3V) &= S_{(15)}V + S_{(13,2)}V + S_{(12,3)}V + S_{(11,4)}V + S_{(11,2,2)}V + S_{(10,5)}V + S_{(10,4,1)}V \\ &\quad + S_{(10,3,2)}V + S_{(9,6)}V + 2S_{(9,4,2)}V + S_{(9,2,2,2)}V + S_{(8,6,1)}V + S_{(8,5,2)}V + S_{(8,4,3)}V \\ &\quad + S_{(8,4,2,1)}V + S_{(8,3,2,2)}V + S_{(7,6,2)}V + S_{(7,5,2,1)}V + S_{(7,4,4)}V + S_{(7,4,3,1)}V \\ &\quad + S_{(7,4,2,2)}V + S_{(7,2,2,2,2)}V + S_{(6,6,3)}V + S_{(6,5,2,2)}V + S_{(6,4,4,1)}V + S_{(6,4,2,2,1)}V \\ &\quad + S_{(5,5,3,1,1)}V + S_{(5,4,4,2)}V. \end{aligned}$$

Since $I_4(Ch_3(V^*))$ contains all the modules with length 4 in $S^4(S^3V)$, by Proposition 5.1.19 any module with length 5 in $S^5(S^3V)$ is in $I_4(Ch_3(V^*)^{(1)}$.

On the other hand, the other modules with length no more than 4 in $S^5(S^3V)$ are not in $I_4(Ch_3(V^*)^{(1)}$: By Proposition 5.1.16, for any module with length no more than 4 in $S^5(S^3V)$, one can find a partial derivative of a highest weight vector of this module such that it is a highest weight vector of a module in $S^4(S^3V)$ but not in $I_4(Ch_3(V^*)$. For most modules, we can check directly, but for some modules, we need to verify carefully. For example, By Proposition 5.1.20, $S_{(5,4,4,2)} \subset (S_{(5,4,2,1)} \oplus S_{(4,4,4)}V)^{(1)}$, but $f_{e_1e_4^2}$ is a highest weight vector of $S_{(4,4,4)}V \subsetneq I_4(Ch_3(V^*))$, so $S_{(5,4,4,2)}$ is not not in $I_4(Ch_3(V^*)^{(1)}$. I conclude

$$\begin{aligned} I_4(Ch_3(V^*))^{(2)} &= S_{(8,2,2,2,2,2)}V \oplus S_{(7,4,2,2,2,1)}V \oplus S_{(6,5,3,2,2,1)}V \oplus S_{(5,5,5,1,1,1)}V, \\ I_4(Ch_3(V^*)^{(3)} &= 0. \end{aligned}$$

$\square$

Therefore by Proposition 5.2.1 and Theorem 5.1.3,

Theorem 5.2.2. (restatement of Theorem 1.2.6) $I_7(\sigma_2(Ch_3(V^*)))= I_4(Ch_3(V^*))^{(3)} = 0$.

Also

Theorem 5.2.3. (restatement of Theorem 1.2.8) $I_8(\sigma_2(Ch_3(V^*))) \supset S_{(5,5,5,5,3,1)}V$.

Proof. By Example 5.1.8, $I_8(\sigma_2(Ch_3(V^*))) = I_5(Ch_3(V^*))^{(3)} \bigcap F_{4,4}^{-1}[I_4(Ch_3(V^*)) \otimes S^4(S^3V) + S^4(S^3V) \otimes I_4(Ch_3(V^*))]$. Since all the modules with 5 columns in $S^5(S^3V)$ are contained in $I_5(Ch_3(V^*))$, by Proposition 5.1.2 and Schur's lemma,

$$S_{(5,5,5,5,3,1)}V \subset I_5(Ch_3(V^*)^{(3)}. \tag{5.7}$$

Consider the map

$$F_{4,4} : S^8(S^3V) \to S^4(S^3V) \otimes S^4(S^3V).$$

Let $I_4(Ch_3(V^*))^c$ denote the complement to $I_4(Ch_3(V^*))$ in $S^4(S^3V)$. Since

$$\begin{aligned} I_4(Ch_3(V^*))^c &= S_{(12)}V + S_{(10,2)}V + S_{(9,3)}V + S_{(8,4)}V + \\ &\quad S_{(8,2,2)}V + S_{(7,4,1)}V + S_{(6,6)}V + S_{(6,4,2)}V + S_{(4,4,4)}V, \end{aligned}$$

and $S_{(5,5,5,5,3,1)}V \nsubseteq S_{(4,4,4)}V \otimes S_{(4,4,4)}V$, by the Littlewood-Richardson rule,

$$S_{(5,5,5,5,3,1)}V \nsubseteq I_4(Ch_3(V^*))^c \otimes I_4(Ch_3(V^*))^c.$$

Therefore by Schur's lemma

$$S_{(5,5,5,5,3,1)}V \subset F_{4,4}^{-1}(I_4(Ch_3(V^*)) \otimes S^4(S^3V) + S^4(S^3V) \otimes I_4(Ch_3(V^*))).$$

The result follows. $\square$

Remark 5.2.4. *Since $\sigma_2(Ch_3(\mathbb{C}^{5*}))$ is a proper subset of $\mathbb{P}S^3(\mathbb{C}^{5*})$, by inheritance* [31]*, the ideal of $\sigma_2(Ch_3(V^*))$ contains modules with length 5. So $S_{(5,5,5,5,3,1)}V$ is not enough to cut out $\sigma_2(Ch_3(V^*))$ set-theoretically. We know that* dim $S_{(5,5,5,5,3,1)}V =$ 1134 *and* codim $\sigma_2(Ch_3(V^*)) = 24$*, therefore $\sigma_2(Ch_3(V^*))$ is very far from being a complete intersection. Obviously $\mathbb{P}S^3(\mathbb{C}^{5*})$ with dimension 34 is in the zero set of $S_{(5,5,5,5,3,1)}V$, while the dimension of $\sigma_2(Ch_3(V^*))$ is 31, the next question is: what is the difference between the dimension of $\sigma_2(Ch_3(V^*))$ and the zero set of $S_{(5,5,5,5,3,1)}V$?*

5.3 The case when the degree is 4

Consider $\sigma_r(Ch_4(V^*)) \subset S^4(V^*)$, where dim $V \geq 4r$, prolongations enable one to find modules in the ideal of $\sigma_r(Ch_4(V^*))$.

Theorem 5.3.1. (restatement of Theorem 1.2.10) *When* dim $V \geq 4r$*,*

$$I_{4r+1}(\sigma_r(Ch_4(V^*))) = I_5(Ch_4(V^*))^{(4r-4)}$$

and

$$S_{(6,6,4^{4r-2})}V \subset I_{4r+1}(\sigma_r(Ch_4(V^*))).$$

Proof. By Proposition 2.5.2, Proposition 2.5.4 and Proposition 2.5.5, $I_4(Ch_4(V^*)) = 0$ and $I_5(Ch_4(V^*)) = S^5(S^4V) - S^4(S^5V)$, so $I_5(Ch_4(V^*))^c = S^4(S^5V)$. By Theorem

5.1.3,

$$I_{4r+1}(\sigma_r(Ch_4V^*)) = I_5(Ch_4V^*)^{(4r-4)}.$$

Consider the polarization map

$$F_{4r-4,5} : S^{4r+1}(S^4V) \to S^{4r-4}(S^4V) \otimes S^5(S^4V),$$

by Proposition 5.1.2,

$$I_5(Ch_4V^*)^{(4r-4)} = F^{-1}_{4r-4,5}(S^{4r-4}(S^4V) \otimes I_5(Ch_4(V^*))).$$

Since $S_{(6,6,6,2)} \subset S^4(S^5V)$ has the lowest highest weight vector with respect to the lexicographic order among all the modules in $S^4(S^5V)$, by the Littlewood-Richardson rule,

$$S_{(6,6,4^{4r-2})}V \subsetneq S^{4r-4}(S^4V) \otimes I_5(Ch_4(V^*))^c = S^{4r-4}(S^4V) \otimes S^4(S^5V).$$

Therefore by Schur's lemma

$$S_{(6,6,4^{4r-2})}V \subset I_5(Ch_4V^*)^{(4r-4)} = I_{4r+1}(\sigma_r(Ch_4(V^*)).$$

$\square$

Remark 5.3.2. *Consider* $r = 2$ *and* $\dim V = 8$*. Since* $\sigma_2(Ch_4\mathbb{C}^{4*}))$ *is a proper subset* $\mathbb{P}S^4(\mathbb{C}^{4*})$*, by inheritance (see [31]), the ideal of* $\sigma_2(Ch_4(V^*))$ *contains modules with length 4. So* $S_{(6,6,4,4,4,4,4,4)}V$ *is not enough to cut out* $\sigma_2(Ch_4(V^*))$ *set-theoretically. We know that* $\dim S_{(6,6,4,4,4,4,4,4)}V = 336$ *and* $\operatorname{codim} \sigma_2(Ch_3(V^*)) =$

272, therefore $\sigma_2(Ch_4(V^*))$ is far from being a complete intersection. Obviously $\mathbb{P}S^4(\mathbb{C}^{7*})$ with dimension 210 is in the zero set of $S_{(6,6,4,4,4,4,4,4)}V$, while the dimension of $\sigma_2(Ch_4\mathbb{C}^{4*}))$ is 57, The next question is: what is the difference between the dimension of $\sigma_2(Ch_4(V^*))$ and the zero set of $S_{(6,6,4,4,4,4,4,4)}V$?

5.4 General case for even degrees

Proposition 5.4.1. [6] *Let λ be a partition with order kd with d odd , then the multiplicity of λ in $S^k(S^dV)$ is less than or equal to the number of semi-standard tableaux of shape λ and content $k \times d$ with the additional property : for each pair $(i,j), 1 \leq i \neq j \leq k$, the set of columns of i is not exactly the columns of j.*

Proposition 5.4.2. [39] *Let λ be a partition with order kd and let u be even, then*

$$\mathrm{mult}(S_\lambda V, S^k(S^dV)) = \mathrm{mult}(S_{\lambda+(u^k)}V, S^k(S^{d+u}V)).$$

Theorem 5.4.3. *The module $S_{((2m+2)^{2m-1},2)}V$ is contained in $S^{2m}(S^{2m+1}V)$, with multiplicity 1, and $S_{((2m+2)^{2m-1},2)}V$ is the smallest module with respect to the lexicographic order among all the modules in the decomposition of $S^{2m}(S^{2m+1}V)$.*

Proof. First, let $\lambda = (\lambda_1, \ldots, \lambda_{2m})$ be a partition with order $4m^2 + 2m$ and smaller than $((2m+2)^{2m-1}, 2)$ with respect to the lexicographic order, then $\lambda_1 \leq 2m+2$ and $\lambda_{2m} \geq 3$. Consider the semi-standard tableaux with content $2m \times (2m+1)$; the first 3 columns must be filled with $\{1, \ldots, 2m\}$. Therefore there are $\binom{\lambda_1-3}{2m-2} \leq 2m-1$ possible sets of columns, but there are $2m$ numbers to be filled in the semi-standard tableaux, so by Proposition 5.4.1, $\mathrm{mult}(S_\lambda V, S^{2m}(S^{2m+1}V)) = 0$.

Second, consider the partition $\lambda = ((2m+2)^{2m-1}, 2)$, by Proposition 5.4.2, I conclude $\mathrm{mult}(S_\lambda V, S^{2m}(S^{2m+1}V)) = \mathrm{mult}(S_{(2m^{2m-1})}V, S^{2m}(S^{2m-1}V))$. By [28] formula (80), $\mathrm{mult}(S_{(2m^{2m-1})}V, S^{2m}(S^{2m-1}V)) = 1$. $\square$

Let $d = 2m \geq 4$ and dim $V \geq 2mr$, consider the variety $\sigma_r(Ch_{2m}(V^*)) \subset S^{2m}V^*$. A partition is an even partition if all the components of the partition are even numbers. When d is even, any even partition with length no more than k has positive plethysm coefficients in $S^k(S^dV)$ [7].

Theorem 5.4.4. (restatement of Theorem 1.2.12) *The isotypic component of the module $S_{((2m+2)^m,(2m)^{2mr-m})}V$ in $S^{2mr+1}(S^{2m}V)$ is contained in $I_{2mr+1}(\sigma_r(Ch_{2m}(V^*)))$. Moreover any module with even partition and smaller than $((2m+2)^{2m-1}, 2)$ (with respect to the lexicographic order) is in $I_{2mr+1}(\sigma_r(Ch_{2m}(V^*)))$.*

Proof. By Theorem 5.4.3, $S_{((2m+2)^{2m-1},2)}V$ is the smallest module (with respect to the lexicographic order) in the decomposition of $S^{2m}(S^{2m+1}V)$. Therefore by Corollary 2.5.3, any module smaller than $S_{((2m+2)^{2m-1},2)}V$ (with respect to the lexicographic order) is not in $I_{2m+1}(Ch_{2m}(V^*))^c \subset S^{2m+1}(S^{2m}V)$.

Consider the polarization map

$$F_{2mr-2m,2m+1} : S^{2mr+1}(S^{2m}V) \to S^{2mr-2m}(S^{2m}V) \otimes S^{2m+1}(S^{2m}V).$$

By Proposition 5.1.2,

$$I_{2m+1}(Ch_{2m}(V^*))^{(2m(r-1))} = F^{-1}_{2mr-2m,2m+1}(S^{2mr-2m}(S^{2m}V) \otimes I_{2m+1}(Ch_{2m}(V^*))).$$

By the Littlewood-Richardson rule,

$$S_{((2m+2)^m,(2m)^{2mr-m})}V \subsetneq S^{2mr-2m}(S^{2m}V) \otimes I_{2m+1}(Ch_{2m}(V^*))^c.$$

Moreover any module in $S^{2mr+1}(S^{2m}V)$ with even partition and smaller than $((2m+2)^{2m-1}, 2)$ is not contained in $S^{2mr-2m}(S^{2m}V) \otimes I_{2m+1}(Ch_{2m}(V^*)))^c$.

Therefore by Schur's lemma the isotypic component of $S_{((2m+2)^m,(2m)^{2mr-m})}V$ is in $F^{-1}_{2mr-2m,2m+1}[S^{2mr-2m}(S^{2m}V)\otimes I_{2m+1}(Ch_{2m}(V^*))] = I_{2m+1}(Ch_{2m}(V^*))^{(2m(r-1))}$.

Moreover any module in $S^{2mr+1}(S^{2m}V)$ with even partition and smaller than $((2m+2)^{2m-1},2)$ (with respect to the lexicographic order) is in $I_{2m+1}(Ch_{2m}(V^*))^{(2m(r-1))}$.

By Corollary 5.1.9, $I_{2m+1}(Ch_{2m}(V^*))^{(2m(r-1))}\subset I_{2mr+1}(\sigma_r(Ch_{2m}(V^*)))$, the results follow. □

5.5 A property of Plethysm coefficients

Lemma 5.5.1. [8, 38, 39] $\mathrm{mult}(S_\lambda V, S^k(S^{2l}V)) = \mathrm{mult}(S_{\lambda^T}V, S^k(\Lambda^{2l}V))$, *and* $\mathrm{mult}(S_\lambda V, S^k(S^{2l+1}V)) = \mathrm{mult}(S_{\lambda^T}V, \Lambda^k(\Lambda^{2l}V))$.

Theorem 5.5.2. *Let d be even, if $S_{(a_1,\cdots,a_p)}\subset S^k(S^dV)$ and $S_{(b_1,\cdots,b_q)}\subset S^l(S^dV)$ with $a_p\geq b_1$, then*

$$S_{(a_1,\cdots,a_p,b_1,\cdots,b_q)}\subset S^{k+l}(S^dV)$$

as long as $\dim V\geq k+l$.

Proof. Let $\lambda=(a_1,\cdots,a_p)$ and $\mu=(b_1,\cdots,b_q)$. By Lemma 5.5.1, $\mathrm{mult}(S_{\lambda^T}V, S^k(\Lambda^dV)) > 0$ and $\mathrm{mult}(S_{\mu^T}V, S^l(\Lambda^dV)) > 0$, so $\mathrm{mult}(S_{\lambda^T+\mu^T}V, S^{k+l}(\Lambda^dV)) > 0$. By Lemma 5.5.1 again,

$$\mathrm{mult}(S_{(\lambda,\mu)}V, S^{k+l}(S^dV)) > 0.$$

□

Remark 5.5.3. *This is false when d is odd: C.Ikenmeyer gave a counter-example for $d=3$. There exists k_0 such that $S_{6^{k_0}}V\subset S^{2k_0}(S^3V)$ but $S_{6^{k_0+1}}V\subsetneq S^{2k_0+2}(S^3V)$.*

6. SUMMARY

In this dissertation, we study chow varieties, their secant varieties and other varieties arising in complexity theory to approach Valiant's conjecture.

I use the polarization of Brill's polynomial map $\mathfrak{B}$ to construct Brill's map $\overline{\mathfrak{B}}$, which is a $GL(V)-$module map, I compute the image of Brill's map to determine Brill's equations as a $GL(V)$-module.

I obtain determinantal equations for Chow varieties and their secant varieties by Koszul Young flattenings, and get a new lower bound for symmetric border rank of square free monomials with odd degree. I compare the flattening rank of a generic polynomial in the Veronese reembeddings of secant varieties of Veronese varieties with that of the perm_n, and prove a complexity lower bound for the permanent.

I use the method of prolongations to obtain equations for secant varieties of Chow varieties as $GL(V)$-modules, The methods I use to compute prolongations are differential operators and plethysm coefficients.

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APPENDIX A

COMPLEXITY THEORY

A.1 Valiant's Conjecture

Definition A.1.1. *An arithmetic circuit $\mathcal{C}$ over $\mathbb{C}$ and the set of variables $\{x_1, \ldots, x_N\}$ is a directed acyclic graph with vertices of in-degree 0 and exactly one vertex of out-degree 0. Every vertex in it with in degree zero is called an input gate and is labeled by either a variable x_i or an element in $\mathbb{C}$. Every other gate is labeled by either $+$ or $\times$, exactly one vertex of out-degree 0.*

A circuit has two complexity measures associated with it: size and depth. The *size* of a circuit is the number of gates in it, and the *depth* of a circuit is the length of the longest directed path in it.

Proposition A.1.2. *On an arithmetic circuit $\mathbb{C}$, each gate computes a polynomial. The polynomial computed by the output gate is denoted by P_C and called the polynomial defined by the circuit.*

Definition A.1.3. *The class* **VP** *consists of sequences of polynomials (p_n) of polynomial of degree $d(n)$ and variables $v(n)$, where $d(n)$ and $v(n)$ are bounded by polynomials in n and such that there exists a sequence of arithmetic circuits $\mathcal{C}_n$ of polynomially bounded size such that $\mathcal{C}_n$ defines p_n.*

Example A.1.4. *The sequence* $(\det_n) \in \mathbf{VP}$, *where det_n denotes the determinant of a $n \times n$ matrix.*

Definition A.1.5. *Consider a sequence $h = (h_n)$ of polynomials in variables $x_1, \ldots, x_n$*

of the form

$$h_n = \sum_{e \in \{0,1\}^n} g_n(e) x_1^{e_1} \cdots x_n^{e_n},$$

where $(g_n) \in \mathbf{VP}$*. The class* $\mathbf{VNP}$ *is defined to be the set of all sequences the form* h*.*

Definition A.1.6. *A problem* P *is hard for a complexity class* $\mathbf{C}$ *if all problems in* $\mathbf{C}$ *can be reduced to* P *(i.e. there is an algorithm to translate any instance of a problem in* $\mathbf{C}$ *to an instance of* P *with comparable input size). A problem* P *is complete for* $\mathbf{C}$ *if it is hard for* $\mathbf{C}$ *and* $P \in \mathbf{C}$*.*

Proposition A.1.7. [47] *The sequence* (perm_n) *is* $\mathbf{VNP}$*-complete.*

Therefore to prove Valiant's Conjecture $\mathbf{VP} \neq \mathbf{VNP}$ [48], we only need to prove there does not exist a polynomial size circuit computing the permanent.

A.2 Shallow circuits

Circuits of bounded depth are called *shallow circuits.* Recently there have been significant advances for circuits of depth 3 [23] and 4 [1, 30, 46] and a special class of circuits of depth 5 [23]. A.Gupta, P.Kamath, N.Kayal and R.Saptharishi [23] showed these shallow circuits could be used to measure the complexity of the permanent to approach Valiant's Conjecture.

Example A.2.1. *Depth 3 circuits are used to compute a polynomial of the form*

$$\sum_{i=1}^{r} \prod_{j=1}^{d} l_j^i,$$

where l_j^i *is an affine linear form. Depth 3 circuits are also called* $\Sigma\Pi\Sigma$ *circuits.*

Example A.2.2. *There is a particular type of depth 5 circuits called* $\Sigma\Lambda\Sigma\Lambda\Sigma$ *circuits, which are used to compute a polynomial of the form*

$$\sum_{i=1}^{r}(\sum_{j=1}^{\rho}(l_j^i)^{\delta_2})^{\delta_1},$$

where l_j^i *is an affine linear form.*

A circuit is *homogeneous* if the polynomial produced by each gate is homogeneous, and otherwise it is *inhomogeneous.*

Remark A.2.3. *A polynomial of the form* $\sum_{i=1}^{r}\prod_{j=1}^{d} l_j^i$, *where* l_j^i *is a linear form, is a general point in the variety* $\sigma_r(Ch_d(V))$, *so the variety* $\sigma_r(Ch_d(V))$ *is associated to homogeneous* $\Sigma\Pi\Sigma$ *circuits. While a polynomial of the form* $\sum_{i=1}^{r}(\sum_{j=1}^{\rho}(l_j^i)^{\delta_2})^{\delta_1}$, *where* l_j^i *is a linear form, is a general point in the variety* $\sigma_\rho(v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}V)))$, *so the variety* $\sigma_\rho(v_{\delta_1}(\sigma_r(v_{\delta_2}(\mathbb{P}V)))$ *is associated to homogeneous* $\Sigma\Lambda\Sigma\Lambda\Sigma$ *circuits.*

A.3 Depth reduction

The following theorem is related to depth reduction, it combines results of [1, 2, 23, 30, 32, 45].

Theorem A.3.1. *Let* $d = n^{O(1)}$ *and let* $P \in S^d\mathbb{C}^n$ *be a polynomial that can be computed by a circuit of size* s. *Then:*

1. P *is computable by a* $\Sigma\Pi\Sigma$ *circuit of size* $2^{O(\sqrt{d\log(n)\log(ds)})}$. *In particular,* $[l^{N-d}P] \in \sigma_r(Ch_N(\mathbb{C}^{n+1}))$ *with* $rN = 2^{O(\sqrt{d\log(n)\log(ds)})}$.

2. P *is computable, for some* $\delta \simeq \sqrt{d}$, *by a homogeneous* $\Sigma\Lambda\Sigma\Lambda\Sigma$ *circuit of size* $2^{O(\sqrt{d\log(n)\log(ds)})}$. *In particular,* $[P] \in \sigma_{r_1}(v_{\frac{d}{\delta}}(\sigma_{r_2}(v_\delta(\mathbb{P}^{n-1}))))$ *with* $r_1r_2(\delta+1) = 2^{O(\sqrt{d\log(n)\log(ds)})}$.

Remark A.3.2. *Theorem A.3.1 implies Theorems1.1.1 and 1.1.2.*