Math 622 Exam 1 practice problems, 2/29/12

J.M. Landsberg

You may use 2 sheets of handwritten notes - no photocopies. All answers must be justified or no credit

Let $\mathbb{R}^n$ be equipped with its standard inner-product: for column vectors v, w, $\langle v, w\rangle = v^T w$ where T denotes transpose. Let

$$CO(n) = \{f : \mathbb{R}^n \to \mathbb{R}^n \mid f \text{ linear}, \ \langle f(v), f(w)\rangle = \lambda_f \langle v, w\rangle, \text{ for some } \lambda_f \in \mathbb{R}_+\}.$$

1. Show $CO(n)$ is a differentiable manifold.
2. Show $CO(n)$ is a group under composition of mappings.
3. Show $CO(n)$ is a Lie group
4. Calculate $T_{Id}CO(n)$
5. Let $\alpha = wzdx + dy + y^2 dz$ and $\beta = wdz$ be 1 forms on $\mathbb{R}^4$. Is it true that for all $p \in \mathbb{R}^4$, there exists an open subset $U \subset \mathbb{R}^2$ and an immersion $f : U \to \mathbb{R}^4$, with $p \in f(U)$, such that $f^*(\alpha) = 0$ and $f^*(\beta) = 0$?
6. Let $\beta = wdz$ be a 1-form on $\mathbb{R}^4$. Is it true that for all $p \in \mathbb{R}^4$, there exists an open subset $U \subset \mathbb{R}^3$ and an immersion $f : U \to \mathbb{R}^4$, with $p \in f(U)$, such that $f^*(\beta) = 0$?
7. Let G be an abelian Lie group. Show that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$, the Lie algebra of G.
8. Does there exist a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_{S^n} \equiv 1$, $f|_{B(\frac{7}{8})} = 0$ and $f|_{\mathbb{R}^n \backslash B(\frac{8}{7})} = 0$? Here S^n is the unit sphere and $B(r)$ is the ball of radius r.